THE HECKE GROUP ALGEBRA OF A COXETER GROUP
AND ITS REPRESENTATION THEORY

FLORENT HIVERT AND NICOLAS M. THIÉRY

Abstract. Let $W$ be a finite Coxeter group. We define its Hecke-group algebra by gluing together appropriately its group algebra and its 0-Hecke algebra. We describe in detail this algebra (dimension, several bases, conjectural presentation, combinatorial construction of simple and indecomposable projective modules, Cartan map) and give several alternative equivalent definitions (as symmetry preserving operator algebra, as poset algebra, as commutant algebra, ...).

In type $A$, the Hecke-group algebra can be described as the algebra generated simultaneously by the elementary transpositions and the elementary sorting operators acting on permutations. It turns out to be closely related to the monoid algebras of respectively nondecreasing functions and nondecreasing parking functions, the representation theory of which we describe as well.

This defines three towers of algebras, and we give explicitly the Grothendieck algebras and coalgebras given respectively by their induction products and their restriction coproducts. This yields some new interpretations of the classical bases of quasi-symmetric and noncommutative symmetric functions as well as some new bases.

Contents

1. Introduction 2
2. Background 4
2.1. Compositions and sets 4
2.2. Coxeter groups and Iwahori-Hecke algebras 4
2.3. Representation theory of the 0-Hecke algebra 5
2.4. Representation theory of the 0-Hecke algebra in type $A$ 6
3. The algebra $\mathcal{H}W$ 6
3.1. Basic properties 8
3.2. $\mathcal{H}W$ as algebra of antisymmetry-preserving operators 10
3.3. Representation theory 12
4. The algebra of non-decreasing functions 20
4.1. Representation on exterior powers, and link with $\mathcal{H}\mathfrak{S}_n$ 20
4.2. Representation theory 22
5. The algebra of non-decreasing parking functions 25
5.1. Representation theory 25
6. Alternative constructions of $\mathcal{H}\mathfrak{S}_n$ in type A 27
7. Research in progress 28
References 29

2000 Mathematics Subject Classification. Primary 16G99; Secondary 05E05, 20C08.

Key words and phrases. representation theory, towers of algebras, Grothendieck groups, Coxeter groups, Hecke algebras, quasi-symmetric and noncommutative symmetric functions.
1. Introduction

Given an inductive tower of algebras, that is a sequence of algebras
\[ A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_n \hookrightarrow \cdots, \]
with embeddings \( A_m \otimes A_n \hookrightarrow A_{m+n} \) satisfying an appropriate associativity condition, one can introduce two Grothendieck rings
\[ G(A) := \bigoplus_{n \geq 0} G_0(A_n) \quad \text{and} \quad K(A) := \bigoplus_{n \geq 0} K_0(A_n), \]
where \( G_0(A) \) and \( K_0(A) \) are the (complexified) Grothendieck groups of the categories of finite-dimensional \( A \)-modules and projective \( A \)-modules respectively, with the multiplication of the classes of an \( A_m \)-module \( M \) and an \( A_n \)-module \( N \) defined by the induction product
\[ [M] \cdot [N] = [\hat{M} \otimes \hat{N}] = [M \otimes N \uparrow_{A_m \otimes A_n}^{A_{m+n}}], \]
where \( \hat{M} \) and \( \hat{N} \) are the (complexified) Grothendieck groups of the categories of finite-dimensional \( A \)-modules and projective \( A \)-modules respectively.

If \( A_{m+n} \) is a projective \( A_m \otimes A_n \)-module, one can define a coproduct on these rings by means of restriction of representations, turning these into coalgebras. Under favorable circumstances the product and the coproduct are compatible turning these into mutually dual Hopf algebras.

The basic example of this situation is the character ring of the symmetric groups (over \( \mathbb{C} \)), due to Frobenius. Here the \( A_n := \mathbb{C}[S_n] \) are semi-simple algebras, so that
\[ G_0(A_n) = K_0(A_n) = R(A_n), \]
where \( R(A_n) \) denotes the vector space spanned by isomorphism classes of indecomposable modules which, in this case, are all simple and projective. The irreducible representations \( [\lambda] \) of \( A_n \) are parametrized by partitions \( \lambda \) of \( n \), and the Grothendieck ring is isomorphic to the algebra \( \text{Sym} \) of symmetric functions under the correspondence \( [\lambda] \leftrightarrow s_\lambda \), where \( s_\lambda \) denotes the Schur function associated with \( \lambda \). Other known examples with towers of group algebras over the complex numbers \( A_n := \mathbb{C}[G_n] \) include the cases of wreath products \( G_n := \Gamma \ltimes S_n \) (Specht), finite linear groups \( G_n := \text{GL}(n, \mathbb{F}_q) \) (Green), etc., all related to symmetric functions (see [Mac95, Zel81]).

Examples involving non-semisimple specializations of Hecke algebras have also been worked out. Finite Hecke algebras of type \( A \) at roots of unity \( (A_n = H_n(\zeta), \zeta^r = 1) \) yield quotients and subalgebras of \( \text{Sym} \) [LLT96]. The Ariki-Koike algebras at roots of unity give rise to level \( r \) Fock spaces of affine Lie algebras of type \( A \) [AK94]. The 0-Hecke algebras \( A_n = H_n(0) \) corresponds to the pair Quasi-symmetric functions / Noncommutative symmetric functions, \( G = \text{QSym}, K = \text{NCSF} \) [KT97]. Affine Hecke algebras at roots of unity lead to the positive halves of the enveloping algebras \( U(\hat{sl}_r) \) and \( U(\hat{sl}_r)^* \) [Ari96], and the case of generic affine Hecke algebras can be reduced to a subcategory admitting as Grothendieck rings the positive halves of the enveloping algebras \( U(\hat{gl}_\infty) \) and \( U(\hat{gl}_\infty)^* \) [Ari96]. Further interesting examples are the tower of 0-Hecke-Clifford algebras [Ols92, BHT04] giving rise to the peak algebras [Ste97], and a degenerated version of the Ariki-Koike algebras [HNT06] giving rise to a colored version of \( \text{QSym} \) and \( \text{NCSF} \).

The original aim of this research was the study of the representation theories of several towers of algebras related to the symmetric groups and their Hecke algebras \( H_n(q) \), in order to derive new examples of Grothendieck algebras and coalgebras.
Along the way, one of these algebras became interesting in itself and for all types. Here is the structure of the paper together with the main results.

In Section 3, we introduce the main object of this paper, namely the Hecke-group algebra $H_W$ of a (finite) Coxeter group $W$. It is constructed as the smallest algebra containing simultaneously the group algebra of $W$ and its 0-Hecke algebra (and in fact any other $q$-Hecke algebra of $W$). It turns out that this algebra has unexpectedly nice properties. We first show that $H_W$ is better understood as the algebra of antisymmetry (or symmetry) preserving operators; this allows us to compute its dimension, and to give explicit bases. We further realize it as the incidence algebra of a pre-order and derive from it its representation theory. In particular, we construct explicitly the projective and simple modules. The Cartan matrix suggests a link between $H_W$ and the incidence algebra of the boolean lattice. We actually show that these algebras are Morita equivalent.

Turning back to type $A$, we get a new tower of algebras $H_{S_n}$. Specifically, each $H_{S_n}$ is the algebra generated by both elementary transpositions and elementary sorting operators acting on permutations of $\{1, \ldots, n\}$. We compute the restrictions and inductions of simple and projective modules. This gives rise to a new interpretation of some bases of quasi-symmetric and noncommutative symmetric functions in representation theory.

In Sections 4 and 5 we turn to the study of two other towers, namely the towers of the monoid algebras of nondecreasing functions $C[NDF_n]$ and of nondecreasing parking functions $C[NDPF_n]$. We prove that those two algebras are the respective quotients of $H_{S_n}$ and $H_n(0)$, through their representations on exterior powers. We deduce the structure of their projective and simple modules, their Cartan matrices, and the induction and restriction rules. We also show that the algebra of nondecreasing parking functions is isomorphic to the incidence algebra of some lattice.

The following diagram summarizes the relations between all the aforementioned towers of algebras, as well as the Temperley-Lieb algebra $TL_n(-1)$ at $q = -1$ and the respective quotients $C[S_n] \otimes \bigwedge^n C^n$ and $H_n(q) \otimes \bigwedge^n C^n$ of $C[S_n]$ and $H_n(q)$ through their representations on exterior powers:

$$
\begin{array}{cccccc}
H_n(-1) & \rightarrow & H_n(0) & \rightarrow & H_n(1) = C[S_n] & \rightarrow & H_n(q) \rightarrow H \otimes S_n \\
C[NDF_n] & \rightarrow & C[S_n] & \rightarrow & C[S_n] \otimes \bigwedge^n C^n & \rightarrow & C[NDF_n]
\end{array}
$$

Finally, in Section 7, we discuss further research in progress, in particular about the links between $H_W$ and the affine Hecke algebras.

Acknowledgments. We would like to thank Jean-Yves Thibon who is at the origin of the present research, and for numerous fruitful discussions.

This paper mostly reports on computation driven research using the package MuPAD-Combinat by the authors of the present paper [HT04]. This package is designed for the computer algebra system MuPAD and is freely available from http://mupad-combinat.sf.net/. Among other things, it allows to automatically compute the dimensions of simple and indecomposable projective modules together with the
matrix of Cartan invariants of a finite dimensional algebra, knowing its multiplication table. We also would like to thank the MuPAD group for their support in the development of MuPAD-Combinat.

2. Background

2.1. Compositions and sets. Let $n$ be a fixed integer. Recall that each subset $S$ of $\{1, \ldots, n-1\}$ can be uniquely identified with a $p$-tuple $I := (i_1, \ldots, i_p)$ of positive integers of sum $n$:

\begin{equation}
S = \{s_1 < s_2 < \cdots < s_p\} \mapsto C(S) := (s_1, s_2 - s_1, s_3 - s_2, \ldots, n - s_p).
\end{equation}

We say that $I$ is a composition of $n$ and we write it by $I \vdash n$. The converse bijection, sending a composition to its descent set, is given by:

\begin{equation}
I = (i_1, \ldots, i_p) \mapsto \text{Des}(I) = \{i_1 + \cdots + i_j \mid j = 1, \ldots, p-1\}.
\end{equation}

The number $p$ is called the length of $I$ and is denoted by $\ell(I)$.

The notions of complement of a set $S^c$ and of inclusion of sets can be transfered to compositions, leading to the complement of a composition $K^c$ and to the refinement order on compositions: we say that $I$ is finer than $J$, and write $I \preceq J$, if and only if $\text{Des}(I) \supseteq \text{Des}(J)$.

2.2. Coxeter groups and Iwahori-Hecke algebras. Let $(W, S)$ be a Coxeter group, that is a group $W$ with a presentation

\begin{equation}
W = \langle S \mid (ss')^{m(s,s')}, \forall s, s' \in S \rangle,
\end{equation}

where each $m(s, s')$ is in $\{1, 2, \ldots, \infty\}$, and $m(s, s) = 1$. The elements $s \in S$ are called simple reflections, and the relations can be rewritten as:

\begin{align}
s^2 &= \text{id}, & \text{for all } s \in S, \\
\frac{s's's's \cdots}{m(s,s')} &= \frac{s's's's \cdots}{m(s,s')}, & \text{for all } s, s' \in S,
\end{align}

Most of the time, we just write $W$ for $(W, S)$. In general, we follow the notations from [BB05], and we refer to this monograph for details on Coxeter groups and their Hecke algebras. Unless stated otherwise, we always assume that $W$ is finite.

The prototypical example of Coxeter group is the $n$-th symmetric group $(W, S) := A_{n-1} = (S_n, \{s_1, \ldots, s_{n-1}\})$, where $s_i$ denotes the elementary transposition which exchanges $i$ and $i+1$. The relations are given by:

\begin{align}
s^2_i &= \text{id}, & \text{for } 1 \leq i \leq n-1, \\
s_is_j &= s_js_i, & \text{for } |i - j| \geq 2, \\
\text{s_is_i+1s_i} &= s_{i+1}s_is_{i+1}, & \text{for } 1 \leq i \leq n-2,
\end{align}

the last two relations are called the braid relations. When we want to write explicitly a permutation $\mu$ in $S_n$, we will use the one line notation, that is the sequence $\mu_1\mu_2\cdots\mu_n := \mu(1)\mu(2)\cdots\mu(n)$.

A reduced word for an element $\mu$ of $W$ is a decomposition $\mu = s_1 \cdots s_k$ of $\mu$ into a product of generators in $S$ of minimal length $l(\mu)$. A (right) descent of $\mu$ is an element $s \in S$ such that $l(\mu s) < l(\mu)$. If $\mu$ is a permutation, this translates into $\mu_i > \mu_{i+1}$. A recoil (or left descent) of $\mu$ is a descent of $\mu^{-1}$. The sets of left and right descents of $\mu$ are denoted respectively by lDes$(\mu)$ and Des$(\mu)$. The Coxeter group $W$ comes equipped with three natural lattice structures. Namely $\mu < \nu$, in
the Bruhat order (resp. (weak) left Bruhat order, resp. (weak) right Bruhat order) if some reduced word for $\mu$ is a subword (resp. a right factor, resp. left factor) of some reduced word for $\nu$. In type $A$, the weak Bruhat orders are the usual left and right permutohedron.

