THE $(1-\mathbb{E})$-TRANSFORM IN COMBINATORIAL HOPF ALGEBRAS

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Abstract. We extend to several combinatorial Hopf algebras the endomorphism of symmetric functions sending the first power-sum to zero and leaving the other ones invariant. As a "transformation of alphabets", this is the $(1-\mathbb{E})$-transform, where $\mathbb{E}$ is the "exponential alphabet", whose elementary symmetric functions are $e_n = \frac{1}{n!}$. In the case of noncommutative symmetric functions, we recover Schocker's idempotents for derangement numbers [Discr. Math. 269 (2003), 239]. From these idempotents, we construct subalgebras of the descent algebras analogous to the peak algebras and study their representation theory. The case of **WQSym** leads to similar subalgebras of the Solomon-Tits algebras. In **FQSym**, the study of the transformation boils down to a simple solution of the Tsetlin library in the uniform case.

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1. Virtual alphabets in combinatorial Hopf algebras

The notion of *virtual alphabet* provides a powerful symbolic notation for dealing with endomorphisms of combinatorial Hopf algebras, at least for those which can be realized in terms of (noncommutative) polynomials. Examples include **Sym** (noncommutative symmetric functions, [12, 19]), **FQSym** (Free quasi-symmetric functions [9, 10], which realize the Malvenuto-Reutenauer algebra [23]), **WQSym** (Word quasi-symmetric functions [18]), **PQSym** (Parking quasi-symmetric functions [28]), and their subalgebras as long as they are stable under the internal product. These algebras can be regarded as generalizations of the Hopf algebra of symmetric functions, for which this formalism was essentially promoted by A. Lascoux [21, 20].

The algebra of symmetric functions is denoted by Sym. Apart from this detail, our notation for symmetric functions follows Macdonald's book [22].

1.1. Virtual alphabets for symmetric functions.

It is convenient to use the generic term *alphabet* to designate the argument of a symmetric function. Indeed, a symmetric function f is characterized by its expression $f(x_1, x_2, \ldots)$ in terms of monomials in an infinite sequence of independent indeterminates x_i, to which various sets or multisets of algebraic expressions (numbers, monomials, cohomology classes, vector bundles, *etc.*) can be substituted. The diversity of possible interpretations suggests to treat as far as possible these arguments as formal symbols (or *letters*, whence the term *alphabet*), and the possible occurence of multiple arguments suggests to extend the usual meaning of this term and to understand it as a *multiset* of symbols.

Actually, a multiset $A = \{a, c, c, c, f, f\}$ is nothing but a formal linear combination of symbols with nonnegative integer coefficients, and it is more convenient to represent it as $A = a + 3c + 2f$. With this notation, the union of multisets becomes just a sum $A + B$, and if we have sums, we can also have differences $A - B$, at least when B is contained in A.

This leads us to the point of this paragraph. When B is not contained in A, let us call the formal combination $A-B$ a *virtual alphabet*. It is easy to define the value of a symmetric function on a virtual alphabet. Indeed, it is well known that any symmetric function f can be expressed in terms of the elementary symmetric functions e_k. The elementary functions $e_k(A)$ of a genuine alphabet A are the coefficients of the product

$$\lambda_t(A) := \prod_{a\in A}(1+ta) = \sum_k e_k(A)t^k \tag{1}$$

and when B is contained in A, those of the alphabet $C = A - B$ are obtained by division of the generating functions:

$$\lambda_t(A-B) = \lambda_t(A)/\lambda_t(B)\,. \tag{2}$$

When B is not contained in A, this still defines coefficients $e_k(A-B)$, which are, by definition, the elementary symmetric functions of the virtual alphabet $A-B$. This defines all symmetric functions of $A-B$, and one has for example the simple expression $p_n(A-B) = p_n(A) - p_n(B)$ for the power sums.

More generally, a virtual alphabet can be defined by *any* sequence $(e_n(A))_{n\geq 1}$, interpreted as its elementary symmetric functions (or as any sequence of independent generators of the algebra of symmetric functions, such as power sums p_n or complete homogeneous functions h_n).

For example, although the exponential function e^t has no zeros in the complex plane, we can introduce a virtual alphabet $\mathbb{E}$ such that

$$e_n(\mathbb{E}) = \frac{1}{n!}, \qquad\qquad \lambda_t(\mathbb{E}) = e^t\,. \tag{3}$$

This is a useful trick, allowing to understand a lot of formulas in combinatorics or analysis as specializations of simple identities on symmetric functions. For example, the exponential generating function of the derangement numbers d_n is just

$$\sigma_t(1-\mathbb{E}) := \lambda_{-t}(1-\mathbb{E})^{-1} = \frac{e^{-t}}{1-t}\,. \tag{4}$$

Also, for a partition λ of n, $n!s_\lambda(\mathbb{E}) = f_\lambda$, the number of standard tableaux of shape λ. Another virtual alphabet of interest is $1-q$, where $p_n(1-q) = 1-q^n$. It plays an essential role in the theory of Hall-Littlewood functions [22].

Our expression of alphabets as formal sums or differences also allows the consideration of products. For genuine alphabets A, B,

$$AB = \{a\,b \mid a\in A,\ b\in B\} = \sum_a a \sum_b b \tag{5}$$

and for virtual alphabets, the symmetric functions of AB are defined by any of the formulas like $p_n(AB) = p_n(A)p_n(B)$, or

$$h_n(AB) = \sum_{\lambda\vdash n} s_\lambda(A)s_\lambda(B)\,, \tag{6}$$

the famous Cauchy identity (the s_λ are the Schur functions). More generally, for any function f, we have: $f(AX) = F(X)*\sigma_1(AX)$, where $*$ denotes the internal product, and $\sigma_t(X) = \sum t^n h_n(X)$ is the generating series of homogeneous complete functions.

Hence, with this formalism, we can consider *transformations of alphabets* on symmetric functions, which are ring homomorphisms mapping any $f(X)$ to $f(\mathbb{E}X)$, $f((1-\mathbb{E})X)$, $f((1-q)X)$, $f(X/(1-q))$, and so on. Remark that in this setting, the virtual alphabet $1-q$ is the inverse of the genuine alphabet $\{1, q, q^2 \ldots\} \equiv 1+q+q^2+\cdots$.

Actually, Sym is a λ-ring [20, 21], and the notion of plethysm allows much more complicated transformations, but in the present paper, we concentrate on the above mentioned ones since these can be defined only in terms of the (combinatorial) Hopf algebra structure.

1.2. **Extension to combinatorial Hopf algebras.** Let X and Y be two infinite alphabets. Identifying expressions like $f(X)g(Y)$ with the tensor product $f \otimes g$, we can regard the transformations

$$\Delta : \ f(X) \longmapsto f(X+Y) \tag{7}$$

and

$$\delta : \ f(X) \longmapsto f(XY) \tag{8}$$

as linear maps $Sym \to Sym \otimes Sym$, that is, as comultiplications. Moreover, both are (obviously) algebra morphisms, and the first one being graded in the sense that

$$\Delta : \ Sym_n \longrightarrow \bigoplus_{i+j=n} Sym_i \otimes Sym_j \tag{9}$$

defines actually a Hopf algebra structure, whose antipode is simply $f(X) \mapsto f(-X)$.

Clearly, the transformations $f(\mathbb{E}X)$, $f((1-\mathbb{E})X)$, $f((1-q)X$ and so on can be defined only in terms of these coproducts, of the antipode, and of the internal product. It turns out that most combinatorial Hopf algebras can be realized in terms of polynomials in some infinite and *totally ordered alphabet*, denoted by $A = \{a_n | n \geq 1\}$ in the case of noncommuting letters, and by $X = \{x_n | n \geq 1\}$ in the case of commuting letters. The basic example is the pair Noncommutative Symmetric Functions – Quasi-symmetric functions $(\mathbf{Sym}, QSym)$ of mutually dual combinatorial Hopf algebras. Sums and differences of alphabets are defined on both sides, and it is possible to make sense of the product XA (see [19]). Hence, our transformations are defined in this case.

The subject of this article is the study of the $(1-\mathbb{E})$-transform in the pair $(\mathbf{Sym}, QSym)$, and its extension to other combinatorial Hopf algebras.

2. The $(1-\mathbb{E})$-transform in **Sym**

2.1. **Background.** Our notations for the Hopf algebra of noncommutative symmetric functions are as in [12, 19]. This Hopf algebra is denoted by **Sym**, or by $\mathbf{Sym}(A)$ if we consider the realization in terms of an auxiliary alphabet. Bases of its homogeneous component $\mathbf{Sym}_n$ are labelled by compositions $I = (i_1, \ldots, i_r)$ of n. The noncommutative complete and elementary functions are denoted by S_n and Λ_n, and the notation S^I means $S_{i_1} \cdots S_{i_r}$. The ribbon basis is denoted by R_I.

The notation $I \vDash n$ means that I is a composition of n. The conjugate composition is denoted by $I^\sim$, the mirror image composition by $\overline{I}$. The *descent set* of I is $\mathrm{Des}\,(I) = \{i_1,\ i_1+i_2, \ldots, i_1+\cdots+i_{r-1}\}$.

The graded dual of **Sym** is $QSym$ (quasi-symmetric functions). The dual basis of (S^I) is (M_I) (monomial), and that of (R_I) is (F_I).

The Hopf structures on **Sym** and $QSym$ allows one to partially extend the λ-ring notation of ordinary symmetric functions (see [19], and [20] for background on the original commutative version). If A and X represent totally ordered sets of noncommuting and commuting variables respectively, the noncommutative symmetric functions of XA are defined by

$$\sigma_t(XA) = \sum_{n\geq 0} t^n S_n(XA) = \prod_{x\in X}^{\rightarrow} \sigma_{tx}(A) = \sum_I t^{|I|} M_I(X) S^I(A)\,. \tag{10}$$

Now, X can be a virtual alphabet, defined by an arbitrary specialization of an independ set of generators of $QSym$. An alternative way to express the transformation of alphabets defined by X is [19]

$$F(XA) = F(A) * \sigma_1(XA)\,, \tag{11}$$

where $*$ is the internal product. Since $\sigma_1(XA)$ is grouplike, the X-transform is a bialgebra morphism, thanks to the splitting formula

$$(F_1F_2\cdots F_r) * G = \mu_r\left[(F_1\otimes\cdots\otimes F_r) * \Delta^r G\right] \tag{12}$$

where in the right-hand side, μ_r denotes the r-fold ordinary multiplication and $*$ stands for the operation induced on $\mathbf{Sym}^{\otimes n}$ by $*$.