For a subset $I$ of $S$, the parabolic subgroup $W_I$ of $W$ is the Coxeter subgroup of $W$ generated by $I$. Left and right cosets representatives for the quotient of $W$ by $W_I$ are given respectively by the recoil class
\begin{equation}
I^W = \{ \mu \in W \mid i\text{Des}(\mu) \cap I = \emptyset \}
\end{equation}
and the descent class
\begin{equation}
W^I = \{ \mu \in W \mid \text{Des}(\mu) \cap I = \emptyset \}.
\end{equation}

For $q$ a complex number, let $\mathcal{H}(W)(q)$ be the (Iwahori-)Hecke algebra of $W$ over the field $\mathbb{C}$. This algebra of dimension $|W|$ has a linear basis $\{T_\mu\}_{\mu \in W}$, and its multiplication is determined by
\begin{equation}
\begin{cases}
T_s T_\mu = (q-1)T_{sw} + (q-1)T_w & \text{if } s \in S \text{ and } \ell(s\mu) < \ell(\mu), \\
T_s T_\nu = T_{s\nu} & \text{if } \ell(\mu) + \ell(\nu) = \ell(\mu\nu).
\end{cases}
\end{equation}

In particular, for any element $\mu$ of $W$, we have $T_\mu = T_{s_1}\cdots T_{s_k}$ where $s_1, \ldots, s_k$ is any reduced word for $\mu$. In fact, $\mathcal{H}(W)(q)$ is the algebra generated by the elements $T_s, s \in S$ subject to the same relations as the $s$ themselves, except that the quadratic relation $s^2 = 1$ is replaced by:
\begin{equation}
T_s^2 = (q-1)T_s + q.
\end{equation}

Setting $q = 1$ yields back the usual group algebra $\mathbb{C}[W]$ of $W$. Similarly, the 0-Hecke algebra $\mathcal{H}(W)(0)$ is obtained by setting $q = 0$ in these relations. Then, the first relation becomes $T_s^2 = -T_s$ [Nor79, KT97]. In this paper, we prefer to use another set of generators $\{\pi_s\}_{s \in S}$ defined by $\pi_i := T_i + 1$. They also satisfy the braid-like relations together with the quadratic relations $\pi_s^2 = \pi_s$. Note that the 0-Hecke algebra is thus a monoid algebra.

2.3. Representation theory of the $0$-Hecke algebra. In this paper, we mostly consider right modules over algebras. Consequently the composition of two endomorphisms $f$ and $g$ is denoted by $fg = g \circ f$ and their action on a vector $v$ is written $v \cdot f$. Thus $g \circ f(v) = g(f(v))$ is denoted $v \cdot fg = (v \cdot f) \cdot g$.

Assume now that $W$ is finite. It is known that $\mathcal{H}(W)(0)$ has $2^{|S|}$ simple modules, all one-dimensional, and naturally labelled by subsets $I$ of $S$ [Nor79]: following the notation of [KT97], let $\eta_I$ be the generator of the simple $\mathcal{H}(W)(0)$-module $S_I$ associated with $I$ in the right regular representation. It satisfies
\begin{equation}
\eta_I \cdot T_i := \begin{cases} 
-\eta_I & \text{if } i \in I, \\
0 & \text{otherwise},
\end{cases}
\end{equation}
or equivalently
\begin{equation}
\eta_I \cdot \pi_i := \begin{cases} 
0 & \text{if } i \in I, \\
\eta_I & \text{otherwise}.
\end{cases}
\end{equation}

The indecomposable projective module $P_I$ associated with $S_I$ (that is such that $S_I = P_I/\text{rad}(P_I)$) can be described as follows: it has a basis $\{b_\mu \mid \text{iDes}(\mu) = I\}$ with the action
\begin{equation}
b_\mu \cdot T_s = \begin{cases} 
-b_\mu & \text{if } s \in \text{Des}(\mu), \\
b_\mu s & \text{if } s \notin \text{Des}(\mu) \text{ and } \text{iDes}(\mu s) = I, \\
0 & \text{otherwise}.
\end{cases}
\end{equation}
2.4. Representation theory of the 0-Hecke algebra in type A. In type A, it is customary to index the projective and simple modules of \( \mathcal{H}_n(0) \) by compositions of \( n \). For notational convenience, and when there is no ambiguity, we simply identify the subset \( I \) of \( S = \{1, \ldots, n-1\} \) and the corresponding composition \( C(I) = (i_1, \ldots, i_k) \) of \( n \).

The Grothendieck rings of \( \mathcal{H}_n(0) \) are naturally isomorphic to the dual pair of Hopf algebras of quasi-symmetric functions \( \text{QSym} \) of Gessel [Ges84] and of noncommutative symmetric functions \( \text{NCSF} \) [GKL+95] (see [KT97]). The reader who is not familiar with those should refer to these papers, as we will only recall the required notations here.

The Hopf algebra \( \text{QSym} \) of quasi-symmetric functions has two remarkable bases, namely the monomial basis \( (M_I)_I \) and the fundamental basis (also called quasi-ribbon) \( (F_I)_I \). They are related by

\[(17) \quad F_I = \sum_{I \succeq J} M_J \quad \text{or equivalently} \quad M_I = \sum_{I \succeq J} (-1)^{i(I) - i(J)} F_J.\]

The characteristic map \( S_I \mapsto F_I \) which sends the simple \( \mathcal{H}_n(0) \)-module \( S_I \) to its corresponding fundamental function \( F_I \) also sends the induction product to the product of \( \text{QSym} \) and the restriction coproduct to the coproduct of \( \text{QSym} \).

The Hopf algebra \( \text{NCSF} \) of noncommutative symmetric functions [GKL+95] is a noncommutative analogue of the algebra of symmetric functions [Mac95]. It has for multiplicative bases the analogues \( (\Lambda^I)_I \) of the elementary symmetric functions \( (e_\lambda)_\lambda \) and as well as the analogues \( (S^I)_I \) of the complete symmetric functions \( (h_\lambda)_\lambda \). The relevant basis in the representation theory of \( \mathcal{H}_n(0) \) is the basis of so called ribbon Schur functions \( (R_I)_I \) which is an analogue of skew Schur functions of ribbon shape. It is related to \( (\Lambda_I)_I \) and \( (S_I)_I \) by

\[(18) \quad S_I = \sum_{I \succeq J} R_J \quad \text{and} \quad \Lambda_I = \sum_{I \succeq J} R_J.\]

Their interpretation in representation theory goes as follows. The complete function \( S^n \) is the characteristic of the trivial module \( S_n \approx P_n \), the elementary function \( \Lambda^n \) being the characteristic of the sign module \( S_1^n \approx P_1^n \). An arbitrary indecomposable projective module \( P_I \) has \( R_I \) for characteristic. Once again the map \( P_I \mapsto R_I \) is an isomorphism of Hopf algebras.

Recall that \( S_I \) is the semi-simple module associated to \( P_I \), giving rise to the duality between \( \mathcal{G} \) and \( \mathcal{K} : \)

\[(19) \quad S_I = P_I/\text{rad}(P_I) \quad \text{and} \quad \langle P_I, S_J \rangle = \delta_{I,J}.\]

This translates into \( \text{QSym} \) and \( \text{NCSF} \) by setting that \( (F_I)_I \) and \( (R_I)_I \) are dual bases, or equivalently that \( (M_I)_I \) and \( (S^I)_I \) are dual bases.

3. The algebra \( \mathcal{H}W \)

Let \( (W, S) \) be a finite Coxeter group. Its group algebra \( \mathbb{C}[W] \) and its 0-Hecke algebra \( \mathcal{H}(W)(0) \) can be realized simultaneously as operator algebras by identifying the underlying vector spaces of their right regular representations. There are several ways to do that, depending on which basis elements of \( \mathcal{H}(W)(0) \) we choose to identify with elements of \( W \). It turns out that the following identification leads to interesting properties.
Namely, consider the plain vector space $CW$. On the first hand, we identify $CW$ with the right regular representation of the algebra $\mathbb{C}[W]$, i.e.: $\mathbb{C}[W]$ acts on $CW$ by multiplication on the right. In type $A$, this is the usual action on positions, where an elementary transposition $s_i$ acts on a permutation $\mu := (\mu_1, \ldots, \mu_n)$ by exchanging $\mu_i$ and $\mu_{i+1}$: $\mu \cdot s_i = \mu s_i$.

On the other hand, we also identify $CW$ with the right regular representation of the 0-Hecke algebra $\mathcal{H}(W)(0)$, i.e.: $\mathcal{H}(W)(0)$ acts on the right on $CW$ by

$$
\mu \cdot \pi_s = \begin{cases} 
\mu & \text{if } \ell(\mu s) < \ell(\mu), \\
\mu s & \text{otherwise}.
\end{cases}
$$

In type $A$, the $\pi_i := \pi_{s_i}$’s are the elementary decreasing bubble sort operators:

$$
\mu \cdot \pi_i = \begin{cases} 
\mu & \text{if } \mu_i > \mu_{i+1}, \\
\mu s_i & \text{otherwise}.
\end{cases}
$$

The following easy lemma will be useful in the sequel.

**Lemma 3.1.** Let $\sigma, \tau \in W$. Then,

(a) There exists $\tau'$ such that $\sigma \cdot \pi_s = \sigma \tau'$ with $\ell(\sigma \tau') = \ell(\sigma) + \ell(\tau')$. Furthermore, $\tau' = 1$ if and only if $\tau \in W_{\text{Des}(\sigma)}$.

(b) There exists $\sigma'$ such that $\sigma \cdot \pi_s = \sigma' \tau$ with $\ell(\sigma' \tau) = \ell(\sigma') + \ell(\tau)$. Furthermore, $\sigma' = 1$ if and only if $\sigma \in W_{\text{Des}(\tau)}$.

**Proof.** Applying $\pi_s$ on an element $\sigma$ either leaves $\sigma$ unchanged if $s \in \text{Des}(\sigma)$, or extends any reduced word for $\sigma$ by $s$ otherwise. (a) follows by induction; in particular $\tau'$ is smaller than $\tau$ in the Bruhat order of $W$.

(b) Since $CW$ is the right regular representation, the linear map

$$
\Phi : \begin{cases} 
\mathbb{C}W & \rightarrow & \mathcal{H}(W)(0) \\
\tau & \mapsto & \pi_\tau
\end{cases}
$$

is a morphism of $\mathcal{H}(W)(0)$-module. Consequently, one has $\pi_{\sigma \cdot \pi_s} = \pi_\sigma \pi_\tau$ which allows us to lift the computation to the 0-Hecke monoid. There $\sigma$ and $\tau$ play a symmetric role, and (b) follows from (a) by reversion of the reduced words. □

**Definition 3.2.** The algebra $\mathcal{H}W$ is the subalgebra of $\text{End}(CW)$ generated by both sets of operators $\{s, \pi_s\}_{s \in S}$.

By construction, the algebra $\mathcal{H}W$ contains both $\mathbb{C}[W]$ and $\mathcal{H}(W)(0)$. In fact, it contains simultaneously all the Hecke algebras: for any values of $q$, $\mathcal{H}(W)(q)$ can be realized by taking the subalgebra of $\mathcal{H}W$ generated by the operators:

$$
T_s := (q-1)(1-\pi_s) + qs, \quad \text{for } s \in S.
$$

A direct calculation shows that the so-defined $T_s$ actually verifies the Hecke relation. Reciprocally, we can recover back $\mathcal{H}W$ by choosing for each $s$ any two generators $T_s(q_1)$ and $T_s(q_2)$ with $q_1 \neq q_2$, because for any $q, q_1, q_2$,

$$
T_s(q) := \frac{q - q_1}{q_2 - q_1} T_s(q_1) + \frac{q - q_2}{q_1 - q_2} T_s(q_2).
$$

Note that setting $T_s(1) := s$ and $T_s(0) := \pi - 1$ this last equation implies the previous one when $q_1 = 1$ and $q_2 = 0$.
Let further \( \pi_s := \pi_s s \) be the operator in \( \mathcal{H}W \) which removes the descent \( s \). In type \( A \), the \( \pi_i := \pi_i s_i \)'s are the \textit{elementary increasing bubble sort operators}:

\[
\mu \cdot \pi_i = \begin{cases} 
\mu & \text{if } \mu_i < \mu_{i+1} , \\
\mu s_i & \text{otherwise}.
\end{cases}
\]

Since \( \pi_s + \pi_s \) is a symmetrizing operator, we have the identity:

\[
\pi_s + \pi_s = 1 + s .
\]

It follows that we can alternatively take as generators for \( \mathcal{H}W \) the operators \( \pi_s \)'s and \( \pi_s \)'s.

In type \( A \), the natural embedding of \( C[S_n] \otimes C[S_m] \) in \( C[S_{n+m}] \) makes \( (H[S_n])_{n \in \mathbb{N}} \) into a tower of algebras, which contains the similar towers of algebras \( (C[S_n])_{n \in \mathbb{N}} \) and \( (H_n(q))_{n \in \mathbb{N}} \).

3.1. Basic properties.

\textbf{Example 3.3.} Much of the structure of \( \mathcal{H}W \) readily appears for \( W := S_2 \). Take the natural basis \( (12, 21) \) of \( C[S_2] \). The matrices of the operators \( 1, s_1, \pi_1, \) and \( \pi_1 \) are respectively:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 \\
1 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix}.
\]

The algebra \( \mathcal{H}S_2 \) is of dimension 3 with basis \( \{1, s_1, \pi_1\} \) and multiplication table:

\[
\begin{array}{ccc}
1 & s_1 & \pi_1 \\
1 & 1 & s_1 & \pi_1 \\
1 & s_1 & 1 & \pi_1 \\
\pi_1 & \pi_1 & \pi_1 = 1 + s_1 - \pi_1 & \pi_1 \\
\end{array}
\]

This algebra can alternatively be described by equations. Namely, take \( f \in \text{End}(C[S_2]) \) with matrix

\[
\begin{pmatrix}
f_{1212} & f_{1221} \\
f_{2112} & f_{2121}
\end{pmatrix};
\]

then, the following properties are equivalent:

- \( f \) belongs to \( \mathcal{H}S_2 \);
- \( f_{2121} - f_{2112} + f_{1221} - f_{1212} = 0 \);
- \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 12 - 21 \) is an eigenvector of \( f \);
- \( (1 - s_1)f(1 + s_1) = 0 \).

In general the relations in the parabolic subalgebra \( C[\pi_i, \pi_i, s_i] \) of \( \mathcal{H}W \) are:

\[
\begin{align*}
\pi_i \pi_i & = \pi_i , & \pi_i \pi_i & = \pi_i , \\
\pi_i \pi_i & = \pi_i , & \pi_i \pi_i & = \pi_i , \\
\pi_i s_i & = \pi_i , & \pi_i s_i & = \pi_i , \\
\pi_i + \pi_i & = 1 + s_i .
\end{align*}
\]

In particular, any two of \( \{\pi_i, \pi_i, s_i\} \) can be taken as generators.
A computer exploration suggests that in type $A$ the dimension of $\mathcal{H}\mathfrak{S}_n$ is given by the following sequence (sequence A000275 of the encyclopedia of integer sequences [Se03]):

$$1, 1, 3, 19, 211, 3651, 90921, 3081513, 136407699, 7642177651, 528579161353, 44237263696473, \ldots$$

These are the numbers $h_n$ of pairs $(\sigma, \tau)$ of permutations without common descents $(\text{Des}(\sigma) \cap \text{Des}(\tau) = \emptyset)$. They were first considered by Carlitz [Car55, CSV76a, CSV76b] as coefficient of the doubly exponential expansion of the inverse Bessel function $J_0$:

$$\sum_{n>0} \frac{h_n}{n!^2} x^n = \frac{1}{J_0(\sqrt{4x})}.$$  

Together with Equation (30), this leads to state the following

**Theorem 3.4.** Let $W$ be a finite Coxeter group. A vector space basis of $\mathcal{H}W$ is given by the family of operators

$$B := \{ \sigma \pi_+ | (\sigma, \pi) \in W^2, \text{ Des}(\sigma) \cap i\text{Des}(\tau) = \emptyset \}.$$  

In particular the dimension of $\mathcal{H}W$ is the number $h$ of pairs of elements of $W$ without common descents.

Our first approach to prove this theorem was to search for a presentation of the algebra. In type $A$, the following relations are easily proved to hold:

$$\begin{align*}
\pi_{i+1}s_i &= \pi_{i+1}\pi_i + s_i\pi_{i+1}\pi_i - \pi_{i+1}\pi_i \pi_i+1, \\
\pi_is_{i+1} &= \pi_i\pi_{i+1} + s_i\pi_{i+1}\pi_i - \pi_i\pi_{i+1}\pi_i, \\
\pi_{i+1}s_i &= s_{i+1}\pi_is_i+1,
\end{align*}$$

and we conjecture that they generate all relations.

**Conjecture 3.5.** A presentation of $\mathcal{H}(\mathfrak{S}_n)$ is given by the defining relations of $\mathfrak{S}_n$ and $\mathcal{H}_n(0)$ together with the relations $s_i\pi_i = \pi_i$ and (33).

Using those relations as rewriting rules yields a straightening algorithm which rewrites any expression in the $s_i$’s and $\pi_i$’s into a linear combination of the $\sigma\pi_\tau$. This algorithm seems, in practice and with an appropriate strategy, to always terminate. However we have no proof of this fact; moreover this algorithm is not efficient, due to the combinatorial explosion of the number and length of words in intermediate results.

**Example 3.6.** In the following computation for $W = \mathfrak{S}_8$, we multiply some element of the Hecke algebra by successive elementary transposition; we use respectively the short hand notation $\pi_{[154]}$ and $\pi_{[154]}$ for the products $s_1s_5s_4$ and $\pi_1\pi_5\pi_4$:

$\begin{align*}
\tau_{[1765432]}\sigma_{[1]} &= \sigma_{[1234567]} \tau_{[675645342312]} - \sigma_{[12345]} \tau_{[7675645342312]} \\
&+ \sigma_{[234567]} \tau_{[675645342312]} - \sigma_{[2345]} \tau_{[7675645342312]} + \sigma_{[2]} \tau_{[76543212]} \\
\tau_{[1765432]}\sigma_{[13]} &= \sigma_{[345672345612345]} \tau_{[67564567345623451234]} - \sigma_{[21345]} \tau_{[7675645342312]} \\
&+ \sigma_{[21]} \tau_{[76543212]} - \sigma_{[34523412345]} \tau_{[67564567345623451234]} \\
&+ \sigma_{[2345671]} \tau_{[675645342312]} - \sigma_{[3452345671234]} \tau_{[67564567345623451234]} \\
&+ \sigma_{[345672345612345]} \tau_{[67564567345623451234]} - \sigma_{[23451]} \tau_{[7675645342312]} \\
&- \sigma_{[345672345612345]} \tau_{[767564567345623451234]} + \sigma_{[21]} \tau_{[76543212]} \\
&+ \sigma_{[34523412345]} \tau_{[767564567345623451234]} + \sigma_{[2]} \tau_{[76543212]} - \sigma_{[34523412345]} \tau_{[7675645342312]} \\
\tau_{[1765432]}\sigma_{[135]} &= \sigma_{[3467456345623456712345]} \tau_{[67564567345623451234]} + 38 \text{ shorter terms} \\
\tau_{[1765432]}\sigma_{[1357]} &= \sigma_{[7654657435623456712345]} \tau_{[7654657435623456712345]} + 116 \text{ shorter terms}
\end{align*}$
Encountering those difficulties does not come as a surprise. The properties of such algebras often become clearer when considering their concrete representations (typically as operator algebras) rather than their abstract presentation. Here, theorem 3.4 as well as the representation theory of $\mathcal{H}W$ follow from the structural characterization of $\mathcal{H}W$ as the algebra of operators preserving certain antisymmetries to be explained below in Section 3.2.

3.1.1. Variants. As mentioned previously, the original goal of the definition of $\mathcal{H}W$ was to put together a Coxeter group $W$ and its 0-Hecke algebra $\mathcal{H}(W)(0)$. Identifying their right regular representation on the canonical basis is just one possible mean. We explore quickly here some variants, and mention alternative constructions of $\mathcal{H}W$ (mostly in type $A$) in Sections 6 and 7.

A first variant is to still consider the right regular actions of $W$ and of $\mathcal{H}(W)(0)$ but this time on $W$ itself. In other words, to consider the monoid $\langle s, \pi_s \rangle_{s \in S}$ generated by the operators $s$ and $\pi_s$. In type $A$, the sizes of those monoids for $n = 1, 2, 3, 4$ are $1, 4, 66, 6264$, which are strictly bigger than the corresponding dimensions of $\mathcal{H}\mathfrak{S}_n$, in particular because we lose the linear relations $1 + \pi_s s = 1 + s$. Incidentally, an interesting question is to find a presentation of this monoid. If instead one takes the monoid $\langle \pi_s, \pi_s \rangle$, the sizes are $1, 3, 23, 477$.

Another natural approach, in type $A$, is to start from the usual action of $\mathfrak{S}_n$ on the ring of polynomials $\mathbb{C}[x_1, \ldots, x_n]$ together with the action of $\mathcal{H}_n(0)$ by isobaric divided differences (see [Las03]). Note that the divided differences being symmetrizing operators, this is in fact more a variant on the adjoint algebra of $\mathcal{H}\mathfrak{S}_n$ (see next section). Again the obtained algebras are bigger: $1, 3, 20, 254, \ldots$, in particular because we lose the two first relations of Equation (33).

3.2. $\mathcal{H}W$ as algebra of antisymmetry-preserving operators. Let $\overrightarrow{s}$ be the right operator in $\text{End}(\mathbb{C}W)$ describing the action of $s_i$ by multiplication on the left (action on values in type $A$). Namely $\overrightarrow{s}$ is defined by

$$\sigma \cdot \overrightarrow{s} := s \sigma.$$

A vector $v$ in $\mathbb{C}W$ is left $s$-symmetric (resp. antisymmetric) if $v \cdot \overrightarrow{s} = v$ (resp. $v \cdot \overrightarrow{s} = -v$). The subspace of left $s$-symmetric (resp. antisymmetric) vectors can be alternatively described as the image (resp. kernel) of the quasi-idempotent (idempotent up to a scalar) operator $1 + \overrightarrow{s}$, or as the kernel (resp. image) of the quasi-idempotent operator $1 - \overrightarrow{s}$.

**Theorem 3.7.** $\mathcal{H}W$ is the subspace of $\text{End}(\mathbb{C}W)$ defined by the $|S|$ idempotent sandwich equations:

$$(1 - \overrightarrow{s}) f (1 + \overrightarrow{s}) = 0, \quad \text{for } s \in S.$$

In other words, $\mathcal{H}W$ is the subalgebra of those operators in $\text{End}(\mathbb{C}W)$ which preserve left antisymmetries.

Note that, $\overrightarrow{s}$ being self-adjoint for the canonical scalar product of $\mathbb{C}W$ (making $W$ into an orthonormal basis), the adjoint algebra of $\mathcal{H}W$ satisfies the equations:

$$(1 + \overrightarrow{s}) f (1 - \overrightarrow{s}) = 0, \quad \text{for } s \in S;$$

thus, it is the subalgebra of those operators in $\text{End}(\mathbb{C}W)$ which preserve left symmetries. Furthermore the group algebra $\mathbb{C}[W]$ of $W$ can be described as the subalgebra of those operators in $\text{End}(\mathbb{C}W)$ which preserve both left symmetries and antisymmetries; it is therefore the intersection of $\mathcal{H}W$ and its adjoint $\mathcal{H}W^\ast$. 

Proof of theorems 3.4 and 3.7. We proceed using three lemmas that occupy the rest of this subsection. We first exhibit a triangularity property of the operators in $B$; this proves that they are linearly independent, so that $\dim \mathcal{H}W \geq h$ (lemma 3.8). Then we prove that the operators in $\mathcal{H}W$ preserve all left antisymmetries (lemma 3.9). Finally we extract from the sandwich equations $\dim \text{End}(CW) - h$ independent linear forms which are annihilated by all left antisymmetry preserving operators in $\text{End}(CW)$ (lemma 3.10). Altogether, it follows simultaneously that $\mathcal{H}W$ has dimension $h$ with $B$ as basis, and that $\mathcal{H}W$ is the full subspace of left antisymmetry preserving operators. □

Let $<$ be any linear extension of the right Bruhat order on $W$. Given an endomorphism $f$ of $CW$, we order the rows and columns of its matrix $M := [f_{\mu\nu}]$ according to $<$, and denote by $\text{init}(f) := \min\{\mu \mid \exists \nu, f_{\mu\nu} \neq 0\}$ the index of the first non zero row of $M$.

**Lemma 3.8.** (a) Let $f := \sigma\pi\tau$ in $B$. Then, $\text{init}(f) = \tau$, and

\[
 f_{\tau\nu} = \begin{cases} 1 & \text{if } \nu \in W_i\text{Des}(\tau)\sigma^{-1}, \\ 0 & \text{otherwise}. \end{cases}
\]

(b) The family $B$ is free.