Thanks to the commutative image homomorphism $\mathbf{Sym} \to Sym$, noncommutative symmetric functions can be evaluated on any element x of a λ-ring, $S_n(x)$ being $S^n(x)$, the n-th symmetric power. Recall that x is said *of rank one* (resp. *binomial*) if $\sigma_t(x) = (1-tx)^{-1}$ (resp. $\sigma_t(x) = (1-t)^{-x}$). The scalar $x=1$ is the only element having both properties. We usually consider that our auxiliary variable t is of rank one, so that $\sigma_t(A) = \sigma_1(tA)$.

The argument A of a noncommutative symmetric function can be a *virtual alphabet*. This means that, being algebraically independent, the S_n can be specialized to any sequence $\alpha_n \in \mathcal{A}$ of elements of any associative algebra $\mathcal{A}$. Writing $\alpha_n = S_n(A)$ defines all the symmetric functions of A. Quasi-symmetric functions of a virtual alphabet X can be defined by a specialization of the algebraic generators of $QSym$, which is more easily done by expressing the noncommutative symmetric functions of XA in terms of A.

The specializations $X = \mathbb{E}$, defined by

$$S_n(\mathbb{E}A) = \frac{1}{n!} S_1(A)^n \tag{13}$$

(so that $F_I(\mathbb{E}) = \frac{1}{n!}$) and $X = \frac{1}{1-q}$, for which

$$S_n\left(\frac{A}{1-q}\right) = \frac{1}{(q)_n} \sum_{I\vDash n} q^{\mathrm{maj}\,(I)} R_I(A) \tag{14}$$

are of special importance. The second one can be used to define the peak algebras and simplifies considerably their investigation [3, 1, 24]. Here, we will study the $(1-\mathbb{E})$- transform by the methods developed in these references.

2.2. **Noncommutative derangement numbers.** A possible motivation for the $(1-\mathbb{E})$-transforms is the combinatorics of derangements. Indeed, as already mentioned, the generating function of the derangement numbers

$$D(t) = \sum_{n\geq 0} d_n \frac{t^n}{n!} = \frac{e^{-t}}{1-t} \tag{15}$$

can be expressed as

$$D(t) = \sigma_1((1-\mathbb{E})t)\,. \tag{16}$$

The specializations of the various bases of symmetric functions at $1-\mathbb{E}$ are obtained by expanding the Cauchy kernel $\sigma_1((1-\mathbb{E})X)$, and, analogously, its quasi-symmetric functions can be defined as the coefficients of the expansion of the non-commutative Cauchy kernel

$$\sigma_1((1-\mathbb{E})A) = e^{-S_1(A)}\sigma_1(A) \tag{17}$$

on any basis of noncommutative symmetric functions.

For example, since the quasi-monomial basis M_I is dual to S^I, we have

$$\sigma_1((1-\mathbb{E})A) = \sum_{n\geq 0} S_n((1-\mathbb{E})A) = \sum_I M_I(1-\mathbb{E})S^I(A), \tag{18}$$

so that

$$S_1((1-\mathbb{E})A) = 0,\ S_2((1-\mathbb{E})A) = S_2(A) - S^{11}(A)/2, \tag{19}$$

$$S_3((1-\mathbb{E})A) = S_3(A) - S^{21}(A) + S_{111}(A)/3, \tag{20}$$

hence impliying

$$M_1(1-\mathbb{E}) = 0,\ M_2(1-\mathbb{E}) = 1,\ M_{11}(1-\mathbb{E}) = -1/2, \tag{21}$$

and

$$M_3(1-\mathbb{E}) = 1,\ M_{21}(1-\mathbb{E}) = -1,\ M_{12}(1-\mathbb{E}) = 0,\ M_{111}(1-\mathbb{E}) = 1/3. \tag{22}$$

Since for $A = 1$, $S_n((1-\mathbb{E})A) = d_n/n!$, these noncommutative symmetric functions might be called noncommutative derangement numbers (see [29] for other examples of *noncommutative combinatorial numbers*).

Another natural noncommutative analogue of d_n is given by

$$\lambda_{-t}(A)(1 - tS_1(A))^{-1} \tag{23}$$

which gives back the d_n for $A = \mathbb{E}$, and $d_n(q)$ for $A = \frac{1}{1-q}$. Its expansion of the ribbon basis is easily seen to be

$$\sum_{n\geq 0} t^n \sum_{i=1}^{[n/2]} \sum_{|J|=n-2i} R_{1^{2i}\triangleright J} \tag{24}$$

(ribbons whose first column is of even height, or permutations whose position of the first local minimum is even; the observation that these permutations are counted by d_n is due to Désarménien, and an explicit bijection has been given by Désarménien and Wachs [6, 7]).

Similarly,

$$\lambda_{u-t}(A)(1-tS_1(A))^{-1} \tag{25}$$

gives a natural noncommutative analog of the generating series $D(t,u) = e^{u-t}/(1-t)$ for permutations by number of fixed points, and the expansion in **FQSym** of

$$\sigma_t(A)(1-t\,S_1(A))^{-1} \tag{26}$$

yields a class of permutations in bijection with arrangements: for $A = \mathbb{E}$, this series is $\frac{e^t}{1-t}$.

Denote for short by a $\sharp$ the $(1-\mathbb{E})$ transform, *i.e.*,

$$F^\sharp := F((1-\mathbb{E})A) = F * \sigma_1^\sharp \tag{27}$$

where

$$\sigma_1^\sharp = \sigma_1((1-\mathbb{E})A) = e^{-S_1}\sigma_1, \tag{28}$$

that is,

$$S_n^\sharp = \sum_{i=0}^{n}(-1)^i \frac{S_1^i}{i!} S_{n-i}. \tag{29}$$

Lemma 2.1. *The $\sharp$-transform is a projector,* i.e.*,*

$$\sigma_1^\sharp * \sigma_1^\sharp = \sigma_1^\sharp\,, \tag{30}$$

or, equivalently,

$$S_n^\sharp * S_n^\sharp = S_n^\sharp \ \text{ for all } n. \tag{31}$$

Proof – Since the $\sharp$ transform is a homomorphism,

$$\begin{aligned} \sigma_1^\sharp * \sigma_1^\sharp &= (e^{-S_1}\sigma_1) * \sigma_1^\sharp = (e^{-S_1} * \sigma_1^\sharp)\,(\sigma_1 * \sigma_1^\sharp) \\ &= e^{-S_1^\sharp}\,\sigma_1^\sharp = \sigma_1^\sharp. \end{aligned} \tag{32}$$

■

Hence, $S_n^\sharp$ corresponds to an idempotent δ_n of the descent algebra Σ_n. We shall see later that the dimension of the representation $\mathbb{C}\mathfrak{S}_n\delta_n$ is the derangement number d_n and that δ_n coincides with Schocker's derangement idempotent [32], whence the name *derangement algebra* below for the corresponding ideal of the descent algebra.

2.3. The small derangement algebra $\mathcal{D}^{(0)}$.

2.3.1. *Definition of $\mathcal{D}^{(0)}$.* Imitating the construction of the peak algebra in [3], we define the *small derangement algebra*, or the *derangement ideal* as

$$\mathcal{D}^{(0)} := \mathbf{Sym}^\sharp = \mathbf{Sym}(A) * \sigma_1^\sharp\,. \tag{33}$$

By definition of the transformation of alphabet, $\mathcal{D}^{(0)}$ is a Hopf subalgebra of **Sym**, and thanks to the second equality, each homogenous component $\mathcal{D}_n^{(0)}$ is a left ideal of $\mathbf{Sym}_n$ for the internal product, explaing the two names for $\mathcal{D}^{(0)}$, depending on which property we want to emphasize.

2.3.2. *Dimension and bases of* $\mathcal{D}_n^{(0)}$. We have $S_1^\sharp = 0$, and the other $S_n^\sharp$ are clearly algebraically independent. Hence, the dimension $d_n^{(0)}$ of $\mathcal{D}_n^{(0)}$ is given by the number of compositions of n with no part equal to 1. These elements satisfy the induction

$$d_0^{(0)} = 1, \quad d_1^{(0)} = 0, \quad d_n^{(0)} = d_{n-1}^{(0)} + d_{n-2}^{(0)} \text{ for } n \geq 2, \tag{34}$$

hence are shifted Fibonacci numbers.

Let us say that a composition is *non-unitary* if it does not contain the part 1. Then, the $S^{I\sharp}$ with I non-unitary form a basis of $\mathcal{D}_n^{(0)}$.

Since the coproducts

$$\Delta S_n^\sharp = \sum_{i+j=n} S_i^\sharp \otimes S_j^\sharp \tag{35}$$

are given by the same formula as those of the S_n, we see that $\mathcal{D}_n^{(0)}$ is isomorphic to the quotient $\mathbf{Sym}'$ of $\mathbf{Sym}$ by the two-sided ideal generated by S_1. Hence, its (graded) dual is isomorphic to the subalgebra $QSym'$ of $QSym$ spanned by the M_I for I non-unitary.

Since $\sigma_1^\sharp$ is a projector, we have

$$f^\sharp * g^\sharp = (f * g)^\sharp \,. \tag{36}$$

$\mathbf{Sym}'$ being stable under the internal product, $\mathcal{D}_n^{(0)}$ is also $*$-isomorphic to $\mathbf{Sym}'_n$.

Recall that the ribbon basis of $\mathbf{Sym}$ is given by

$$R_I = \sum_{J \leq I} S^J \tag{37}$$

where $\leq$ is the reverse refinement order. If I is non-unitary, so are all $J \leq I$, thus

$$Q_I := R_I^\sharp \qquad (1 \notin I) \tag{38}$$

is a basis of $\mathcal{D}_n^{(0)}$.