**Proof.** (a) is a direct corollary of Lemma 3.1 (b).

(b) follows by triangularity: the operator $\sigma\pi\tau$ has coefficient $m_{\tau\sigma^{-1}} = 1$, whereas any other operator $\sigma'\pi'\tau$ such that $D(\sigma') \cap R(\tau) = \emptyset$, $\tau' \leq \tau$ and $\sigma' \neq \sigma$ has coefficient $m_{\tau\sigma^{-1}} = 0$. □

**Lemma 3.9.** The operators in $\mathcal{H}W$ preserve all left antisymmetries.

**Proof.** It is sufficient to prove that the generators $s$ and $\pi_s$ of $\mathcal{H}W$ preserve any left antisymmetry. For a generator $s$, this is obvious since the actions of $S$ and $s$ commute. Let now $v$ be an $s'$-antisymmetric vector; without loss of generality, we may assume that $v = (1 - s')\sigma$ where $\sigma$ is some permutation without recoil at position $s'$. We use the same linear isomorphism $\Phi$ as in lemma 3.1 to lift the computation to the 0-Hecke algebra and use its associativity:

\[
 v \cdot \pi_s = \Phi^{-1}(((1 - \pi_{s'})\pi_\sigma)\pi_s) \\
 = \Phi^{-1}((1 - \pi_{s'})\pi_\sigma)\pi_s \\
 = \begin{cases} 0 & \text{if } s \in i\text{Des}(\sigma \cdot \pi_s), \\ (1 - s')\pi_\sigma & \text{otherwise}. \end{cases}
\]

Therefore, $v \cdot \pi_s$ is again $s'$-antisymmetric. □

We now turn to the explicit description of the sandwich equations. Given an endomorphism $f$ of $CW$, denote by $(f_{\mu\nu})_{\mu\nu}$ the coefficients of its matrix in the natural group basis. Given two elements $\mu, \nu$ in $W$ and a simple reflection $s \in S$, let $R_{\mu,\nu,i}$ be the linear form on $\text{End}(CW)$ which computes the $(\mu, \nu)$ coefficient of the matrix of $(1 - \frac{s}{s'})f(1 + \frac{s}{s'})$:

\[
 R_{\mu,\nu,i}(f) := ((1 - \frac{s}{s'})f(1 + \frac{s}{s'}))_{\mu,\nu} = f_{\mu,\nu} - f_{\mu,\nu} + f_{\mu,\nu} - f_{\mu,\nu}.
\]

By construction, $R_{\mu,\nu,i}$ annihilates any operator which preserves $s$-antisymmetries.
Given a pair \((\mu, \nu)\) of elements of \(W\) having at least one descent in common, set 
\[ R_{\mu,\nu} = R_{\mu,\nu,s} \]
where \(s\) is the smallest common descent of \(\mu\) and \(\nu\) (the choice of the common descent \(s\) is, in fact, irrelevant for our purposes). Finally, let

\[ R := \{ R_{\mu,\nu} \mid \text{Des}(\mu) \cap \text{Des}(\nu) \neq \emptyset \} . \]

For example in type \(A\) we have:

\begin{align*}
\text{for } n = 1: & \quad R = \{ \} \\
\text{for } n = 2: & \quad R = \{ R_{21,21,1} \} = \{ f_{21,21} - f_{21,12} + f_{12,21} - f_{12,12} \} \\
\text{for } n = 3: & \quad R = \left\{ R_{213,213,1,1}, R_{213,312,1,1}, R_{213,321, \ldots, R_{321,321,1}} \right\} \\
\end{align*}

For \(n = 2\), the reader will recognize the linear relation described in example 3.3.

**Lemma 3.10.** The \(|W|^2 - h\) linear forms in \(R\) are linearly independent.

**Proof.** Take some linear form \(R_{\mu,\nu} = R_{\mu,\nu,i}\) in \(R\), and represent it as the \(|W| \times |W|\) array of its values on the elements of the canonical basis of \(\text{End}(W)\), with the rows and columns sorted as previously. For example, here is the array for \(R_{213,312} = R_{213,312,1,1}\) in type \(A_2\):

\[
\begin{array}{cccccc}
123 & 132 & 213 & 231 & 312 & 321 \\
0 & -1 & 0 & 0 & 1 & 0 & 123 \\
0 & 0 & 0 & 0 & 0 & 0 & 132 \\
0 & -1 & 0 & 0 & 1 & 0 & 213 \\
0 & 0 & 0 & 0 & 0 & 0 & 231 \\
0 & 0 & 0 & 0 & 0 & 0 & 312 \\
0 & 0 & 0 & 0 & 0 & 0 & 321 \\
\end{array}
\]

This array has a coefficient 1 at position \((\mu, \nu)\). Since \(s\) is both a descent of \(\mu\) and \(\nu\), \(s\mu < \mu\) and \(sv < \nu\); so the three other non-zero coefficients are either strictly higher or strictly to the left in the array. Furthermore, no other linear form in \(R\) has a non-zero coefficient at position \((\mu, \nu)\). Hence, by triangularity the linear forms in \(R\) are linearly independent. \(\square\)

3.3. **Representation theory.** Due to the particular structure of \(\mathcal{H}W\) as an operator algebra, the easiest thing to start with is the study of projective modules. Along the way we define a particular basis of \(\mathbb{C}W\) which plays a key role for the representation theory.

3.3.1. **Projective modules.** Recall that \(\mathcal{H}W\) is the algebra of operators preserving left antisymmetries. Thus, given \(I \subset S\), it is natural to introduce the \(\mathcal{H}W\)-submodule

\[ P_I := \bigcap_{s \in I} \ker(1 + \overline{s}) . \]

of the vectors in \(\mathbb{C}W\) which are \(s\)-antisymmetric for all \(s \in I\). For example, \(P_S\) is one dimensional, and spanned by \(\sum_{\nu \in W} (-1)^{\ell(\nu)} \nu\), whereas \(P_\emptyset = \mathbb{C}W\).

The goal of this section is to prove that the family of modules \((P_I)_{I \subset S}\) forms a complete set of representatives of the indecomposable projective modules of \(\mathcal{H}W\). First, we need a more practical definition of \(P_I\).
Lemma 3.11. Let \( I \subset S \). Then \( P_I \) is the \( \mathcal{H}W \)-submodule of \( CW \) generated by
\[
(v_I := \sum_{\nu \in W_I} (-1)^{\ell(\nu)} \nu,
\]
or equivalently the \( \mathbb{C}[W] \)-submodule generated by \( v_I \).

Proof. First, it is clear that \( v_I \) belongs to \( P_I \). Since \( CW \) is the right regular representation of \( W \), we may temporarily identify \( CW \) and \( \mathbb{C}[W] \). There, it is well known that \( v_I \) is an idempotent (up to a scalar factor). Take in general \( u \in P_I \). For any \( s \in I \), one has \( (1+s)u = 0 \), that is \( su = -u \). It follows that \( v_Iu = |W_I|u \), and we can conclude that \( P_I = v_I \cdot \mathbb{C}[W] \). \( \square \)

Actually, as we will see later, it is also an idempotent in the 0-Hecke algebra and even in the generic Hecke algebra.

For each \( \sigma \in W \), define \( v_\sigma := v_{S \setminus i \text{Des}(\sigma)} \cdot \sigma \). Note that \( \sigma \) is the element of minimal length appearing in \( v_\sigma \). By triangularity, it follows that the family \( (v_\sigma)_{\sigma \in \sigma_n} \) forms a vector space basis of \( CW \). See Figure 2 for an example.

The usefulness of this basis comes from the fact that it is compatible with the module structure.

Proposition 3.12. For any \( I \subset S \), the module \( P_I \) is of dimension \( |W| \), and
\[
\{ v_I \cdot \sigma \mid \sigma \in I \} \quad \text{and} \quad \{ v_\sigma \mid \sigma \in I \}
\]
are both vector space bases of \( P_I \).

Proof. First note that, by the same triangularity argument, the first family is free as well. Furthermore, by the previous lemma, \( P_I \) is spanned by all the vectors \( v_I \cdot \sigma \) with \( \sigma \in W \). Take \( \sigma \notin I \); then \( \sigma \) is of the form \( \sigma = s\sigma' \) with \( s \in I \) and \( \ell(\sigma') < \ell(\sigma) \), and it follows that \( v_I \cdot \sigma = -v_I \cdot \sigma' \). Applying induction on the length yields that the first family is a basis. By dimension count, the second family (each element \( v_\sigma \) of which is in \( P_{S \setminus i \text{Des}(\sigma)} \subset P_I \)) is also a basis of \( P_I \). \( \square \)

Corollaries 3.13. \( P_I \) is generated by \( v_I \), either as \( \mathcal{H}W \), \( W \), or \( \mathcal{H}(W)(0) \)-module. \( P_J \subset P_I \) if and only if \( I \subset J \).

Proof. If a finite Coxeter group, the sets \( \{ w \mid \text{Des}(w) = I \} \) are never empty; therefore \( P_J \subset P_I \) whenever \( I \subset J \). \( \square \)

In type \( A \), the recoil class \( I \) is the set of the shuffles of the words \( 1 \cdots a_1, a_1 + 1 \cdots a_2, \ldots \), where \( I \) is the set \( \{ a_1 < a_2 < \cdots \} \). As a consequence

Corollary 3.14. In type \( A \) the module \( P_I \) is of dimension \( \frac{n!}{i_1!i_2!\cdots i_k!} \), where \( (i_1, \ldots, i_k) \) is the composition associated to \( I \).

The following proposition elucidates the structure of \( P_I \) as \( W \) and \( \mathcal{H}(W)(0) \)-module.

Proposition 3.15. Let \( -1 \) denote the sign representation of \( W \) as well as the corresponding representation of the Hecke algebra \( \mathcal{H}(W)(0) \) (sending \( T_s \) to \( -1 \), or equivalently \( \pi_s \) to 0).

(a) As a \( W \)-module, \( P_I \approx (-1)^{\frac{|W|}{|W_I|}} \).

(b) As a \( \mathcal{H}(W)(0) \)-module, \( P_I \approx (-1)^{\frac{\mathcal{H}(W)(0)}{\mathcal{H}(W_I)(0)}} \); it is a projective module.

(c) The \( P_I \)'s are non isomorphic as \( \mathcal{H}(W)(0) \)-modules and thus as \( \mathcal{H}W \)-modules.
Figure 1. Some projective modules of $\mathcal{H}_n$, described on their \{$v_I \cdot \sigma \mid \sigma \in \mathcal{W}$\} bases. Note that those are not quite combinatorial modules, as coefficients $-1$ or $0$ may occur; in the later case, the corresponding edges are not drawn. Those pictures have been produced automatically, using MuPAD-Combinat, graphviz, and dot2tex.
Proof. First, for function \( e \) Corollary 3.16. In type \( P \) characteristic of the modules symmetric function (resp. a non-commutative symmetric function). Here is the \( (\) resp. \( H \)\( I \)) the complement of \( M \). In particular, the \( M \) subset, \( (49) \) \( M \) \( = \) \( \sum_{\nu \in W_I} (-1)^{\ell(\nu)} \nu \pi_s = 0 \) for any \( s \in W_I \).

The \( P_I \)'s being projective \( \mathcal{H}(W)(0) \)-modules, it is sufficient to prove that the associated semi-simple modules (obtained by factoring out by their radicals) are pairwise non-isomorphic. Namely, consider for each \( I \)
\[
(47) \quad M_I := \left( -1 \right)^{\ell(\tau)} \Lambda_{H_I} / \text{rad} \left( \Lambda_{H_I} \right) ,
\]
writing for short \( H := \mathcal{H}(W)(0) \) and \( H_I := \mathcal{H}(W_I)(0) \). Since \( -1 \) is simple as \( H_I \)-module,
\[
(48) \quad M_I = \left( -1 \right)^{\ell(\tau)} \Lambda_{H_I} / \text{rad} \left( \Lambda_{H_I} \right) .
\]
Recall [Nor79] that \( H / \text{rad}(H) \) (resp. \( H_I / \text{rad}(H_I) \)) is the commutative algebra over \( (T_s)_{s \in S} \) (resp. \( (T_s)_{s \in I} \)). Hence, a simple \( H / \text{rad}(H) \)-module \( S^0_J \) is characterized by the set \( J \in S \) of \( s \) such that \( \pi_s(S^0_J) = 0 \). Since there is a simple \( H \)-module for each subset,
\[
(49) \quad M_I \cong \bigoplus_{J \supseteq I} S^0_J .
\]
In particular, the \( M_I \)'s are pairwise non-isomorphic \( H \)-modules, as desired. \( \square \)

As a consequence of the preceding proof, if we denote \( P^0_I \) the projective \( \mathcal{H}(W)(0) \)-module associated with \( S^0_J \), then, as an \( \mathcal{H}(W)(0) \)-module
\[
(50) \quad P_I \cong \bigoplus_{J \supseteq I} P^0_J .
\]

In the following corollary, we focus on the particular type \( A \), where a \( \mathfrak{S}_n \)-module (resp. \( H_n(0) \)-module) is characterized by its so-called characteristic, which is a symmetric function (resp. a non-commutative symmetric function). Here is the characteristic of the modules \( P_I \):

**Corollary 3.16.** In type \( A \), the characteristic of the \( \mathfrak{S}_n \)-module \( P_I \) is the symmetric function \( e_K := e_{k_1} \cdots e_{k_l} \), where \( K := (k_1, \ldots, k_l) \) is the composition associated to the complement of \( I \).

The characteristics of the \( H_n(0) \)-module \( P_I \) is the noncommutative symmetric function \( \Lambda^K := \Lambda_{k_1} \cdots \Lambda_{k_l} \), where \( K := (k_1, \ldots, k_l) \) is the composition associated to the complement of \( I \).

3.3.2. \( \mathcal{H}W \) as an incidence algebra. Since \( \{ v_\sigma \}_{\sigma \in W} \) is a basis of \( \mathcal{C}W \), we may consider for each \( (\sigma, \tau) \) in \( W^2 \) the matrix element \( e_{\sigma, \tau} \in \text{End}(\mathcal{C}W) \) which maps \( v_\sigma \) to \( v_\tau \) if \( \sigma' = \sigma \) and to 0 otherwise.

We can now prove the main result of this section.