2.4. **Algebraic structure of** $(\mathcal{D}_n^{(0)}, *)$**.** Since the construction of $\mathcal{D}^{(0)}$ mimicks the one of the peak ideal, obtained in [3] as the image of the $(1-q)$-tranform at $q=-1$, one may expect that $\mathcal{D}^{(0)}$ shares many properties with the peak ideal. There are some differences however. Whilst the peak ideal has no unity for the internal product $*$, we have:

Proposition 2.2. *For all* n, $(\mathcal{D}_n^{(0)}, *)$ *is a unital algebra with* $S_n^\sharp$ *as neutral element.*

Proof – From Lemma 2.1, we already know that $\sigma_1^\sharp$ is neutral on the right. To prove that it is neutral on the left, let us consider its action on the generating series of a basis of $\mathcal{D}^{(0)}$

$$\sigma_1(X \cdot (1 - \mathbb{E})A) = \sum_I M_I(X) S^{I\sharp}(A) \,. \tag{39}$$

We have

$$\sigma_1^\sharp * \sigma_1(X \cdot (1 - \mathbb{E})A) = \sigma_1^\sharp * \sigma_1(XA) * \sigma_1^\sharp \tag{40}$$

and

$$\begin{aligned}\sigma_1^\sharp * \sigma_1(XA) &= \left(e^{-S_1(A)}\sigma_1(A)\right) * \sigma_1(XA)\\ &= \mu\left[(e^{-S_1(A)} \otimes \sigma_1(A)) *_2 (\sigma_1(XA) \otimes \sigma_1(XA))\right]\\ &= (e^{-S_1(A)} * \sigma_1(XA))\,(\sigma_1(A) * \sigma_1(XA))\\ &= e^{-S_1(XA)}\sigma_1(XA)\end{aligned} \tag{41}$$

so that

$$\begin{aligned}\sigma_1^\sharp * \sigma_1(X\cdot(1-\mathbb{E})A) &= \left[e^{-S_1(XA)}\sigma_1(XA)\right] * \sigma_1^\sharp\\ &= e^{-S_1(X\cdot(1-\mathbb{E})A)}\,\sigma_1(X\cdot(1-\mathbb{E})A)\\ &= \sigma_1(X\cdot(1-\mathbb{E})A)\end{aligned} \tag{42}$$

since $S_1(X\cdot(1-\mathbb{E})A) = 0$. ∎

2.5. **Representation theory of $\mathcal{D}_n^{(0)}$.** Now that we know that $\mathcal{D}_n^{(0)}$ is unital, we can investigate its representation theory. We first look at the idempotents.

2.5.1. *Idempotents in $\mathcal{D}_n^{(0)}$.* Recall that the *(right) Zassenhaus idempotents* ζ_n are defined as the homogeneous elements of degree n (that is, $\zeta_n \in \mathbf{Sym}_n$) satisfying (see [19]):

$$\sigma_1 =: \prod_{k\geq 1} e^{\zeta_k} = e^{\zeta_1}e^{\zeta_2}e^{\zeta_3}\dots. \tag{43}$$

Obviously, $\zeta_1 = S_1$ so that $\zeta_1^\sharp = 0$. Since the $\sharp$-transform is multiplicative,

$$\sigma_1^\sharp = e^{\zeta_2^\sharp}e^{\zeta_3^\sharp}\dots \tag{44}$$

but also

$$\sigma_1^\sharp = e^{-\zeta_1}\sigma_1 = e^{\zeta_2}e^{\zeta_3}\dots \tag{45}$$

so that $\zeta_i^\sharp = \zeta_i$. Extracting the term of degree n, we have

$$S_n^\sharp = \sum_{\substack{|\lambda|=n\\ 1\notin\lambda}} \frac{\zeta^\lambda}{m_\lambda}, \tag{46}$$

where for a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$, $\zeta^\lambda := \zeta_{\lambda_1}\cdots\zeta_{\lambda_r}$, and $m_\lambda := \prod_{i\geq 1} m_i(\lambda)!$, where $m_i(\lambda)$ the multiplicity of i in λ.

We shall make use of the notation $e_\lambda := \frac{\zeta^\lambda}{m_\lambda}$. For a composition I, we set $\zeta^I = \zeta_{i_1}\dots\zeta_{i_r}$ and $m_I = m_\lambda$ if $I\downarrow = \lambda$.

2.5.2. *A basis of idempotents.* The ζ_n are primitive elements with commutative image p_n/n, hence are Lie idempotents [19]. As with any sequence of Lie idempotents, we can construct an idempotent basis of $\mathbf{Sym}_n$ from the ζ_n.

We need the following lemma from [24], easily derived from the splitting formula (compare [19, Lemma 3.10]).

Recall that the radical of $(\mathbf{Sym}_n, *)$ is $\mathcal{R}_n = \mathcal{R}\cap\mathbf{Sym}_n$, where $\mathcal{R}$ is the kernel of the commutative image $\mathbf{Sym} \to Sym$.

Lemma 2.3. *Denote by $\mathfrak{S}(J)$ the set of distinct rearrangements of a composition J. Let $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_s)$ be two compositions of n. Then,*

(i) if $\ell(J) < \ell(I)$ then $\zeta^I * \zeta^J = 0$.

(ii) if $\ell(J) > \ell(I)$ then $\zeta^I * \zeta^J \in \mathrm{Vect}\,\langle \zeta^K \,:\, K \in \mathfrak{S}(J)\rangle \cap \mathcal{R}$. More precisely,

$$\zeta^I * \zeta^J = \sum_{\substack{J_1,\dots J_r \\ |J_k| = i_k}} \langle J\,,\, J_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup J_r\rangle \Gamma_{J_1} \cdots \Gamma_{J_r} \tag{47}$$

where for a composition K of k, $\Gamma_K := \zeta_k * \zeta^K$, and $\sqcup\!\sqcup$ denotes the shuffle products of compositions regarded as words over the positive integers.

(iii) if $\ell(J) = \ell(I)$, then $\zeta^I * \zeta^J \neq 0$ only for $J \in \mathfrak{S}(I)$, in which case $\zeta^I * \zeta^J = m_I\, \zeta^I$.

■

Corollary 2.4. *(i) The elements*

$$e_I = \frac{1}{m_I} \zeta^I\,, \quad I \vDash n\,, \tag{48}$$

are all idempotents and form a basis of $\mathbf{Sym}_n$. This basis contains in particular a complete set of minimal orthogonal idempotents, e_λ of $\mathbf{Sym}_n$.

(ii) The e_I such that I does not have a part equal to 1 form a basis of $\mathcal{D}_n^{(0)}$.

(iii) The e_λ with no part equal to 1 in λ form a complete set of minimal orthogonal idempotents of $\mathcal{D}_n^{(0)}$.

■

2.6. **Cartan invariants of $\mathcal{D}_n^{(0)}$.** By (iii) of lemma 2.3, the indecomposable projective module $P_\lambda = \mathcal{D}_n^{(0)} * e_\lambda$ contains the e_I for $I \in \mathfrak{S}(\lambda)$. For $I \notin \mathfrak{S}(\lambda)$, (i) and (ii) imply that $e_I * e_\lambda$ is in $\mathrm{Vect}\,\langle \zeta^K \,:\, K \in \mathfrak{S}(\lambda)\rangle$. Hence, this space coincides with P_λ. So, we get immediately an explicit decomposition

$$\mathcal{D}_n^{(0)} = \bigoplus_{\lambda \vdash n,\ 1 \notin \lambda} P_\lambda\,, \qquad P_\lambda = \bigoplus_{I \in \mathfrak{S}(\lambda)} \mathbb{C} e_I\,. \tag{49}$$

The Cartan invariants

$$c_{\lambda,\mu} = \dim\,(e_\mu * \mathcal{D}_n^{(0)} * e_\lambda) \tag{50}$$

are now easily obtained. The above space is spanned by the

$$e_\mu * e_I * e_\lambda = e_\mu * e_I\,, \quad I \in \mathfrak{S}(\lambda)\,. \tag{51}$$

From (ii) of Lemma 2.3, this space has the dimension of the space $[S^\mu(L)]_\lambda$, spanned by all symmetrized products of Lie polynomials of degrees $\mu_1, \mu_2, \dots$ formed from $\zeta_{i_1}, \zeta_{i_2}, \dots$, as in the classical result of Garsia-Reutenauer for the descent algebra [15].

In the following examples, partitions are ordered by reverse lexicographic order. For $n \leq 4$, the Cartan matrix of $\mathcal{D}_n^0$ is trivial: it is the identity matrix of size $d_n^{(0)}$, since there is at most one value in a partition of n with no part one. For n up to 7, the Cartan invariants are given by the matrices (60) and (61) at $q = 1$. Indeed, the q-analogues defined from the Loewy series can be explicitely calculated.

2.7. **Quiver and q-Cartan invariants (Loewy series).** We shall use the following modified refinement order on partitions: we write $\lambda \prec_p \mu$, and say that λ is p-finer than μ, if each part of μ is either a part of λ or a sum of distinct parts of λ. Hence, λ covers μ iff μ is obtained from λ by merging two distinct parts.

Still relying upon point (ii) of the lemma, we see that $c_{\lambda,\mu} = 0$ if λ is not p-finer than (or equal to) μ, and that if μ is obtained from λ by adding up two parts λ_i, λ_j, $e_\mu * e_I = 0$ if $\lambda_i = \lambda_j$ and is a nonzero element of the radical otherwise.

The q-analogues of the Cartan invariants

$$c_{\lambda,\mu} := \sum_k q^k \dim\, \left[(e_\mu * \operatorname{rad}^k \mathcal{D}_n^{(0)} * e_\lambda)/\operatorname{rad}^{k+1} \mathcal{D}_n^{(0)}\right] \tag{52}$$

can now be obtained from Proposition 2.2 and the following lemma.

Lemma 2.5. *Let A be an associative algebra. If e is an idempotent of A such that e is neutral in $B = Ae$, then*

$$\operatorname{rad}^k B = e(\operatorname{rad}^k A)e. \tag{53}$$

Proof – Let $x \in \operatorname{rad} B$. There exists an integer n such that $(xB)^n = 0$. Then,

$$(xeAe)^n = (xA)^n e = 0, \tag{54}$$

so that, as well

$$(xA)^n exA = (xA)^n xA = (xA)^{n+1} = 0. \tag{55}$$

Thus, the right ideal xA is nilpotent, which proves that $x \in \operatorname{rad} A$. Since $x = exe$, $x \in e \operatorname{rad} Ae$ and we have shown that $\operatorname{rad} B \subseteq e \operatorname{rad} Ae$.

Conversely, if $x \in \operatorname{rad} A$, so that $x^n = 0$ for a certain n, then $(exe)^n = x^n e = 0$, whence $exe \in \operatorname{rad} B$, which proves the claim for $k = 1$.

Now, if $x \in \operatorname{rad}^k B$, $x = x_1 \dots x_k$ with $x_i = ey_i e$, for some $y_i \in \operatorname{rad} A$. Hence

$$x = ey_1 e \dots ey_k e = ey_1 y_2 \dots y_r e \in e \operatorname{rad}^k Ae. \tag{56}$$

Conversely, any x of the form $ey_1 \dots y_r e$ with $y_i \in \operatorname{rad} A$ can be rewritten as $x = ey_1 e \dots ey_k e \in \operatorname{rad}^k B$. ∎

Applying this to $A = \mathbf{Sym}_n$, and $e = S_n^\sharp$, we obtain from the known description of the q-Cartan matrices of $\mathbf{Sym}_n$ [31]:

Theorem 2.6. *(i) In the quiver of $\mathcal{D}_n^{(0)}$, there is an arrow $\lambda \to \mu$ iff μ is obtained from λ by adding two distinct parts,*
(ii) The q-Cartan invariants of $\mathcal{D}_n^{(0)}$ are given by

$$c_{\lambda,\mu}(q) = c_{\lambda,\mu}\, q^{\ell(\lambda)-\ell(\mu)}. \tag{57}$$

if λ is finer than or equal to μ, and $c_{\lambda,\mu}(q) = 0$ otherwise. ∎

The result can also be derived as follows: In [4], it is shown that the powers of the radical of $\mathbf{Sym}_n$ for the internal product coincide with the homogeneous component of degree n of the lower central series of $\mathbf{Sym}$ for the external product:

$$\mathcal{R}^{*j} = \gamma^j(\mathbf{Sym}) \tag{58}$$

where $\gamma^j(\mathbf{Sym})$ is the ideal of $\mathbf{Sym}$ generated by the commutators $[\mathbf{Sym}, \gamma^{j-1}(\mathbf{Sym})]$.

Since $\mathcal{D}^{(0)}$ is a free associative algebra over a sequence of primitive elements $(\zeta_k)_{k\geq 2}$ with the same internal product as in $\mathbf{Sym}$, the argument of [4] can be reproduced *verbatim*, and we see that

$$(\operatorname{rad}\mathcal{D}^{(0)})^{*j} = \gamma^j(\mathcal{D}^{(0)}) = \mathcal{R}^{*j} \cap \mathcal{D}^{(0)}\,. \tag{59}$$

This shows that, in $\mathcal{D}^{(0)}$ as well as in $\mathbf{Sym}$, for λ finer than μ, $e_\mu * e_I$ is nonzero modulo rad^{*2} iff μ is obtained from λ by summing two distinct parts. And more generally, $e_\mu * e_I$ is in rad^{*k} and nonzero modulo rad^{*k+1} iff $\ell(\lambda) - \ell(\mu) = k$.

2.8. **Tables of the q-Cartan invariants of $\mathcal{D}_n^{(0)}$.** The labels for row and columns of the q-Cartan matrices, namely partitions with no part one, are in reverse lexicographic order. In the following matrices, the zero entries are represented by dots to enhance readability.

For $n \leq 4$, the q-Cartan matrix of $\mathcal{D}_n^0$ is the identity matrix of size $d_n^{(0)}$. The first non-trivial example arises for $n = 5$. The q-Cartan matrices of $\mathcal{D}_5^{(0)}$, $\mathcal{D}_6^{(0)}$, and $\mathcal{D}_7^{(0)}$ are respectively

$$\begin{pmatrix} 1 & q \\ . & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & q & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & q & q & q^2 \\ . & 1 & . & q \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix} \tag{60}$$

and those of $\mathcal{D}_8^{(0)}$ and $\mathcal{D}_9^{(0)}$ are

$$\begin{pmatrix} 1 & q & q & . & q^2 & q^2 & . \\ . & 1 & . & . & q & . & . \\ . & . & 1 & . & . & q & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & q & q & q & q^2 & 2q^2 & . & q^3 \\ . & 1 & . & . & q & q & . & q^2 \\ . & . & 1 & . & . & q & . & . \\ . & . & . & 1 & . & q & . & . \\ . & . & . & . & 1 & . & . & q \\ . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \end{pmatrix} \tag{61}$$

On these matrices, one can read the quiver of $\mathcal{D}_n^{(0)}$. Note that it is a subquiver of the quiver of $\mathbf{Sym}_n$ (see [31]), since one cannot create parts 1 by merging parts of non-unitary partitions.

2.9. **The (large) derangement algebra $\mathcal{D} = \mathcal{D}^{(\infty)}$.** Pursuing the analogy with the peak algebra, let us define

$$\mathcal{D} := \bigoplus_{k\geq 0} S_k\mathcal{D}^{(0)}\,. \tag{62}$$

Note already that $\mathcal{D}$ is not a subalgebra of $\mathbf{Sym}$. It is only a sub-coalgebra. Moreover, we have:

Theorem 2.7. *Each homogeneous component $\mathcal{D}_n$ of $\mathcal{D}$ is stable under $*$. It is a unital algebra, since it contains S_n, the neutral element of $*$ in $\mathbf{Sym}$.*

We shall prove a slightly more general result, interpolating between $\mathcal{D}^{(0)}$ and $\mathcal{D}$.

2.10. **A filtration of $\mathcal{D}$.** Define $\mathcal{D}_n^{(k)}$ by

$$\mathcal{D}_n^{(k)} := \bigoplus_{j=0}^{k} S_j \mathcal{D}_{n-j}^{(0)}. \tag{63}$$

For $k=0$, this is $\mathcal{D}_n^{(0)}$ and for $k \geq n$, one recovers $\mathcal{D}_n$.

The following alternative definition of the filtration will be useful in the sequel:

Lemma 2.8.

$$\mathcal{D}_n^{(k)} = \bigoplus_{j=0}^{\min(n,k)} S_1^j \mathcal{D}_{n-j}^{(0)}. \tag{64}$$

Proof – Expanding $\sigma_1 = e^{S_1}\sigma_1^\sharp$, we see that

$$S_k \equiv \frac{S_1^k}{k!} \pmod{\bigoplus_{j<k} S_j \mathcal{D}_{n-j}^{(0)}}. \tag{65}$$

∎

2.11. **Dimensions of the $\mathcal{D}_n^{(k)}$.** From (63), we see that the dimension $d_n^{(k)}$ of $\mathcal{D}_n^{(k)}$ is

$$d_n^{(k)} = \sum_{i \leq k} d_{n-i}^{(0)}. \tag{66}$$

For $k=\infty$, these are the usual Fibonacci numbers.

The first values are given in the following table:

(67)

n	0	1	2	3	4	5	6	7	8	9
$d_n^{(0)}$	1	0	1	1	2	3	5	8	13	21
$d_n^{(1)}$	1	1	1	2	3	5	8	13	21	34
$d_n^{(2)}$	1	1	2	2	4	6	10	16	26	42
$d_n^{(3)}$	1	1	2	3	4	7	11	18	29	47
$d_n^{(\infty)}$	1	1	2	3	5	8	13	21	34	55

2.12. **The complete picture.**

Theorem 2.9. *For all n and k, each homogeneous component $\mathcal{D}_n^{(k)}$ of $\mathcal{D}^{(k)}$ is stable under $*$. It is a unital algebra and the neutral element is*

$$P_n^{(k)} := \sum_{i=0}^{\min(k,n)} \frac{S_1^i}{i!} S_{n-i}^\sharp. \tag{68}$$

The theorem is a consequence of the following lemma:

Lemma 2.10. *Let f and g be in* **Sym**. *Then*

$$\left(\frac{S_1^m}{m!} f^\sharp\right) * \left(\frac{S_1^n}{n!} g^\sharp\right) = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{S_1^n}{n!}(f^\sharp * g^\sharp) & \text{otherwise.} \end{cases} \tag{69}$$

Proof – Replacing the right factor by its generating series, we have

$$\begin{aligned}(S_1^m f^\sharp) * (e^{S_1} g^\sharp) &= \sum_{(g)} (S_1^m * (e^{S_1} g^\sharp_{(1)}))\, (f^\sharp * (e^{S_1} g^\sharp_{(2)})) \\ &= \sum_{(g)} (e^{S_1} g^\sharp_{(1)} * S_1^m)\, (f^\sharp * (e^{S_1} g^\sharp_{(2)})),\end{aligned} \tag{70}$$

since S_1^m is central for $*$. Now, $(e^{S_1} S^{I^\sharp} * S_1^m) = 0$ if I is not empty since $S_n^\sharp * S_1^n = S_1^n * S_n^\sharp = 0$ for $n \geq 1$ as $e^{-S_1} * \sigma_1^\sharp = 1$. So the whole sum reduces to

$$S_1^m (f^\sharp * (e^{S_1} g^\sharp)) \tag{71}$$

Thus,

$$f^\sharp * \left(e^{S_1} g^\sharp\right) = f * \sigma_1^\sharp * \left(e^{S_1} g^\sharp\right) \tag{72}$$

Consider now the generic case $g = \sigma_1(XA)$:

$$\begin{aligned}\sigma_1^\sharp * \left(e^{S_1} \sigma_1(XA)^\sharp\right) &= \left(e^{-S_1} \sigma_1\right) * \left(e^{S_1} \sigma_1(XA)^\sharp\right) \\ &= \left(e^{-S_1} * e^{S_1} \sigma_1(XA)^\sharp\right) \left(e^{S_1} \sigma_1(XA)^\sharp\right) \\ &= \left(e^{S_1} * e^{-S_1}\right) \left(\sigma_1(XA)^\sharp * e^{-S_1}\right) \left(e^{S_1} \sigma_1(XA)^\sharp\right) \\ &= e^{-S_1} \left(e^{-S_1} * \sigma_1(XA)^\sharp\right) \left(e^{S_1} \sigma_1(XA)^\sharp\right) \\ &= e^{-S_1} e^{-S_1(XA)^\sharp} e^{S_1} \sigma_1(XA)^\sharp \\ &= e^{-S_1} e^{S_1} \sigma_1(XA)^\sharp = \sigma_1(XA)^\sharp\end{aligned} \tag{73}$$

where the third and fourth equalities come from the fact that e^{-S_1} is central for $*$ and idempotent. Multiplying by f on the left yields

$$f^\sharp * \left(e^{S_1} g^\sharp\right) = f * g^\sharp, \tag{74}$$

whence the statement. ∎

Proof – [of the theorem] From Lemma 2.8, we have $P_n^{(k)} \in \mathcal{D}_n^{(k)}$. Moreover, since the $S_1^k S^{I^\sharp}$ with I non-unitary form a basis of $\mathcal{D}_n^{(k)}$ adapted to the direct sum decomposition, that $P_n^{(k)}$ is neutral on both sides is equivalent to the already known fact that $S_n^\sharp$ is neutral in $\mathcal{D}_n^{(0)}$. ∎

Corollary 2.11. *The map* $\phi_m : \mathcal{D}_k^{(0)} \to \mathcal{D}_{k+m}^{(m)}$ *defined by*

$$\phi_m(f) = \frac{S_1^m}{m!} f \tag{75}$$

is a (non-unital) monomorphism of algebras.

Corollary 2.12. *As an* algebra, $\mathcal{D}_n$ *is isomorphic to the direct sum*

$$\mathcal{D}_n \sim \bigoplus_{k=0}^{n} \mathcal{D}_{n-k}^{(0)}. \tag{76}$$

Proof – As a vector space, $\mathcal{D}_n$ is the direct sum of the spaces $V_n^{(k)} = S_1^k \mathcal{D}_{n-k}^{(0)}$. By Lemma 2.10, $V_n^{(k)} * V_n^{(\ell)} = 0$ for $k \neq \ell$, and by Corollary 2.11, each $V_n^{(k)}$ is a subalgebra isomorphic to $D_{n-k}^{(0)}$. ∎

2.13. **Representation theory of $\mathcal{D}_n^{(k)}$.** We are now in a position to deduce the representation theory of the $\mathcal{D}_n^{(k)}$ from that of $\mathcal{D}_n^{(0)}$. We can extend Corollary 2.4 to all values of k, as a direct consequence of Lemma 2.3.