**Theorem 3.17.** For each \( (\sigma, \tau) \) in \( W^2 \), let \( e_{\sigma, \tau} \in \text{End}(\mathcal{C}W) \) defined by
\[
(51) \quad e_{\sigma, \tau}(\sigma') = \delta_{\sigma', \sigma} v_\tau .
\]
Then the family \( \{ e_{\sigma, \tau} \mid \text{iDes}(\sigma) \supset \text{iDes}(\tau) \} \) is a vector space basis of \( \mathcal{H}W \).
\[ v_{123} = 123 - 213 - 132 + 231 + 312 - 321 \]
\[ v_{213} = 213 - 312 \]
\[ v_{132} = 132 - 231 \]
\[ v_{312} = 312 - 321 \]
\[ v_{321} = 321 \]

**Figure 2.** The basis \((v_\sigma)_\sigma\) of \(\mathbb{C}\mathfrak{S}_3\), together with the graph structure which makes \(\mathcal{H}\mathfrak{S}_3\) into an incidence algebra. Underlined: the recoils of the permutations.

**Proof.** This family is free by construction. It has the appropriate size because for any finite Coxeter group \(|\{\sigma \mid \text{iDes}(\sigma) = I\}| = |\{\sigma \mid \text{iDes}(\sigma) = S \setminus I\}|\) (a bijection between the two sets is given by \(u \mapsto \omega u\), where \(\omega\) is the maximal element of \(W\); see [BB05, Exercise 10 p. 57]).

It remains to check that any of its element \(e_{\sigma,\tau}\) preserves \(i\)-left antisymmetries and therefore is indeed in \(\mathcal{H}W\). Take \(i \in S\), and consider the basis of \(P_{(i)}\) of proposition 3.12: \(\{v^i_\sigma \mid i \notin \text{iDes}(\sigma')\}\); an element of this basis is either killed by \(e_{\sigma,\tau}\) or sent to another element of this basis. Therefore, \(P_{(i)}\) is stable by \(e_{\sigma,\tau}\). \(\square\)

Recall (see [Sta97]) that the incidence algebra \(\mathbb{C}[P]\) of a poset \((P, \preceq)\) is the algebra whose basis elements \(e_{u,v}\) are indexed by the couples \((u, v) \in P^2\) such \(u \preceq v\), and whose multiplication rule is given by:

\[
e_{u,v} \cdot e_{u',v'} = \delta_{v,u'} e_{u,v}' .
\]

Here we need a slightly more general extension of this notion where \(P\) is not a partially ordered set but only a pre-order (not necessarily anti-symmetric).

**Corollary 3.18.** \(\mathcal{H}W\) is isomorphic to the incidence algebra of the pre-order \((W, \preceq)\) where

\[
\sigma \preceq \sigma' \quad \text{whenever} \quad \text{iDes}(\sigma) \supset \text{iDes}(\sigma') .
\]

**Proof.** By construction, the \(e_{\sigma,\tau}\) satisfy the usual product rule of incidence algebras:

\[
e_{\sigma,\tau} e_{\sigma',\tau'} = \delta_{\tau,\sigma'} e_{\sigma,\tau'} .
\]

This is sufficient since \((e_{\sigma,\tau})\) is a basis. \(\square\)

The representation theory of \(\mathcal{H}W\) (projective and simple modules and the Cartan matrix) will follow straightforwardly from this corollary. Furthermore, a good way to think about the pre-order \(\preceq\) is to view it as the transitive closure of a graph \(G\) which essentially encodes the action of the generators of \(\mathcal{H}W\). Namely, define \(G\) as the graph with vertex set \(W\) and where \(\sigma \rightarrow \sigma'\) is an edge whenever there exists \(s \in S\) such that \(\sigma' = \sigma s\) and \(\text{iDes}(\sigma) \supset \text{iDes}(\sigma')\) (cf. Figure 2).

**Corollary 3.19.**

(a) The ideal \(e_\sigma \mathcal{H}W\) is isomorphic to \(P_{S \setminus \text{iDes}(\sigma)}\) as a right \(\mathcal{H}W\)-module;

(b) The idempotents \(e_\sigma := e_{\sigma,\sigma}\) give a maximal decomposition of the identity into orthogonal idempotents in \(\mathcal{H}W\);
Hence we get the full description of the projective $HW$-modules.

**Corollary 3.20.** The family of modules $(P_I)_{I \subseteq S}$ forms a complete set of representatives of the indecomposable projective modules of $HW$.

*Proof.* Follows from (a) and (b) and Proposition 3.15 (c). \qed

### 3.3.3. Cartan matrix and the boolean lattice.

We now turn to the description of the Cartan matrix. For any $I \subseteq S$, let $\alpha(I)$ be the shortest permutation such that \( i \text{Des}(\alpha) = S \setminus I \) (the choice of the *shortest* is in fact irrelevant). For any $(I,J)$ such that $I \subset J \subset S$, define $e_{I,J} := e_{\alpha(I),\alpha(J)}$.

\[
e_{I,J}(v_\sigma) = \begin{cases} v_J & \text{if } \sigma = \alpha(I), \\ 0 & \text{otherwise.} \end{cases}
\]

(55)

**Remark 3.21.** Using the incidence algebra structure, the sandwich $e_I \mathcal{H} W e_J$ is $\mathbb{C}$. $e_{I,J}$ if $I \subset J$ and is trivial otherwise.

By Corollary 3.19, $\text{Hom}(P_I, P_J)$ is isomorphic to the sandwich $e_I \mathcal{H} W e_J$. Therefore we have the following corollary:

**Corollary 3.22.**

\[
\dim \text{Hom}(P_I, P_J) = \dim e_I \mathcal{H} W e_J = \begin{cases} 1 & \text{if } I \supset J, \\ 0 & \text{otherwise.} \end{cases}
\]

(56)

In other words, the Cartan matrix of $HW$ is the incidence matrix of the boolean lattice. This suggests the existence of a close relation between $HW$ and the incidence algebra of the boolean lattice.

Recall that an algebra is called *elementary* (or sometimes *reduced*) if its simple modules are all one dimensional. Starting from an algebra $A$, it is possible to get a canonical elementary algebra by the following process. Start with a maximal decomposition of the identity $1 = \sum_i e_i$ into orthogonal idempotents. Two idempotents $e_i$ and $e_j$ are *conjugate* if $e_i$ can be written as $ae_jb$ where $a$ and $b$ belongs to $A$, or equivalently, if the projective modules $e_i A$ and $e_j A$ are isomorphic. Select an idempotent $e_c$ in each conjugacy classes $c$ and put $e := \sum e_c$. Then, it is well known [CR90] that the algebra $e Ae$ is elementary and that the functor $M \mapsto Me$ which sends a right $A$-module to a $e Ae$-module is an equivalence of category. Recall finally that two algebras $A$ and $B$ such that the categories of $A$-modules and $B$-modules are equivalent are said *Morita equivalent*. Thus $A$ and $e Ae$ are Morita-equivalent.

**Corollary 3.23.** Let $e$ be the idempotent defined by $e := \sum_{I \subseteq S} e_I$. Then the algebra $e \mathcal{H} W e$ is isomorphic to the incidence algebra $\mathbb{C}[B_S]$ of the boolean lattice $B_S$ of subsets of $S$. Consequently, $\mathcal{H} W$ and $\mathbb{C}[B_S]$ are Morita equivalent.

Actually, the previous construction is fairly general when $A$ is the incidence algebra of a pre-order $\preceq$, the construction of $e Ae$ boils down to taking the incidence algebra of the canonical order $\leq$ associated to $\preceq$ by contracting its strongly connected components, or equivalently by picking any representative. For $\mathcal{H} W$, the strongly connected components of the pre-order $(W, \preceq)$ are by construction $((\sigma \mid i \text{Des}(\sigma) = I))_{I \subseteq S}$, and the associated order is simply the boolean lattice $B_S$ of subsets $I$ of $S$.  


3.3.4. Simple modules. The simple modules are obtained as quotients of the projective modules by their radical: \( S_I := P_I / \sum_{J \subsetneq I} P_J \). This has the natural effect of making a vector \( v \) in \( S_I \) both \( i \)-antisymmetric for \( i \in I \) and \( i \)-symmetric for \( i \not\in I \).

**Theorem 3.24.** The modules \( (S_I)_{I \subseteq S} \) form a complete set of representatives of the simple modules of \( \mathcal{H}W \). Moreover, the projection of the family \( \{v_{s_I} \mid i\text{Des}(\sigma) = I \} \) in \( S_I \) forms a vector space basis of \( S_I \).

**Proof.** This is a consequence of Proposition 3.22. Indeed, since there is no nontrivial morphism from \( P_I \) to itself, the radical of \( P_I \) is the sum of the images of all morphisms from the \( P_J \) for \( J \neq I \) to \( P_J \). But there exists up to constant at most one such morphism, that is when \( I \supset J \), then \( P_I \subset P_J \). Hence \( \text{rad}(P_I) = \sum_{J \subsetneq I} P_J \).

As a consequence, together with Proposition 3.12, we get the given basis. \( \square \)

The following proposition elucidates the structure of \( S_I \) as \( W \) and \( \mathcal{H}(W)(0) \)-module.

**Proposition 3.25.** Let \( K \) be the composition associated to \( I \).

(a) As a \( W \)-module, \( S_I \) is isomorphic to the subspace of \( \mathbb{C}W \) of vectors which are simultaneously \( i \)-antisymmetric for \( i \) in \( I \) and \( i \)-symmetric for \( i \) in \( S \setminus I \).

In type \( A \) this is the usual Young’s representation \( V_K \) indexed by the ribbon \( K \), and each \( v_\sigma \), with \( \text{Des}(\sigma) = I \) corresponds to the basis element of \( V_K \) indexed by the standard ribbon tableaux associated to \( \sigma^{-1} \). Its character is the ribbon Schur symmetric function \( s_K \).

(b) As a \( \mathcal{H}(W)(0) \)-module, \( S_I \) is the indecomposable projective module indexed by \( S \setminus I \). In type \( A \), its character is the noncommutative symmetric function \( R_K \).

**Proof.** (a) This description of \( S_I \) is clear by construction. In type \( A \) this is a well known description of the ribbon representation.

(b) For \( \sigma \) such that \( i\text{Des}(\sigma) = S \setminus I \), define \( b_\sigma \) as the image of \( v_I \cdot \sigma \) by the canonical quotient map. Then, \( \{b_\sigma \mid i\text{Des}(\sigma) = S \setminus I \} \) is a basis of \( S_I \). Furthermore, one can easily check that:

\[
 b_\sigma \pi_s = \begin{cases} 
 b_\sigma & \text{if } s \in \text{Des}(s), \\
 b_{\sigma s} & \text{if } s \not\in \text{Des}(s) \text{ and } i\text{Des}(\sigma s) = S \setminus I, \\
 0 & \text{otherwise}.
\end{cases}
\]

One recognizes the usual description of the projective \( \mathcal{H}(W)(0) \)-module \( P_{S \setminus I} \) of Section 2.3.

Note that \( S_I \) is also isomorphic to the simple module \( S_{S \setminus I} \) of the transpose algebra \( \mathcal{H}W^* \).

3.3.5. Induction, restriction, and Grothendieck rings. In this subsection, we concentrate on the tower of type-\( A \) algebras \( (\mathcal{H}\mathfrak{S}_n)_n \).

Let \( \mathcal{G} := \mathcal{G}(\mathfrak{S}_n)_n \) and \( \mathcal{K} := \mathcal{K}(\mathfrak{S}_n)_n \) be respectively the Grothendieck rings of the characters of the simple and projective modules of the tower of algebras \( (\mathcal{H}\mathfrak{S}_n)_n \). Let furthermore \( C \) be the Cartan map from \( \mathcal{K} \) to \( \mathcal{G} \). It is the algebra and coalgebra morphism which gives the projection of a module onto the direct sum of its composition factors. It is given by

\[
 C(P_I) = \sum_{I \supset J} S_J.
\]
Since the indecomposable projective modules are indexed by compositions, it comes out as no surprise that the structure of algebras and coalgebras of $G$ and $K$ are isomorphic to $\text{QSym}$ and $\text{NCSF}$, respectively. However, we do not get Hopf algebras, because the structures of algebras and coalgebras are not compatible.

**Proposition 3.26.** The following diagram gives a complete description of the structures of algebras and of coalgebras on $G$ and $K$:

\[
\begin{array}{ccc}
(\text{QSym}, \cdot) & \xrightarrow{\chi(S_I) \mapsto M_I} & (G, \cdot) \\
\downarrow & & \downarrow \\
(NCSF, \Delta) & \xrightarrow{\chi(S_I) \mapsto R_I} & (G, \Delta)
\end{array}
\]

\[
\begin{array}{ccc}
(G, \cdot) & \xrightarrow{C} & (K, \cdot) \\
\downarrow & & \downarrow \\
(NCSF, \Delta) & \xrightarrow{\chi(P_I) \mapsto F_I} & (\text{QSym}, \cdot)
\end{array}
\]

**Proof.** The bottom line is already known from Proposition 3.15 and the fact that, for all $m$ and $n$, the following diagram commutes:

\[
\begin{array}{c}
\mathcal{H}_m(0) \otimes \mathcal{H}_n(0) \\
\downarrow \\
\mathcal{H}S_m \otimes \mathcal{H}S_n
\end{array} \xrightarrow{\chi(S_I) \mapsto R_I} \begin{array}{c}
\mathcal{H}_m(0) \\
\downarrow \\
\mathcal{H}S_m + \mathcal{H}S_n
\end{array}
\]

Thus the map which sends a module to the characteristic of its restriction to $\mathcal{H}_n(0)$ is a coalgebra morphism. The isomorphism from $(K, \cdot)$ to $\text{QSym}$ is then obtained by Frobenius duality between induction of projective modules and restriction of simple modules. And the last case is obtained by applying the Cartan map $C$. □

It is important to note that the algebra $(G, \cdot)$ is not the dual of the coalgebra $(K, \Delta)$ because the dual of the restriction of projective modules is the so-called *co-induction* of simple modules which, in general, is not the same as the induction for non self-injective algebras.