Corollary 2.13. *Let e_I be the idempotents of* **Sym** *defined in Corollary 2.4.*
(i) The e_J such that $J = (1^j, I)$ with $j \leq k$ and I does not contain a part 1 form a basis of $\mathcal{D}_n^{(k)}$.
(ii) The principal idempotents of $\mathcal{D}_n^{(k)}$ are the e_λ such that $m_1(\lambda) \leq k$. ∎

We can now state the general result on the representation theory of $\mathcal{D}_n^{(k)}$:

Corollary 2.14. *The irreducible representations of $\mathcal{D}_n^{(k)}$ are of dimension 1 and parametrized by partitions of n with at most k parts equal to 1.* ∎

Note that the principal idempotents of $\mathcal{D}_n$ are also a complete set of minimal orthogonal idempotents of $\mathbf{Sym}_n$.

2.14. **q-Cartan matrix of $\mathcal{D}_n^{(k)}$.** We order the labels of the rows and columns of the q-Cartan matrices, namely partitions with at most k ones, first by their number of ones and then in reverse lexicographic order. So, for example, with $n = 5$ and $k = 3$, the order is

$$[5,\ 32,\ 41,\ 221,\ 311,\ 2111]. \tag{77}$$

With this convention, the q-Cartan matrix of $\mathcal{D}_n^{(k)}$ is the block-diagonal matrix obtained by putting on the diagonal the q-Cartan matrices of $\mathcal{D}_{n-i}^{(0)}$ for $i \leq \min(k, n)$. Indeed, from the previous results, one easily sees that

Lemma 2.15.

$$\mathcal{D}_n^{(k)} = P_n^{(k)} * \mathbf{Sym}_n * P_n^{(k)}. \tag{78}$$

Since each $P_n^{(k)}$ is a sum of orthogonal idempotents (the $S_1^i S_{n-i}^\sharp / i!$), this proves that the q-Cartan matrix of $\mathcal{D}_n^{(\infty)}$ is deduced from the q-Cartan matrix of $\mathbf{Sym}_n$ by putting to 0 the entries whose row and column do not have the same number of ones.

2.15. **Block projectors.** We have seen that each space $S_1^j \mathcal{D}_{n-j}^{(0)}$ has as unit a part of $P_n^{(k)}$, namely

$$D_{n,k} = \frac{S_1^k}{k!} S_{n-k}^\sharp \tag{79}$$

Their double generating series is

$$D(t, u; A) := \sum_{n \geq 0} \sum_{k=0}^{n} t^{n-k} u^k D_{n,k}(A) = e^{(u-t)S_1(A)} \sigma_t(A), \tag{80}$$

that is, the noncommutative analog of the generating function of $d_{n,k}$, the number of permutations in $\mathfrak{S}_n$ with exactly k fixed points:

$$D(t,u) = \sum_{n\geq 0} \frac{1}{n!} \sum_{k=0}^{n} d_{n,k} u^k t^{n-k} = \frac{e^{u-t}}{1-t}. \tag{81}$$

2.16. **q-dimension polynomials.** If one sums up the entries of the q-Cartan matrix of $\mathcal{D}_n^{(\infty)}$, one gets the following polynomials in q, refining the Fibonacci numbers:

$$1,\ 2,\ 3,\ 5,\ q+7,\ 2\,q+11,\ q^2+5\,q+15,\ 3\,q^2+9\,q+22,\ldots \tag{82}$$

better represented in the following triangle:

$$\begin{array}{ccccccc}
1 & & & & & & \\
2 & & & & & & \\
3 & & & & & & \\
5 & & & & & & \\
7 & 1 & & & & & \\
11 & 2 & & & & & \\
15 & 5 & 1 & & & & \\
22 & 9 & 3 & & & & \\
30 & 17 & 7 & 1 & & & \\
42 & 28 & 16 & 3 & & & \\
56 & 47 & 31 & 9 & 1 & & \\
77 & 73 & 58 & 21 & 4 & & \\
101 & 114 & 102 & 47 & 12 & 1 & \\
135 & 170 & 175 & 94 & 32 & 4 & \\
176 & 253 & 286 & 183 & 74 & 14 & 1 \\
231 & 365 & 461 & 333 & 162 & 40 & 5
\end{array} \tag{83}$$

The first column (constant terms of the polynomials) corresponds to the size of the matrix, hence the number of partitions of n. The second column is sequence A000097 of [34]: from the characterization of the quiver of $\mathbf{Sym}_n$, we also have that the second column gives to the number of ways of selecting two different parts different from 1, in all partitions of n. Finally, it is also equal to the number of ways of selecting two different parts, in all partitions of $n-2$, hence justifying that the number of arrows in the quiver of $\mathcal{D}_n^{(\infty)}$ is equal to the number of arrows in the quiver of $\mathbf{Sym}_{n-2}$.

3. The $(1-\mathbb{E})$ transform in $\mathbf{WQSym}^*$

3.1. **Word quasi-symmetric functions.** A word u over $\mathbb{N}^*$ is said to be *packed* if the set of letters occuring in u is an interval of $\mathbb{N}^*$ containing 1. The algebra $\mathbf{WQSym}(A)$ (Word Quasi-Symmetric functions) is defined as the subalgebra of $\mathbb{K}\langle A\rangle$ based on *packed words* and spanned by the elements

$$\mathbf{M}_u(A) := \sum_{\mathrm{pack}\,(w)=u} w, \tag{84}$$

where $\mathrm{pack}\,(w)$ is the *packed word* of w, that is, the word obtained by replacing all occurrences of the k-th smallest letter of w by k. For example,

$$\mathrm{pack}\,(871883319) = 431442215. \tag{85}$$

This is the invariant algebra of the quasi-symmetrizing action of $\mathfrak{S}(A)$ on $\mathbb{K}\langle A\rangle$ [10]. Packed words can be identified with set compositions in an obvious way, and geometrically, they can be interpreted as facets of the permutohedron: a packed word $w = w_1 \dots w_n$ with largest entry ℓ can be identified with the set composition $[P_1, \dots, P_\ell]$ where $P_j = \{i \leq n \mid w_i = j\}$. For example, 431442215 corresponds to $[\{3,8\},\{6,7\},\{2\},\{1,4,5\},\{9\}]$.

Let $\mathbf{N}_u = \mathbf{M}_u^*$ be the dual basis of $(\mathbf{M}_u)$. It is known that **WQSym** is a self-dual Hopf algebra [18, 27] and that on the graded dual $\mathbf{WQSym}^*$, an internal product $*$ may be defined by

$$\mathbf{N}_u * \mathbf{N}_v = \mathbf{N}_{\mathrm{pack}\,(u,v)}, \tag{86}$$

where the packing of biwords is defined with respect to the lexicographic order on biletters, so that, for example,

$$\mathrm{pack}\begin{pmatrix}42412253\\53154323\end{pmatrix} = 62513274. \tag{87}$$

This product is induced from the internal product of parking functions [26, 25, 28] and allows one to identify the homogeneous components $\mathbf{WQSym}_n$ with the (opposite) Solomon-Tits algebras, in the sense of [30].

The (opposite) Solomon descent algebra, realized as $\mathbf{Sym}_n$, is embedded in the (opposite) Solomon-Tits algebra realized as $\mathbf{WQSym}_n^*$ by

$$S^I = \sum_{\mathrm{ev}\,(u)=I} \mathbf{N}_u, \tag{88}$$

where $\mathrm{ev}\,(u)$ is the evaluation of u.

From now on, we shall denote $\mathbf{WQSym}^*$ by $\mathcal{W}$.

3.2. **Idempotents of $\mathcal{W}$.**

3.2.1. *The semi-simple quotient.* It is known that the radical of $\mathcal{W}_n$ is spanned by the differences

$$\mathbf{N}_u - \mathbf{N}_v, \tag{89}$$

where $v = \sigma(u)$ for some permutation σ of the support of u, $\mathrm{supp}(u) = \{i \mid |u|_i \neq 0\}$. This is easily seen: Equation (86) implies that the $\mathbf{N}_u - \mathbf{N}_v$ are nilpotent of order 2 if $v = \sigma(u)$, and that their span is an ideal $\mathcal{R}_n$. Moreover, any product of n such factors $\mathbf{N}_{u_i} - \mathbf{N}_{v_i}$ vanishes since the product of two such factors is either strictly finer than u_i or zero, so that $\mathcal{R}_n$ is nilpotent. The quotient $W_n/\mathcal{R}_n$ is semi-simple. It can be identified with the (commutative) algebra of set partitions with $\wedge$ (the inf for the refinement order on set partitions) as product. Indeed, packed words encode

set compositions, $u = u_1 \dots u_n$ corresponding to the set composition of $[n]$ in which i belongs to the block u_i, *e.g.*,

$$u = 21231 \Longleftrightarrow [\{2,5\},\{1,3\},\{4\}]. \tag{90}$$

and the left action of permutations amounts to permuting the blocks, *e.g.*, with $\sigma = 231$,

$$\sigma(21231) = 32312 \Longleftrightarrow [\{4\},\{2,5\},\{1,3\}]. \tag{91}$$

Hence, the idempotents of a complete family of $\mathcal{W}_n$ are parametrized by set partitions of $[n]$.

3.2.2. *The idempotents of Saliola.* In [31], Saliola has given a general recipe for constructing such complete sets. Given a packed word u, denote by $\Pi(u)$ the set partition obtained by forgetting the order among the blocks of the set compositions encoded by u.

For each set partition π of $[n]$, choose a linear combination

$$l_\pi = \sum_{\Pi(u)=\pi} c_u \mathbf{N}_u, \tag{92}$$

where the coefficient c_u depends only on the evaluation $\mathrm{ev}\,(u)$ of u, and

$$\sum_{\Pi(u)=\pi} c_u = 1. \tag{93}$$

Start with the initial condition

$$e_{\{1\},\{2\},\dots,\{n\}} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathbf{N}_\sigma, \tag{94}$$

hence equal to $S_1^n/n!$ in $\mathbf{Sym}$, and define by the induction

$$e_\pi = l_\pi * (\mathbf{N}_{1^n} - \sum_{\pi' > \pi} e_{\pi'}) \tag{95}$$

where $\mathbf{N}_{1^n} = S_n$ is the identity of $*$, and $\geq$ is the refinement order. Then, the e_π form a complete set of orthogonal idempotents of $\mathcal{W}_n$.

3.2.3. *A non-recursive construction.* Families of Saliola idempotents can be computed for all $\mathcal{W}_n$ simultaneously, in a non-recursive way from families of idempotents of the descent algebra $\mathbf{Sym}_n$ constructed by the method developed in [19]. Recall that the starting point of this construction is a sequence of Lie idempotents $\gamma_n \in \mathbf{Sym}_n$, that is, an arbitrary sequence of primitive elements whose commutative image in Sym is p_n/n.

Then, if we decompose the identity S_n of $\mathbf{Sym}_n$ as

$$S_n = \sum_{I \vDash n} c_I \gamma^I, \tag{96}$$

the elements

$$e_\lambda := \sum_{I \downarrow n} c_I \gamma^I, \tag{97}$$

with λ a partition of n form a complete family of orthogonal idempotents of $\mathbf{Sym}_n$.

Let us fix such a family, and define for each set partition π of $[n]$

$$l_\pi = \sum_{\Pi(u)=\pi} c_{\mathrm{ev}\,(u)} \mathbf{N}_u. \tag{98}$$

These elements satisfy Saliola's conditions: obviously, $c_{\mathrm{ev}\,(u)}$ depends only on $\mathrm{ev}\,(u)$, and

$$\sum_{\Pi(u)=\pi} c_{\mathrm{ev}\,(u)} = \prod_i m_i(\lambda)! \sum_{I\downarrow\lambda} c_I \tag{99}$$

is the coefficient of p_λ/z_λ in the commutative image of S_n, which is h_n, so it is 1. Hence, the sequence (γ_n) determines idempotents e_π of the $\mathcal{W}_n$ by the recursion (95). But we can also compute these directly as follows:

Theorem 3.1. *The idempotents e_π are given by the internal products*

$$e_\pi = l_\pi * e_\lambda. \tag{100}$$

Proof – Let $\tilde{e}_\pi = l_\pi * e_\lambda$. For $\pi = \{\{1\}, \dots, \{n\}\}$, we have $l_\pi = S_1^n/n!$, $e_\lambda = S_1^n/n!$, so that $\tilde{e}_\pi = S_1^n/n!$.

Let $l_\lambda = \sum_{\Lambda(\pi)=\lambda} l_\pi$, where $\Lambda(\pi)$ is the integer partition recording the block lengths of π. We have

$$l_\lambda = \sum_{I\downarrow\lambda} c_I S^I \equiv e_\lambda \mod \bigoplus_{l(I)>l(\lambda)} \mathbb{K}\gamma^I, \tag{101}$$

so that $e_\lambda = l_\lambda * e_\lambda$ in $\mathbf{Sym}$.

We want to show that $\tilde{e}_\pi = e_\pi$. For that, recall from [31] that, if $\Pi(u) \not\leq \pi$, $\mathbf{N}_u * e_\pi = 0$, so that $l_\pi * e_{\pi'} = 0$ if $\pi \not> \pi'$, and $\Lambda(\pi') \neq \Lambda(\pi)$. This implies in particular that $e_\lambda = \sum_{\Lambda(\pi)=\lambda} e_\pi$. Indeed, this is true for $\lambda = 1^n$, and, by induction, $e_\pi = l_\pi * (\mathbf{N}_{1^n} - \sum_{l(\pi')>l(\pi)} e_{\pi'})$, since $\pi' > \pi$ implies $l(\pi') > l(\pi)$. Hence

$$e_\pi = l_\pi * (\mathbf{N}_{1^n} - \sum_{l(\lambda')>l(\lambda)} e_{\lambda'}) = l_\pi * \sum_{l(\lambda')\leq l(\lambda)} e_{\lambda'}. \tag{102}$$

Summing over π, we get

$$\sum_{\Lambda(\pi)=\lambda} e_\pi = l_\lambda * \sum_{l(\lambda')\leq l(\lambda)} e_{\lambda'} = e_\lambda. \tag{103}$$

Now,

$$\begin{aligned} e_\pi &= l_\pi * (\mathbf{N}_{1^n} - \sum_{\pi'>\pi} e_{\pi'}) = l_\pi * \sum_{\pi'\not>\pi} e_{\pi'} \\ &= l_\pi * (e_\lambda + \sum_{\pi'\not>\pi;\Lambda(\pi')\neq\Lambda(\pi)} e_{\pi'}) = l_\pi * e_\lambda = \tilde{e}_\pi. \end{aligned} \tag{104}$$

■

3.3. **The $(1-\mathbb{E})$-transform in $\mathcal{W}$.** The embedding (88) of **Sym** in $\mathcal{W}$ can be defined on the generators as

$$S_n \longmapsto \mathbf{N}_{1^n}\,. \tag{105}$$

It is clearly a bialgebra morphism. The element

$$\sigma_1^\sharp = e^{-\mathbf{N}_1} \sum_{n\geq 0} \mathbf{N}_{1^n} \tag{106}$$

is well-defined in $\mathcal{W}$, and so is the $\sharp$-transform

$$F^\sharp := F * \sigma_1^\sharp\,. \tag{107}$$

3.4. **Bases of $\mathcal{W}^\sharp$.** Let us say that a packed word u is *non-unitary* (and *unitary* otherwise) if no letter occurs exactly once in u. These words correspond to set compositions without singletons.

Proposition 3.2. *The $\mathbf{N}_u^\sharp$ with u non-unitary form a basis of $\mathcal{W}^\sharp$.*

Proof – Let us say that v is finer than u (and write $v > u$) if the set composition encoded by v is finer than the set composition encoding v. Then,

$$\mathbf{N}_u * \sigma_1^\sharp = \mathbf{N}_u + \sum_v c_{uv}\mathbf{N}_v, \tag{108}$$

where $v > u$ or v is unitary. Hence, the $\mathbf{N}_v^\sharp$ with u non-unitary are linearly independent. ∎

3.5. **Algebraic structure of $\mathcal{W}^\sharp$.** Let $\mathcal{J}$ be the two-sided ideal of $\mathcal{W}$ generated by the $\mathbf{N}_u$ such that u has at least a letter occuring exactly once. The product rule (86) shows that $\mathcal{J}$ is an ideal for the internal product as well. Hence, the projection

$$\pi :\ \mathcal{W} \longrightarrow \mathcal{W}/\mathcal{J} \tag{109}$$

is a morphism for $*$. Its restriction to $\mathcal{W}^\sharp$ is then an isomorphism, and clearly,

$$\pi(\sigma_1^\sharp) = \sigma_1\,. \tag{110}$$

Since σ_1 is neutral in $\mathcal{W}/\mathcal{J}$, we have:

Proposition 3.3. *$\sigma_1^\sharp$ is neutral in $\mathcal{W}^\sharp$.* ∎

Note that this proof would apply to **Sym** as well. To summarize,

Proposition 3.4. *$\mathcal{W}^\sharp$ is isomorphic to $\mathcal{W}/\mathcal{J}$ as a Hopf algebra, and each $\mathcal{W}_n^\sharp$ is $*$-isomorphic to $\mathcal{W}_n/\mathcal{J}_n$, with $D_n = S_n^\sharp = \mathbf{N}_{1^n}^\sharp$ as neutral element.* ∎

3.6. **Representation theory of $\mathcal{W}^\sharp$.** We can now apply Lemma 2.5 with $A = \mathcal{W}_n$, $B = \mathcal{W}_n^\sharp$ and $e = \mathbf{S}_n^\sharp$.

Thus

$$\operatorname{rad}^k \mathcal{W}_n^\sharp = D_n * (\operatorname{rad}^k \mathcal{W}_n) * D_n. \tag{111}$$

The irreducible representations of $\mathcal{W}_n$, which are one-dimensional are parametrized by set partitions of $[n]$.

The q-Cartan matrices and quiver of $\mathcal{W}_n$ have been determined in [32]:

$$c_{\alpha,\beta}(q) = c_{\alpha,\beta} q^{l(\alpha)-l(\beta)}, \tag{112}$$

where $l(\pi)$ is the number of blocks of a set partition π, and the Cartan invariant $c_{\alpha,\beta}$ is 0 if α is not finer than β, and otherwise

$$c_{\alpha,\beta} = \prod_i (m_i - 1)!, \tag{113}$$

where for each block B_i of β, m_i is the number of blocks of α into which B_i has been split.

For example, with $\alpha = 12|3|4|56|7$ and $\beta = 1234|567$, we have: $c_{\alpha,\beta} = (3-1)!(2-1)! = 2$, $l(\alpha) = 5$, $l(\beta) = 2$, so that $c_{\alpha,\beta}(q) = 2q^3$.

Theorem 3.5. *The q-Cartan matrix of $\mathcal{W}_n^\sharp$ is the restriction to rows and columns indexed by non-unitary set partitions of $[n]$ of (112). In particular, the vertices of the quiver of $\mathcal{W}_n^\sharp$ are the non-unitary set partitions, and there is an arrow $\alpha \to \beta$ whenever β is obtained from α by merging two blocks.*

3.7. **Analogue of $\mathcal{D}$ in $\mathcal{W}$.** Let $\mathcal{V}_n^{(0)} = \mathcal{W}_n^\sharp$ and

$$\mathcal{V}_n^{(k)} = \bigoplus_{k=0}^{n} D_{n,k} * \mathcal{W}_n * D_{n,k}\,. \tag{114}$$

Then, as in the case of **Sym**, each $\mathcal{V}_n^{(k)}$ is a unital subalgebra of $\mathcal{W}_n^\sharp$.

4. The $(1-\mathbb{E})$-transform in **FQSym**

4.1. **Definition.** Recall that **FQSym** is based on permutations, that in the mutually dual bases $\mathbf{F}_\sigma = \mathbf{G}_{\sigma^{-1}}$, the internal product is defined by

$$\mathbf{F}_\sigma * \mathbf{F}_\tau = \mathbf{F}_{\sigma\tau} \quad \text{or equivalently} \quad \mathbf{G}_\sigma * \mathbf{G}_\tau = \mathbf{G}_{\tau\sigma}\,, \tag{115}$$

and that **Sym** is embedded into **FQSym** by $S_n = \mathbf{G}_{12\ldots n}$. The transformation can therefore be defined by

$$\mathbf{F}_\sigma^\sharp := \mathbf{F}_\sigma * \sigma_1^\sharp\,. \tag{116}$$

Since the splitting formula remains valid in **FQSym** when the right factor of the internal product is in **Sym** [10], this is again a Hopf algebra morphism.

As we shall see below, in **FQSym**, the idempotent $D_n = S_n^\sharp$ as well as the other $D_{n,k}$ defined in (79) admit an interesting interpretation.