Finally, we briefly describe the Grothendieck rings for the adjoint algebra $\mathcal{H}W^*$ which preserves symmetries. The projective modules are defined by the two following equivalent formulas:

\[
P_I := \bigcap_{x \in I} \ker(1 - x) = \left( \sum_{\nu \in W_I} \nu \right) \cdot \mathcal{H}W^*.
\]

As $W$-module (resp. $\mathcal{H}(W)(0)$-module), they are isomorphic to the modules induced by the trivial modules of $W_I$ (resp. $\mathcal{H}(W_I)(0)$) whose Frobenius characteristic are complete symmetric functions. The rest of our arguments can be adapted easily, yielding the following diagram:

\[
\begin{array}{ccc}
(\text{QSym}, \cdot) & \xrightarrow{\chi(S_I) \mapsto X_I} & (G, \cdot) \\
\downarrow & & \downarrow \\
(NCSF, \Delta) & \xrightarrow{\chi(S_I) \mapsto R_I} & (G, \Delta)
\end{array}
\]

\[
\begin{array}{ccc}
(G, \cdot) & \xrightarrow{C} & (K, \cdot) \\
\downarrow & & \downarrow \\
(NCSF, \Delta) & \xrightarrow{\chi(P_I) \mapsto F_I} & (\text{QSym}, \cdot)
\end{array}
\]

where $(X_I)_I$ is the dual basis of the elementary basis $(\Lambda_I)_I$ of $\text{NCSF}$. Thus we have a representation theoretical interpretation of many bases of $\text{NCSF}$ and $\text{QSym}$. 

4. The algebra of non-decreasing functions

**Definition 4.1.** Let NDF\(_n\) be the set of non-decreasing functions from \(\{1, \ldots, n\}\) to itself. Its cardinality is given by:

\[
\binom{2n-1}{n-1} = \sum_{k=1}^{n} \binom{n}{k} \binom{n-1}{k-1},
\]

where, on the right, functions are counted according to the size \(k\) of their images.

The composition and the neutral element \(\text{id}_n\) make NDF\(_n\) into a monoid and we denote by \(\mathbb{C}[\text{NDF}_n]\) its monoid algebra. The monoid NDF\(_n \times \text{NDF}_m\) can be identified as the submonoid of NDF\(_{n+m}\) whose elements stabilize both \(\{1, \ldots, n\}\) and \(\{n+1, \ldots, n+m\}\). This makes \((\mathbb{C}[\text{NDF}_n])_n\) into a tower of algebras.

The semi-group properties of NDF\(_n\) have been studied (see e.g. [GH92], where \(\mathcal{O}_n\) coincides with NDF\(_n\) striped of the identity). In particular [GH92, Theorem 4.8], one can take as idempotent generators for NDF\(_n\) the functions \(\pi_i\) and \(\pi\_i\) defined by:

\[
\begin{align*}
\pi_i(i+1) &:= i \quad \text{and} \quad \pi_i(j) := j, \text{ for } j \neq i + 1, \\
\pi_i(i) &:= i + 1 \quad \text{and} \quad \pi_i(j) := j, \text{ for } j \neq i.
\end{align*}
\]

The functions \(\pi_i\) satisfy the braid relations, together with a new relation:

\[
\pi_i^2 = \pi_i \quad \text{and} \quad \pi_{i+1}\pi_i\pi_{i+1} = \pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i.
\]

This readily defines a morphism \(\phi : \pi_{\mathcal{H}_n(0)} \rightarrow \pi_{\mathbb{C}[\text{NDF}_n]}\) of \(\mathcal{H}_n(0)\) into \(\mathbb{C}[\text{NDF}_n]\). Its image is the monoid algebra of non-decreasing parking functions which will be discussed in Section 5 and of which Equation (65) actually gives a presentation. The same properties hold for the operators \(\pi_i\)'s. Although this is not a priori obvious, it will turn out that the two morphisms \(\phi : \pi_{\mathcal{H}_n(0)} \rightarrow \pi_{\mathbb{C}[\text{NDF}_n]}\) and \(\varphi : \pi_{\mathcal{H}_n(0)} \rightarrow \pi_{\mathbb{C}[\text{NDF}_n]}\) are compatible, making \(\mathbb{C}[\text{NDF}_n]\) into a quotient of \(\mathcal{H}\text{S}_n\) (Proposition 4.3). This will be used in Subsection 4.2 to deduce the representation theory of \(\mathbb{C}[\text{NDF}_n]\).

4.1. Representation on exterior powers, and link with \(\mathcal{H}\text{S}_n\). We now want to construct a suitable faithful representation of \(\mathbb{C}[\text{NDF}_n]\) where the existence of the epimorphism from \(\mathcal{H}\text{S}_n\) onto \(\mathbb{C}[\text{NDF}_n]\) becomes clear.

The natural representation of \(\mathbb{C}[\text{NDF}_n]\) is obtained by taking the vector space \(\mathbb{C}^n\) with canonical basis \(e_1, \ldots, e_n\), and letting a function \(f\) act on it by \(e_i \cdot f = e_{f(i)}\). For \(n > 2\), this representation is a faithful representation of the monoid NDF\(_n\) but not of the algebra, as \(\dim \mathbb{C}[\text{NDF}_n] = \binom{2n-1}{n-1} \gg n^2\). However, since NDF\(_n\) is a monoid, the diagonal action on exterior powers

\[
(x_1 \wedge \cdots \wedge x_k) \cdot f := (x_1 \cdot f) \wedge \cdots \wedge (x_k \cdot f)
\]

still defines an action. Taking the exterior powers \(\wedge^k \mathbb{C}^n\) of the natural representation gives a new representation, whose basis \(\{e_S := e_{s_1} \wedge \cdots \wedge e_{s_k}\}\) is indexed by subsets \(S = \{s_1, \ldots, s_k\}\) of \(\{1, \ldots, n\}\). The action of a function \(f\) in NDF\(_n\) is simply given by (note the absence of sign!):

\[
e_{S,f} := \begin{cases} 
e_{f(S)} & \text{if } |f(S)| = |S|, \\ 0 & \text{otherwise.} \end{cases}
\]
We call representation of $\mathbb{C}[\text{NDF}_n]$ on exterior powers the representation of $\mathbb{C}[\text{NDF}_n]$ on $\bigoplus_{k=1}^{n} \wedge^k \mathbb{C}^n$, which is of dimension $2^n - 1$ (it turns out that we do not need to include the component $\wedge^0 \mathbb{C}^n$ for our purposes).

**Lemma 4.2.** For $n > 0$, the representation $\bigoplus_{k=1}^{n} \wedge^k \mathbb{C}^n$ of $\mathbb{C}[\text{NDF}_n]$ is faithful.

**Proof.** We exhibit a triangularity property. For a function $f$ in NDF$_n$, let $\text{im} f := f(\{1, \ldots, n\})$ be its image set, and $R(f)$ be the preimage of $\text{im} f$ defined by $R(f) := \{ \min \{ x \mid f(x) = y \} \mid y \in S \}$. Put a partial order on $\mathbb{C}[\text{NDF}_n]$ by setting $f \prec g$ if $\text{im} f = \text{im} g$ and $R(f)$ is lexicographically smaller than $R(g)$. Fix a function $f$. If the representation matrix of a function $g$ has coefficient 1 on row $\text{im} f$ and column $R(f)$, that is if $g(R(f)) = \text{im} f$, then $g \preceq f$.

Remark: for $n > 0$, the component $\wedge^0 \mathbb{C}^n$ is not needed because $\text{im} f$ and $R(f)$ are non empty. □

We now realize the representation of $\mathbb{C}[\text{NDF}_n]$ on the $k$-th exterior power as a representation of $\mathcal{H}\mathcal{S}_n$. Let us start from the $k$-th exterior product $\wedge^k \mathbb{C}^n$ of the natural representation of $\mathcal{S}_n$ on $\mathbb{C}^n$. It is isomorphic to $V_{(n-k,1,\ldots,1)} \oplus V_{(n-k+1,1,\ldots,1)}$, where $V_{\lambda}$ denotes the irreducible representation of $\mathcal{S}_n$ indexed by the partition $\lambda$. It can be realized as a submodule of the regular representation of $\mathcal{S}_n$ using the classical Young construction by mean of the row-symmetrizers and column-antisymmetrizers on the skew ribbon

$$
\begin{array}{cccc}
\vdots \\
1 \\
k+1 & \cdots & n
\end{array}
$$

This only involves left antisymmetries on the values $1, \ldots, k-1$ and symmetries on the values $k+1, \ldots, n-1$. Therefore, $\wedge^k \mathbb{C}^n$ can alternatively be realized as the $\mathcal{H}\mathcal{S}_n$-module

$$
P_n^k := \frac{P_{\{1,\ldots,k-1\}}}{\sum_{s=k+1}^{n-1} P_{\{1,\ldots,k-1,s\}}}
$$

(68)

(an element of $P_{\{1,\ldots,k-1\}}$ is by construction left antisymmetric on the values $1, \ldots, k-1$, and the quotient by each $P_{\{1,\ldots,k-1,s\}}$ makes it $s$-left symmetric). In general, this construction turns any $\mathcal{S}_n$-module indexed by a shape made of disconnected rows and columns into an $\mathcal{H}\mathcal{S}_n$-module; it does not apply to shapes like $\lambda = (2,1)$, as they involve symmetries or antisymmetries between non-consecutive values, like 1 and 3.

A basis of $P_n^k$ indexed by subsets of size $k$ of $\{1, \ldots, n\}$ is obtained by taking for each such subset $S$ the image in the quotient $P_n^k$ of

$$
es_S := \sum_{\sigma(S) = \{1, \ldots, k\}} (-1)^{k(\sigma)} \sigma.
$$

(70)

It is straightforward to check that the actions of $\pi_i$ and $\pi_i$ of $\mathcal{H}\mathcal{S}_n$ on $e_S$ of $P_n^k$ coincide with the actions of $\pi_i$ and $\pi_i$ of $\mathbb{C}[\text{NDF}_n]$ on $e_S$ of $\wedge^k \mathbb{C}^n$ (justifying a posteriori the identical notations). In the sequel, we identify the modules $P_n^k$ and
$\bigwedge^k \mathbb{C}^n$ of $\mathcal{H}\mathcal{S}_n$ and $\mathbb{C}[\text{NDF}_n]$, and we call representation on exterior powers of $\mathcal{H}\mathcal{S}_n$ its representation on $\bigoplus_{k=1}^n \bigwedge^k \mathbb{C}^n$.

Using Lemma 4.2 we are in position to state the following

**Proposition 4.3.** $\mathbb{C}[\text{NDF}_n]$ is the quotient of $\mathcal{H}\mathcal{S}_n$ obtained by considering its representation on exterior powers. The restriction of this representation of $\mathcal{H}\mathcal{S}_n$ to $\mathbb{C}[\mathcal{S}_n]$, $\mathcal{H}_n(0)$, and $\mathcal{H}_n(-1)$ yield respectively the usual representation of $\mathcal{S}_n$ on exterior powers, the algebra $\text{NDPF}_n$ of non-decreasing parking functions (see Section 5), and the Temperley-Lieb algebra $\mathcal{TL}_n(-1)$.

**Proof.** The only case which remains is $q = -1$. Recall that [TL71] the Temperley-Lieb algebra $\mathcal{TL}_n(q)$ is the quotient of the Hecke algebra by the relations

$$ e_i e_{i+1} e_i = q e_i , $$

where $e_i = T_i + q$. As $\text{NDPF}_n$, its dimension is the Catalan number $C_n$.

The algebra $A$ is generated by the operators $e_i = \pi_i - \pi_i$, which satisfy the relations:

$$ e_i^2 = 0 , $$

$$ e_i e_{i+1} e_i = - e_i . $$

Therefore $A$ is a quotient of the Temperley-Lieb algebra $\mathcal{TL}_n(-1)$. We now prove that the quotient is trivial by exhibiting $C_n$ elements of $A$ satisfying a triangularity property w.r.t. those of $\text{NDPF}_n$.