4.2. **The Tsetlin library (uniform case).** The $(1-\mathbb{E})$-transform in **FQSym** is related to a classical problem in probability theory known as the *Tsetlin library* (see *e.g.,* [2]). This is a Markov chain on $\mathfrak{S}_n$, defined by a shelf of n books, which are randomly picked by users and them put back at the left of the shelf after use. In the uniform case (when all books are picked with the same probability), the determination of the stationary distribution amounts to the diagonalisation of the linear operator on $\mathbb{C}\mathfrak{S}_n$

$$t_n(f) = f\tau_n \tag{117}$$

where

$$\tau_n = 12\dots n + 2134\dots n + 3124\dots n + \cdots + n12\dots n-1 \in \mathbb{C}\mathfrak{S}_n\,. \tag{118}$$

This problem is also an ingredient of the proof of Hivert's conjecture by Garsia and Wallach [16]. It can be solved in many different ways. The following one is quite natural in the context on Noncommutative Symmetric Functions.

We start with the observation that τ_n is in the descent algebra Σ_n. Indeed, $\tau_n = D_{\subseteq\{1\}}$ (the sum of permutations having at most a descent at the first position), so that its representation as a noncommutative symmetric function is $S^{1,n-1}$, a rather well-understood element.

From this remark, we obtain immediately the eigenvalues of t_n. Indeed, according to Proposition 3.12 of [19], these are the scalar products $\langle h_{n-1}h_1, p_\lambda\rangle$ of ordinary symmetric functions. Clearly, the scalar product evaluates to $m_1(\lambda)$, so that the spectrum is $0, 1, 2, \dots, n-2, n$.

Let us now construct the spectral projectors. To this aim, we shall need to evaluate some polynomials in t_n. Let us set $T_n = S^{1,n-1}$ and consider the generating function

$$T = \sum_{n\geq 0} T_n = S_1\sigma_1\,. \tag{119}$$

Since the internal products $T_n * T_m$ are (by definition) 0 for $m \neq n$, we have

$$\sum_n T_n^{*r} = T^{*r} \tag{120}$$

and using iteratively the splitting formula ([19], Prop. 2.1)

$$T * T^{*(r-1)} = (S_1\sigma_1) * T^{*(r-1)} = \mu[(S_1\otimes\sigma_1) *_2 \Delta(T^{*(r-1)})], \tag{121}$$

we get the expression

$$T^{*r} = B_r(S_1)\sigma_1, \tag{122}$$

where $B_r(x)$ are the Bell polynomials (this is the obvious noncommutative analogue of the classical formula for the Kronecker powers of the representation of $\mathfrak{S}_n$ by permutation matrices).

Using the fact that the coefficients of B_n are the Stirling numbers of the second kind $S(n,k)$, we obtain in **Sym**

$$T * (T-1) * (T-2) * \cdots * (T-k+1) = S_1^k\sigma_1\,, \tag{123}$$

and in particular, in degree n,

$$T_n * (T_n - 1) * (T_n - 2) * \cdots * (T_n - n + 1) = S_1^n \tag{124}$$

and as well

$$T_n * (T_n - 1) * (T_n - 2) * \cdots * (T_n - n + 2) = S_1^n \,, \tag{125}$$

and since it is plain that $S_1^n * T_n = nS_1^n$, so that $S_1^n * (T_n - n) = 0$, the minimum polynomial of T_n is

$$P_n(x) = (x - n) \prod_{k=0}^{n-2} (x - k) \,. \tag{126}$$

This shows that T_n is semisimple, and allows an easy construction of the spectral projectors.

Let us start with the kernel. The projector is given by $f \mapsto f * D_n$ where

$$D_n = \frac{(T_n - 1) * (T_n - 2) * \cdots * (T_n - n + 2) * (T_n - n)}{(-1)(-2)\cdots(2 - n)(-n)} \tag{127}$$

but since $T_n - n + 1$ is invertible, one can take as well

$$D_n = \frac{(-1)^n}{n!} (T_n - 1)_{*n} \tag{128}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, and the star means evaluation with the internal product. This is a better choice, since we have now a simple generating series for all these projectors,

$$\sum_{n \geq 0} D_n x^n = e^{-xS_1} \cdot \sigma_x = \sigma_x((1 - \mathbb{E})A) \,. \tag{129}$$

Indeed, we have $(x-1)_n = (x)_n - n(x-1)_{n-1}$, so that

$$\frac{(-1)^n}{n!} (T-1)_{*n} = \frac{(-S_1)^n}{n!} \sigma_1 - \frac{(-1)^{n-1}}{(n-1)!} (T-1)_{*\ n-1} = \sum_{k=0}^{n} \frac{(-S_1)^k}{k!} \sigma_1 \,. \tag{130}$$

The same reasoning shows that the projectors $D_{n,k}$ on the eigenspaces of k are given by the generating series

$$D(t,u) = \sum_{n \geq 0} \sum_{k=0}^{n} t^{n-k} u^k D_{n,k} = e^{(u-t)S_1} \sigma_1 \,, \tag{131}$$

which is $\sigma_1(tA - (t-u)\mathbb{E}A)$, so that these elements coincide with those defined by (79).

4.3. **Characters of the associated modules.** The Frobenius characteristic of the left ideal of $\mathbb{C}\mathfrak{S}_n$ generated by the idempotents $\delta_{n,k}$ corresponding to $D_{n,k}$ via the identification $\sigma \leftrightarrow \mathbf{F}_\sigma$ can now be calculated as follows (compare [32, Cor. 4.2]).

Since $\delta_{n,k}$ is an idempotent, its characteristic $\mathrm{ch}\,(\delta_{n,k})$ coincides with its cycle index $Z(\delta_{n,k})$. By the Gessel-Reutenauer formula [17], the coefficient of p_μ in $Z(\delta_{n,k})$ is

equal to $\langle \underline{D_{n,k}}, L_\mu \rangle$, where $\underline{F}$ means the commutative image of the noncommutative symmetric function F, and for $\nu = 1^{n_1}2^{n_2}\ldots$,

$$L_\nu = h_{n_1}[\ell_1]h_{n_2}[\ell_2]\cdots, \quad \ell_n = \frac{1}{n}\sum_{d|n}\mu(d)p_{n/d}^d\,, \tag{132}$$

μ denoting here the Moebius function. Hence, the generating function of all the cycle indexes is

$$Z_Y(t,u) = \langle D(t,u;X), L(X,Y)\rangle_X = \langle D(t,u;X), L(Y,X)\rangle_X \tag{133}$$

by the symmetry formula [33]

$$L(X,Y) = L(Y,X) = \sum_\mu L_\mu(X)p_\mu(Y) = \prod_{n\geq 1}\sigma_{p_n(X)}[\ell_n(Y)]\,. \tag{134}$$

Plugging this last expression into the scalar product and dualizing, we obtain

$$\prod_{n\geq 1}\sigma_{p_n(t+(u-t)\mathbb{E})}[\ell_n(Y)] = \frac{\sigma_1((u-t)Y)}{1-tp_1(Y)}\,. \tag{135}$$

In particular, specializing at $Y=\mathbb{E}$ gives that the dimension of $\mathbb{C}\mathfrak{S}_n\delta_{n,k}$ is $d_{n,k}$, the number of permutations in $\mathfrak{S}_n$ with exactly k fixed points.

It is also easy to obtain the expansion of $Z_Y(t,u)$ as a combination of the $L_\mu(Y)$. Indeed, writing $D(t,u;x) = \sigma_1((u-t)\mathbb{E}X+tX)$, we have $\langle D(t,u), p_\mu\rangle = u^{m_1}\prod_{i\geq 2}t^{m_i}$, where m_i is the multiplicity of the part i in μ. Hence,

$$Z(\delta_{n,k}) = \sum_{m_1(\mu)=k} L_\mu \tag{136}$$

We note that this is the quasi-symmetric generating function of the permutations with exactly k fixed points. Note that $Z(D(t)$ is the commutative image of the generating series of desarrangements (23).

4.4. **q-derangement numbers.** From the above considerations, one can easily derive a (known) closed formula for the q-derangement numbers (compare [32, Theorem 4.5])

$$d_n(q) := \sum_{\sigma\in D_n} q^{\mathrm{maj}\,\sigma}\,. \tag{137}$$

Indeed,

$$\begin{aligned} d_n(q) &= \left\langle \sum_{\sigma\in D_n}\mathbf{F}_\sigma, \sum_{\tau\in\mathfrak{S}_n} q^{\mathrm{maj}\,(\tau)}\mathbf{G}_\tau\right\rangle_{\mathbf{FQSym}} \\ &= \sum_{\sigma\in D_n}\sum_{|I|=n} q^{\mathrm{maj}\,(I)}\langle F_{C(\sigma)}, R_I\rangle = \left\langle \sum_{m_1(\mu)=0} L_\mu, K_n(q)\right\rangle\,. \end{aligned} \tag{138}$$

Hence,

$$\begin{aligned}\sum_{n\geq 0} d_n(q)\frac{x^n}{(q)_n} &= \left\langle \sigma_1\left[\sum_{n\geq 2} x^n l_n\right], \sigma_1\left(\frac{X}{1-q}\right)\right\rangle \\ &= \left\langle \frac{\lambda_{-x}}{1-xp_1}, \sigma_1\left(\frac{X}{1-q}\right)\right\rangle = \lambda_{-x}\left(\frac{1}{1-q}\right)\left(1-\frac{x}{1-q}\right)^{-1} \\ &= \frac{1-q}{1-x-q}\prod_{n\geq 0}(1-xq^n)\end{aligned} \tag{139}$$

so that finally [35]

$$d_n(q) = [n]!\sum_{k=0}^{n}\frac{(-1)^k}{[k]!}q^{\binom{k}{2}}\,. \tag{140}$$

4.5. **Characters from Lie idempotents.** The expression

$$\mathrm{ch}(\delta_{n,k}) = \sum_{m_1(\lambda)=k} L_\lambda\,, \tag{141}$$

is also a consequence of the following type of expressions

$$D_{n,k} = \sum_{m_1(\lambda)=k} E_\lambda(\pi) \tag{142}$$

in the notation of [19, Theorem 3.16], for some sequence π_n of Lie idempotents in descent algebras. Indeed, [19, Theorem 3.21], implies then that the character is given by (141). We have already seen one such expression with $\pi_n = \zeta_n$, the Zassenhaus idempotents. We can also write one involving the Hausdorff series. Writing as usual

$$\Phi = \sum_{n\geq 1}\phi_n = \log\sigma_1 \tag{143}$$

(the Solomon idempotents), we have

$$\sum_{n\geq 0} D_n = e^{-\phi_1}e^{\Phi} = e^{H(-\phi_1,\Phi)} \tag{144}$$

where H is the Hausdorff series. Taking $\pi_1 = \phi_1$ and $\pi_n = H_n(-\phi_1,\Phi)$ for $n\geq 2$, we obtain a sequence of Lie idempotents (see, *e.g.*, [19], Theorem 3.1), from which it is easy to build a decomposition of the identity

$$\sigma_1 = e^{\pi_1}\exp\left\{\sum_{n\geq 2}\pi_n\right\}\,, \tag{145}$$

and more explicitely,

$$S_n = \sum_{r+s=n}\frac{1}{r!s!}\sum_{\ell(J)=r,|J|=n-s,1\notin J}\pi^{1^s,J}. \tag{146}$$

This gives in particular the decomposition (142), with, for a partition λ such that $m_1(\lambda) = 0$,

$$E_\lambda(\pi) = \frac{1}{\ell(\lambda)!} \sum_{I \downarrow \lambda} \pi^I \,, \tag{147}$$

where $I \downarrow \lambda$ means that the nondecreasing rearrangement of the composition I is the partition λ.