Let $\leq$ be the pointwise partial order on $\text{NDF}_n$ such that $f \leq g$ if and only if $f(i) \leq g(i)$ for all $i$. The following properties are easily verified:

$$ f \pi_i \leq f \leq f \pi_i , $$

$$ f \leq g \implies f \pi_i \leq g \pi_i , $$

$$ f \leq g \implies f \pi_i \leq g \pi_i . $$

Take $i_1, \ldots, i_k$ such that the product $\pi_{i_1} \cdots \pi_{i_k}$ is reduced in $\text{NDPF}_n$, and consider a function

$$ f = \pi_{i_1} \cdots \pi_{i_k} \pi_{i_{k+1}} \cdots \pi_{i_{k+1}} \cdots $$

of $\text{NDF}_n$ appearing in the expansion of the product

$$ e_{i_1} \cdots e_{i_k} = (\pi_{i_1} - \pi_{i_1}) \cdots (\pi_{i_k} - \pi_{i_k}) . $$

Clearly, $\pi_{i_1} \cdots \pi_{i_k} \leq f$. Furthermore, if equality holds then

$$ \pi_{i_1} \cdots \pi_{i_k} \leq \pi_{i_1} \cdots \pi_{i_k} \pi_{i_{k+1}} \cdots \leq f = \pi_{i_1} \cdots \pi_{i_k} . $$

Since the product is reduced, $k = k$, and therefore, $\pi_{i_1} \cdots \pi_{i_k}$ appears with coefficient 1 in $e_{i_1} \cdots e_{i_k}$. □

### 4.2. Representation theory.

In this section, we derive the representation theory of $\text{NDF}_n$ from that of $\mathcal{H}\mathcal{S}_n$. An alternative more combinatorial approach would be to construct by inclusion/exclusion a graded basis $(\text{gr}_f)_{f \in \text{NDPF}_n}$ of $\mathbb{C}[\text{NDPF}_n]$ such that:

$$ \text{gr}_f \, \text{gr}_g = \begin{cases} \text{gr}_g & \text{if } |\text{im } f| = |\text{im } g| = |\text{im } f \circ g|, \\ 0 & \text{otherwise.} \end{cases} $$

Then, looking at the principal modules $\text{gr}_f \, \mathbb{C}[\text{NDPF}_n] = \mathbb{C} \{ \text{gr}_g \mid \text{fibers}(g) = \text{fibers}(f) \}$ splits the regular representation into a direct summand of $\binom{n-1}{k-1}$ copies of $P_n^k$.
each $k$. Combinatorial proofs for the Cartan matrix and the simple modules are then lengthy but straightforward.

4.2. Projective modules, simple modules, and Cartan matrix. Let $\delta$ be the usual homology border map ($\delta^2 = 0$):

\begin{equation}
\delta : \bigg\{ \begin{array}{c}
P^k_n \\
S := \{s_1, \ldots, s_k\}
\end{array} \rightarrow \begin{array}{c}
P^{k-1}_n \\
\sum_{i\in\{1,\ldots,k\}} (-1)^{k-i} S\setminus\{s_i\}
\end{array},
\end{equation}

which induces the following exact sequence of morphisms of $\mathbb{C}[\text{NDF}_n]$-modules:

\begin{equation}
0 \rightarrow P^n_0 \xrightarrow{\delta} \cdots \xrightarrow{\delta} P^{k+1}_n \xrightarrow{\delta} P^k_n \xrightarrow{\delta} P^{k-1}_n \xrightarrow{\delta} \cdots \xrightarrow{\delta} P^1_n \xrightarrow{\delta} P^0_n \rightarrow 0 /
\end{equation}

For $k = 1, \ldots, n$, set $S^k_n := P^k_n / \ker\delta$, so that we have the short exact sequence:

\begin{equation}
0 \rightarrow S^{k+1}_n \rightarrow P^n_n \rightarrow S^k_n \rightarrow 0.
\end{equation}

In particular, $\dim S^k_n = \binom{n-1}{k-1}$ since $\dim P^k_n = \binom{n}{k}$. The following proposition states that the morphism of Equation (79) is essentially the only non trivial morphisms between the $(P^k_n)_{k=1,\ldots,n}$.

**Proposition 4.4.** Let $k$ and $l$ be two integers in $\{1, \ldots, n\}$. Then,

\begin{equation}
\dim\text{Hom}(P^k_n, P^l_n) = \begin{cases} 
1 & \text{if } k \in \{l, l+1\}, \\
0 & \text{otherwise}.
\end{cases}
\end{equation}

**Proof.** From the Cartan matrix of $\mathcal{HS}_n$ (see Proposition 3.22) we can deduce the dimension of $\text{Hom}(P^k_n, P^l_n)$. Indeed, any non trivial $\mathbb{C}[\text{NDF}_n]$-morphism $\phi$ from $P^k_n$ to $P^l_n$ is a $\mathcal{HS}_n$-morphism, and thus can be lifted to a non trivial $\mathcal{HS}_n$-morphism $\psi$ from $P^k_{\{1,\ldots,k-1\}}$ to the projective module $P^l_{\{1,\ldots,l-1\}}$.

If $k < l$, $\dim\text{Hom}(P^k_{\{1,\ldots,k-1\}}, P^l_{\{1,\ldots,l-1\}}) = 0$, and therefore $\dim\text{Hom}(P^k_n, P^l_n) = 0$. Otherwise, $\dim\text{Hom}(P^k_{\{1,\ldots,k-1\}}, P^l_{\{1,\ldots,l-1\}}) = 1$. If $k > l + 1$, then $P^k_{\{1,\ldots,k-1\}}$ is mapped by $\psi$ to its unique copy in $P^l_{\{1,\ldots,l+1\}}$, and is therefore killed in the quotient

\begin{equation}
P^l_n := P^l_{\{1,\ldots,l-1\}} / \sum_{s=l+1}^{n-1} P^l_{\{1,\ldots,l-1,s\}},
\end{equation}

therefore $\dim\text{Hom}(P^k_n, P^l_n) = 0$. In the remaining cases we can conclude that $\text{Hom}(P^k_n, P^l_n) = \mathbb{C}.\text{id}$, and $\text{Hom}(P^k_n, P^{k-1}_n) = \mathbb{C}.\delta$. \hfill $\square$

We are now in position to describe the projective and simple modules.

**Proposition 4.5.** The modules $(P^k_n)_{k=1,\ldots,n}$ form a complete set of representatives of the indecomposable projective modules of $\mathbb{C}[\text{NDF}_n]$.

The modules $(S^k_n)_{k=1,\ldots,n}$ form a complete set of representatives of the simple modules of $\mathbb{C}[\text{NDF}_n]$.

**Proof.** It follows from Proposition 4.4 that the modules $(P^k_n)_{k=1,\ldots,n}$ are both indecomposable ($\dim\text{Hom}(P^k_n, P^k_n) = 1$) and non isomorphic ($\dim\text{Hom}(P^k_n, P^l_n) = 0$ if $k < l$). It remains to prove (i) that each of them is projective and (ii) that we obtain all the projective modules this way. It then follows from the description of the morphisms between the $P^k_n$ that the $(S^k_n)_{k=1,\ldots,n}$ form a complete set of representatives of the simple modules.
We first achieve (i) by constructing explicitly an idempotent $e^k_n$ such that the principal ideal $e^k_n \mathbb{C}[\text{NDF}_n]$ is isomorphic to $P^k_n$. Define the idempotent $e^k_n$ as follows:

\begin{equation}
 e^k_n := \pi_n - 1 \pi_n - 2 \cdots \pi_{k+1} \pi_k (1 - \pi_{k-1})(1 - \pi_{k-2}) \cdots (1 - \pi_2)(1 - \pi_1).
\end{equation}

To prove that $e^k_n$ is indeed an idempotent, it is sufficient to use the presentation of $\mathbb{C}[\text{NDF}_n]$ given in Equation (65) to check that

\begin{equation}
 e^k_n \pi_i = \begin{cases} 
 e^k_n & \text{if } i > k, \\
 0 & \text{otherwise}.
\end{cases}
\end{equation}

Alternatively, we could have shown that $e^k_n$ is the image in $\mathbb{C}[\text{NDF}_n]$ of the classical hook idempotent of the 0-Hecke algebra

\begin{equation}
 \prod_j \pi_j \prod_j (1 - \pi_j),
\end{equation}

where the left (resp. on the right) product ranges over a reduced word of the maximal permutation of the parabolic subgroup $\mathfrak{S}_{k,...,n-1}$ (resp. $\mathfrak{S}_{1,...,k-1}$).

We want now to prove that $P^k_n$ is isomorphic to $e^k_n \mathbb{C}[\text{NDF}_n]$. Let us denote $B^k_n$ the set of functions in $\text{NDF}_n$ which are injective on $\{1,\ldots,k\}$ and such that $f(i) = f(k)$ for all $i \geq k$. It is clear that for any function $f$ of $\text{NDF}_n$, the actual value of $e^k_n f$ depends only on the set $f\{1,\ldots,k\}$. Moreover, for any $f \in B^k_n$,

\begin{equation}
 e^k_n f = f + \sum \text{functions strictly smaller than } f,
\end{equation}

where the order considered on the set of functions is the product order on the tuple $(f(1),\ldots,f(n))$ (i.e. $f \geq g$ iff $f(i) \geq g(i)$ for all $i$). It follows by triangularity that $\{e^k_n f \mid f \in B^k_n\}$ is a basis for $e^k_n \mathbb{C}[\text{NDF}_n]$.

Now the linear map

\begin{equation}
 \phi^k_n : \begin{cases} 
 e^k_n \mathbb{C}[\text{NDF}_n] & \rightarrow P^k_n \\
 e^k_n f & \mapsto \{f(1),\ldots,f(k)\}
\end{cases},
\end{equation}

is in fact a morphism of $\mathbb{C}[\text{NDF}_n]$-module. Indeed,

\begin{equation}
 \phi^k_n(e^k_n f \pi_i) = \phi^k_n(e^k_n(f \pi_i)) = \{f(1)\pi_i,\ldots,f(k)\pi_i\} = \{f(1),\ldots,f(k)\} \pi_i,
\end{equation}

and the same holds for $\pi_i$. Since the cardinal of $B^k_n$ is the same as the dimension of $P^k_n$, on gets that $\phi^k_n$ is in fact an isomorphism. Therefore $P^k_n$ is projective.

To derive (ii), note that the faithful $\mathbb{C}[\text{NDF}_n]$-module $\bigoplus_{k=1}^n \bigwedge^k \mathbb{C}^n = \bigoplus_{k=1}^n P^k_n$ (Lemma 4.2) is now projective. And it is a general property that any projective module of a finite dimensional algebra $A$ occurs as a submodule of any faithful projective module $M$ of $A$ (the right regular representation of $A$ is a sub-module of a certain number of copy of $M$, and the decomposition of any of its projective module is known to be unique up to an isomorphism).

\[\Box\]

Hence Proposition 4.4 actually gives the Cartan matrix of $\mathbb{C}[\text{NDF}_n]$. 


4.2.2. Induction, restriction, and Grothendieck rings.

**Proposition 4.6.** The restriction and induction of indecomposable projective modules and simple modules are described by:

\[
\begin{align*}
  P_{n_1+n_2}^{k_1+k_2} [\mathbb{C}[\text{NDF}_{n_1+n_2}]] & \approx \bigoplus_{k_1+k_2=k, k_1 \leq n_1, k_2 \leq n_2} P_{n_1}^{k_1} \otimes P_{n_2}^{k_2} \\
  S_{n_1+n_2}^{k_1+k_2} [\mathbb{C}[\text{NDF}_{n_1} \otimes \mathbb{C}[\text{NDF}_{n_2}]] & \approx \bigoplus_{k_1+k_2=n, k_1 \leq n_1, k_2 \leq n_2} S_{n_1}^{k_1} \otimes S_{n_2}^{k_2} \\
  S_{n_1+n_2}^{k_1+k_2} [\mathbb{C}[\text{NDF}_{n_1+n_2}]] & \approx S_{n_1+n_2}^{k_1+k_2}
\end{align*}
\]

Those rules yield structures of commutative algebras and cocommutative coalgebras on the Grothendieck rings of \( \mathbb{C}[\text{NDF}_n] \). However, we do not get Hopf algebras, because the structures of algebras and coalgebras are not compatible (the coefficient of \( \chi(P^i_1) \otimes \chi(P^i_1) \) differs in \( \Delta(\chi(P^i_1)\chi(P^i_1)) \) and \( \Delta(\chi(P^i_1))\Delta(\chi(P^i_1))) \).

**Proposition 4.7.** The Grothendieck rings \( \mathcal{G} \) and \( \mathcal{K} \) of \( \mathbb{C}[\text{NDF}_n] \) can be realized as quotients or subcoalgebras of \( \text{Sym}, \text{QSym}, \) and \( \text{NCSF} \), as described in the following diagram:

\[
\begin{array}{cccccc}
\text{(Sym,)} & \overset{h}{\longrightarrow} & \chi \left( S_1^{(i)} \right) & \left( \mathcal{G}, \right) & \overset{c}{\longrightarrow} & \left( \mathcal{K}, \right) \overset{R}{\longrightarrow} \chi \left( P_1^{(i)} \right) & \left( \text{NCSF,} \right) \\
\text{(QSym,} \Delta) & \overset{F}{\longrightarrow} & \chi \left( S_1^{(i)} \right) & \left( \mathcal{G}, \Delta \right) & \overset{C}{\longrightarrow} & \left( \mathcal{K}, \Delta \right) & \overset{c}{\longrightarrow} \chi \left( P_1^{(i)} \right) \overset{F}{\longrightarrow} \chi(\Delta) \end{array}
\]

5. The algebra of non-decreasing parking functions

**Definition 5.1.** A nondecreasing parking function of size \( n \) is a nondecreasing function \( f \) from \( \{1, 2, \ldots, n\} \) to \( \{1, 2, \ldots, n\} \) such that \( f(i) \leq i \), for all \( i \leq n \).

The composition of maps and the neutral element \( \text{id}_n \) make the set of nondecreasing parking function of size \( n \) into a monoid denoted \( \text{NDPF}_n \).

Parking functions where introduced in [KW66]. It is well known that the nondecreasing parking functions are counted by the Catalan numbers \( C_n = \frac{1}{n+1} \binom{2n}{n} \). It is also clear that \( \text{NDPF}_n \) is the sub-monoid of \( \text{NDF}_n \) generated by the \( \pi_i \)'s.

5.1. Representation theory

5.1.1. Simple modules. The goal of the sequel is to study the representation theory of \( \text{NDPF}_n \), or equivalently of its algebra \( \mathbb{C}[\text{NDPF}_n] \). The following remark allows us to deduce the representations of \( \mathbb{C}[\text{NDPF}_n] \) from the representations of \( \mathcal{H}_n(0) \).

**Proposition 5.2.** The kernel of the algebra epi-morphism \( \phi : \mathcal{H}_n(0) \rightarrow \mathbb{C}[\text{NDPF}_n] \) defined by \( \phi(\pi_i) = \pi_i \) is a sub-ideal of the radical of \( \mathcal{H}_n(0) \).
Proof. It is well known (see [Nor79]) that the quotient of \( \mathcal{H}_n(0) \) by its radical is a commutative algebra. Consequently, \( \pi_i \pi_{i+1} \pi_i - \pi_i \pi_{i+1} = [\pi_i \pi_{i+1}, \pi_i] \) belongs to the radical of \( \mathcal{H}_n(0) \).

As a consequence, taking the quotient by their respective radical shows that the projection \( \phi \) is an isomorphism from \( \mathcal{H}_n(0) / \text{rad}(\mathcal{H}_n(0)) \) to \( \mathbb{C}[\text{NDPF}_n] / \text{rad}(\mathbb{C}[\text{NDPF}_n]) \). Moreover \( \mathbb{C}[\text{NDPF}_n] / \text{rad}(\mathbb{C}[\text{NDPF}_n]) \) is isomorphic to the commutative algebra generated by the \( \pi_i \) such that \( \pi_i^2 = \pi_i \). As a consequence, \( \mathcal{H}_n(0) \) and \( \mathbb{C}[\text{NDPF}_n] \) share, roughly speaking, the same simple modules:

**Corollary 5.3.** There are \( 2^{n-1} \) simple \( \mathbb{C}[\text{NDPF}_n] \)-modules \( S_I \), and they are all one dimensional. The structure of the module \( S_I \), generated by \( \eta_I \), is given by

\[
\begin{cases}
\eta_I \cdot \pi_i = 0 & \text{if } i \in I, \\
\eta_I \cdot \pi_i = \eta_I & \text{otherwise}.
\end{cases}
\]

**5.1.2. Projective modules.** The projective modules of NDPF\(_n\) can be deduced from the ones of NDF\(_n\).

**Theorem 5.4.** Let \( I = \{s_1, \ldots, s_k\} \subset \{1, \ldots, n-1\} \). Then, the principal submodule

\[
P_I := (e_1 \wedge e_{s_1+1} \wedge \cdots \wedge e_{s_k+1}) \cdot \mathbb{C}[\text{NDPF}_n] \subset \bigwedge^{k+1} \mathbb{C}^n
\]

is an indecomposable projective module. Moreover, the set \( (P_I)_{I \subset n} \) is a complete set of representatives of indecomposable projective modules of \( \mathbb{C}[\text{NDPF}_n] \).

This suggests an alternative description of the algebra \( \mathbb{C}[\text{NDPF}_n] \). Let \( G_{n,k} \) be the lattice of subsets of \( \{1, \ldots, n\} \) of size \( k \) for the product order defined as follows. Let \( S := \{s_1 < s_2 < \cdots < s_k\} \) and \( T := \{t_1 < t_2 < \cdots < t_k\} \) be two subsets. Then,

\[
S \leq_G T \quad \text{if and only if} \quad s_i \leq t_i, \text{ for } i = 1, \ldots, k.
\]

One easily sees that \( S \leq_G T \) if and only if there exists a nondecreasing parking function \( f \) such that \( e_S = e_T \cdot f \). This lattice appears as the Bruhat order associated to the Grassman manifold \( G^m_k \) of \( k \)-dimensional subspaces in \( \mathbb{C}^n \).

**Theorem 5.5.** There is a natural algebra isomorphism

\[
\mathbb{C}[\text{NDPF}_n] \approx \bigoplus_{k=0}^{n-1} \mathbb{C}[G_{n-1,k}].
\]

In particular the Cartan map \( C : \mathcal{K} \to \mathcal{G} \) is given by the lattice \( \leq_G \):

\[
C(P_I) = \sum_{J, \text{Des}(J) \leq_G \text{Des}(I)} S_J.
\]

On the other hand, due to the commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}_m(0) \otimes \mathcal{H}_n(0) & \xrightarrow{\imath} & \mathcal{H}_{m+n}(0) \\
\downarrow & & \downarrow \\
\text{NDPF}_m \otimes \text{NDPF}_n & \xrightarrow{\imath} & \text{NDPF}_{m+n}
\end{array}
\]

it is clear that the restriction of simple modules and the induction of indecomposable projective modules follow the same rule as for \( \mathcal{H}_n(0) \). The induction of simple...
modules can be deduced via the Cartan map, giving rise to a new basis $G_I$ of NCSF. The restriction of indecomposable projective modules leads to a new operation on compositions, which seems not to be related to anything previously known. All of this is summarized by the following diagram:

\[
\begin{align*}
\text{(NCSF, .)} & \xrightarrow{\chi(S_I)\rightarrow G_I} \text{(G, .)} \xleftarrow{\chi(P_I)\rightarrow R_I} \text{(NCSF, .)} \\
\text{(QSym, \Delta)} & \xrightarrow{\chi(S_I)\rightarrow F_I} \text{(G, \Delta)} \xleftarrow{\chi(P_I)\rightarrow ???} ???
\end{align*}
\]

(100)

6. Alternative constructions of $\mathcal{H}\mathfrak{S}_n$ in Type A

In type $A$, the actions of the operators $s_i$ and $\pi_i$ on permutations of $\mathfrak{S}_n$ extend straightforwardly to an action on the set $A^n$ of words $w$ of length $n$ over any totally ordered alphabet $A$ by

\[
w \cdot \pi_i = \begin{cases} 
w & \text{if } w_i \geq w_{i+1}, \\
w_{s_i} & \text{otherwise.} \end{cases}
\]

(101)

We may again construct the algebra $\mathbb{C}[s_i, \pi_i]$, and wonder whether it is strictly larger than $\mathcal{H}\mathfrak{S}_n$.

**Theorem 6.1.** Let $A$ be a totally ordered alphabet of size at least $n$. Then the subalgebra of $\text{End}(\mathbb{C}.A^n)$ generated by both sets of operators $\{s_i, \pi_i\}_{i=1,\ldots,n-1}$ is isomorphic to $\mathcal{H}\mathfrak{S}_n$.

This theorem is best restated and proved using a commuting property with the monoid of non decreasing functions. Recall that the evaluation $e(w)$ of a word $w$ on an alphabet $A$ is the function which counts the number of occurrences in $w$ of each letter $a \in A$; for example, the evaluation of a permutation is the constant function $a \mapsto 1$. For a given evaluation $e$, write $C_e$ the subspace of $\mathbb{C}.A^n$ spanned by the words with evaluation $e$, and $p_e$ the orthogonal projection on $C_e$. Let finally $\text{End}_e(\mathbb{C}.A^n) = \bigoplus_e \text{End}(C_e)$ denote the algebra of evaluation preserving endomorphisms of $\mathbb{C}.A^n$.

**Theorem 6.2.** Let $A$ be a totally ordered alphabet of size at least $n$. Consider the monoid $\text{NDF}_A$ of non decreasing functions from $A$ to $A$, acting on values on the words of $A^n$. Then the commutant of $\text{NDF}_A$ in $\text{End}_e(\mathbb{C}.A^n)$ coincides with $\mathbb{C}[s_i, \pi_i]$ and is isomorphic to $\mathcal{H}\mathfrak{S}_n$.

Alternatively, $\mathbb{C}[s_i, \pi_i]$ is the commutant of the subalgebra of $\text{End}(\mathbb{C}.A^n)$ generated by both $\text{NDF}_A$ and $(p_e)_e$.

**Proof.** First it is clear that both $s_i$ and $\pi_i$ preserve the evaluation and commute with non decreasing functions; so $\mathbb{C}[s_i, \pi_i]$ is a subset of the commutant.

Taking $n$ distinct letters in $A$ which we may call $1 < \cdots < n$ yields a component $C_{\text{std}}$ isomorphic to $\mathcal{H}\mathfrak{S}_n$, which we call standard component. The restriction of $\mathbb{C}[s_i, \pi_i]$ on $C_{\text{std}}$ is of course $\mathcal{H}\mathfrak{S}_n$.

Fix now some operator $f \in \text{End}_e(\mathbb{C}.A^n)$ which commutes with $\text{NDF}_A$. For each $i$, take a function $\pi_i$ in $\text{NDF}_A$ such that for $1 \leq j < k \leq n$, $\pi_i(j) = \pi_i(k)$ if and only if $j = i$ and $k = i + 1$. A vector $v$ in the standard component is in the kernel of $\pi_i$ if and only if $v$ is $i$-left antisymmetric. Therefore, the restriction on $C_{\text{std}}$ of $f$ preserves $i$-antisymmetries, and thus coincides with some operator $g$ of $\mathcal{H}\mathfrak{S}_n$. 
On the other hand, for any component $C_e$ there exists a non decreasing function $f_e$ which maps $C_{\text{std}}$ onto $C_e$. Since $f$ commutes with $f_e$, the action of $f$ on $C_e$ is determined by its action of $C_{\text{std}}$, that is by $g$. \hfill $\Box$

Note: it is in fact sufficient to consider just $p_{\text{std}}$ instead of $(p_e)_e$ in theorem 6.2.

The argument can in fact be generalized to any subset $W$ of words containing some words with all letters distinct, and stable simultaneously by the right action of $S_n$ and by the left action of some monoid of non decreasing functions large enough to contain analogues of the $\pi_i$’s and $f_e$’s. The following proposition gives two typical examples of that situation (A function $f$ from $\{1, \ldots, n\}$ to itself is \textit{initial} if there exists $k \leq n$ such that $\text{im}(f) = \{1, \ldots, k\}$; for parking functions see [KW66]).

**Proposition 6.3.** Let $A$ be the totally ordered alphabet $\{1 < \cdots < n\}$.

(a) Consider the monoid $\text{NDPF}_n$ of non decreasing parking functions, acting on the left on the set $\text{PF}_n$ of parking functions from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$. Then the commutant of $\text{NDPF}_n$ in $\text{End}_c(\mathbb{C}.\text{PF}_n)$ coincides with $\mathbb{C}[s_i, \pi_i]$ and is isomorphic to $\mathcal{H}\mathcal{S}_n$.

(b) Consider the monoid $\text{NDInit}_n$ of non decreasing initial functions, acting on the left on the set $\text{Init}_n$ of initial functions from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$. Then the commutant of $\text{NDInit}_n$ in $\text{End}_c(\mathbb{C}.\text{Init}_n)$ coincides with $\mathbb{C}[s_i, \pi_i]$ and is isomorphic to $\mathcal{H}\mathcal{S}_n$.

7. Research in progress

A first direction of research concerns the links between Hecke group algebras and affine Hecke algebras [HST08a, HST08b]. It turns out that for any Weyl group, $\mathcal{H}W$ is the natural quotient of the generic untwisted affine Hecke algebra $\mathcal{H}(W)(q)$, through its level zero action. On one hand, this may shed some new light on the properties of the Hecke group algebras, and in particular their representation theory seem also to generalize nicely to infinite Coxeter groups, up to some little adaptations. First, when a parabolic subgroup $W_I$ is infinite, it is not possible to realize the projective module $P_I$ inside $\mathbb{C}[W]$, or at least not without an appropriate completion (because of the infinite alternating sum). Reciprocally, $P_{S\setminus I}$ is not distinct anymore from $\bigcup_{J \supset I} P_J$ (there are no element of $W$ with descent set $I$; cf also proposition 3.12). This suggests that $\mathcal{H}W$ is Morita equivalent to the poset algebra of some convex subset of the boolean lattice.

A last direction of research is the generalization of sections 4 and 5. This essentially boils down to the following question: what is the natural definition of the representation on exterior powers for a general Coxeter group? One such attempt
in type $B$ gives rise to some tower of self-injective monoid of signed non-decreasing parking functions whose sizes appear to be given by sequence A086618 of the encyclopedia of integer sequences [Se03].

REFERENCES