4.6. **Eigenbases of t_n.** From Proposition 7.4 of [11], we know the image a projector of the type (147). It is formed of weighted symmetrizations of Lie elements. With the above π_n, the distribution is uniform, so that the kernel consists in ordinary symmetrized products of Lie elements. Concretely, a basis of $\operatorname{Ker} t_n$ in $\mathbb{C}\mathfrak{S}_n$ is for example

$$(\gamma_1\theta_{\lambda_1}, \gamma_2\theta_{\lambda_2}, \cdots, \gamma_r\theta_{\lambda_r}) \tag{148}$$

where $(a, b, c) = abc + acb + bac + bca + cab + cba$, and so on (symmetrized products), the γ_k are the minimal representatives of the cycles of a derangement, $\theta_n = [[\cdots[1, 2], 3], \cdots n]$ is a Dynkin element, and λ runs over partitions without part 1.

For example, a basis of $\operatorname{Ker} t_4$ of dimension $d_4 = 9$ is given by the elements $[[[1, a], b]c]$ with abc running over permutations of 234, for L_4, and by the three symmetrized products $([1, 2], [3, 4])$, $([1, 3], [2, 4])$, and $([1, 4], [2, 3])$ for L_{22}.

Bases of the other eigenspaces are obtained by the same process, using weighted symmetrizations as indicated in [11]. Indeed, Equation (146) shows that a basis of the eigenspace with eigenvalue s is given by

$$(\gamma_1\theta_{\lambda_1}, \gamma_2\theta_{\lambda_2}, \cdots, \gamma_r\theta_{\lambda_r}) \cdot (j_1 \sqcup\!\sqcup j_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup j_s) \tag{149}$$

where γ_k are the minimal representatives of the cycles of length at least 2 a permutation of cycle type $(\lambda, 1^s)$ having s fixed points $j_1, j_2, \ldots, j_s$.

To continue with $n = 4$, a basis of the 1 eigenspace is $[[i, j], k] \cdot l$ ($i < j, k$, $ijkl$ a permutation of 1234), dimension 8, and a basis of the 2-eigenspace is given by

$$[i, j] \cdot (kl + lk) \;\; i < j, \;\; k < l$$

where $ijkl$ is a permutation of 1234 (dimension 6). Finally, the 4-eigenspace is one dimensional and generated by the full symmetrizer.

Using the ζ_n instead of the π_n, we can replace symmetrized products by ordinary products of homogeneous Lie polynomials taken in nondecreassing order of the degrees.

The idempotents $\delta_{n,k}$ have been first studied by M. Schocker [32] (apparently unaware of previous works on the subject and of their relation with the Tsetlin library).

4.7. **A basis of $\mathbf{FQSym}^\sharp$.** We have seen that the $(1 - \mathbb{E})$-transform is a bialgebra morphism in **FQSym**. Hence, its image $\mathbf{FQSym}^\sharp$ is a Hopf subalgebra. The $\mathfrak{S}_n$-module $\Delta_{n,k}$ can be identified with

$$\mathbf{FQSym}^\sharp_n = \mathbf{FQSym}_n * D_{n,k}, \tag{150}$$

so that

$$\dim \mathbf{FQSym}_n^\sharp = d_n. \tag{151}$$

It is therefore desirable to find a basis of $\mathbf{FQSym}^\sharp$ labeled by derangements, or some other set of permutations naturally in bijection with these. As we shall see, the natural transformation involved here is simply a version of Foata's first fundamental transformation [14].

Let γ_n be the cycle

$$\gamma_n := n\,1\,2\ldots n-1, \tag{152}$$

so that

$$\mathbf{S}^{\gamma_n} = T_n = S_1 S_{n-1} = R_n + R_{1,n-1} = \sum_{\sigma \in 1 \sqcup\!\sqcup 23\ldots n} \mathbf{F}_\sigma. \tag{153}$$

Since the $\sharp$-transform is a morphism for the product of $\mathbf{FQSym}$,

$$\mathbf{S}^{\gamma_n} * \sigma_1^\sharp = S_1^\sharp S_{n-1}^\sharp = 0, \tag{154}$$

and, for any permutation $\sigma \in \mathfrak{S}_n$,

$$(\mathbf{F}_\sigma * \mathbf{S}^{\gamma_n})^\sharp = \mathbf{F}_\sigma * \mathbf{S}^{\gamma_n} * \sigma^\sharp = 0. \tag{155}$$

Recall that i is a *left-right minimum* of σ if

$$\sigma_j > \sigma_i \text{ for all } j < i. \tag{156}$$

Let X_n be the set of permutations of $\mathfrak{S}_n$ such that $\sigma \cdot 0$ does not have two consecutive left-right minima (that is, σ does not end by 1 and does not have two consecutive LR-minima), and let $Y_n = \mathfrak{S}_n \backslash X_n$.

Lemma 4.1. *For $\sigma \in Y_n$, write*

$$\sigma \cdot 0 = \cdots \sigma_i \sigma_{i+1} \cdots \tag{157}$$

where i and $i+1$ is the first pair of consecutive LR-minima, and let

$$\sigma' = \sigma_i \cdot \sigma_1 \cdots \hat{\sigma_i} \cdots \sigma_n \tag{158}$$

be the permutation obtained by moving σ_i at the first position, leaving the remaining letters unchanged, and removing the zero in the end. Then

$$\mathbf{F}_{\sigma'} * \mathbf{S}^{\gamma_n} = \mathbf{F}_\sigma + \sum_{\tau \in T} \mathbf{F}_\tau, \tag{159}$$

where the permutations of T are lexicographically smaller than σ.

Proof – The expression

$$\mathbf{F}_{\sigma'} * \mathbf{S}^{\gamma_n} = \sum_{\tau \in \sigma_i \sqcup\!\sqcup \sigma_1 \ldots \hat{\sigma_i} \ldots \sigma_n} \mathbf{F}_\tau \tag{160}$$

contains $\mathbf{F}_\sigma$ and, since σ_i is a LR-minimum, σ is the maximal element of the previous sum. ∎

For a permutation σ with LR-minima $i_1, \ldots, i_p$, let

$$\phi(\sigma) = (\sigma_1 \ldots \sigma_{i_1 - 1})(\sigma_{i_1} \ldots \sigma_{i_2 - 1}) \ldots (\sigma_{i_p} \ldots \sigma_n), \tag{161}$$

where each parenthesis represents a cycle. For example, with $\sigma = 62781453$,

$$\phi(\sigma) = (6)(278)(1453) = 47153682. \tag{162}$$

This is Foata's first fundamental transformation (up to reversing the order on the integers), hence a bijection. Clearly, $\phi(\sigma)$ has fixed points whenever $\sigma \in Y_n$, so ϕ induces a bijection between X_n and derangements of $\mathfrak{S}_n$.

From Lemma 4.1, we see that the elements

$$(\mathbf{F}^\sharp_\sigma)_{\sigma\in X_n} \tag{163}$$

span $\mathbf{FQSym}^\sharp_n$. Since $|X_n| = d_n = \dim \mathbf{FQSym}^\sharp_n$, we have finally

Theorem 4.2. *The $\mathbf{F}^\sharp_\sigma$ for $\sigma \in X_n$ form a basis of $\mathbf{FQSym}^\sharp$.* ∎

The sets X_n have an interesting structure.

Theorem 4.3. *The set X_n is an ideal of the left weak order on $\mathfrak{S}_n$. Its maximal elements are the left-shifted concatenations*

$$w_I := w_{i_1} \blacktriangleright \ldots \blacktriangleright w_{i_r}, \tag{164}$$

where $w_i := 1\,i\,i-1,\ldots,2$, composition I has no part 1, and $\alpha \blacktriangleright \beta = \alpha[\ell] \cdot \beta$ if $\beta \in \mathfrak{S}_\ell$.

Proof – To show that X_n is an ideal, we will prove that if s_i denotes the elementary transposition $(i, i+1)$, then $\sigma \in Y_n$ and $\mathrm{inv}\,(s_i\sigma) = \mathrm{inv}\,(\sigma) + 1$, implies $s_i\sigma \in Y_n$.

If $\sigma_k = r > s = \sigma_{k+1}$ are consecutive LR-minima of σ, they will remain so for $s_i\sigma$, unless $i = r-1$, r, $s-1$, or s. Since $s_i\sigma$ has one inversion more than σ, we can exclude the case $i = r-1$: r being a LR-minimum, $r-1$ cannot be to the left of r in σ. We can also exclude $i = s-1$ for the same reason. If $i = r$, then r is exchanged with $r+1$, which has to be to its right in σ, so that again σ_k and σ_{k+1} are consecutive LR-minima in $s_i\sigma$. The same reasoning applies with $i = s$. Hence Y_n is a coideal, and consequently X_n is an ideal.

Now, the elements w_I are clearly in X_n when I has no part 1, and any exchange of consecutive values creating an inversion in such a w_I would create a pair of consecutive LR-minima. So these w_I are maximal elements of the ideal X_n.

Conversely, consider $\sigma \in X_n$ maximal. Then consider the suffix s of σ beginning with 1. The maximality condition of σ implies that if t belongs to that suffix, then $t-1$ also belongs to it. So this prefix is a permutation of an $\mathfrak{S}_{|s|}$, then should be $1|s|\ldots 2$. The same now works by induction on the permutation τ defined by $\sigma = \tau \blacktriangleright s$. ∎

For example, with $n = 5$, we get the following three maximal elements of X_n:

$$15432,\ 35412,\ 45132. \tag{165}$$

The same proof can be adapted to the case of permutations that are images by ϕ of permutations with at most k fixed points. Let $X_n^{(k)}$ be the image by ϕ of permutations with at most k fixed points. Then $X_n^{(k)}$ is the set of permutations with at most k consecutive LR-minima.

Theorem 4.4. *The set $X_n^{(k)}$ is a ideal of the left weak order on $\mathfrak{S}_n$. Its maximal elements are the w_I where I runs over compositions with*

- *either $k-1$ ones and the remaining parts equal to 2,*
- *or exactly k ones.*

Proof – The fact that $X_n^{(k)}$ is an ideal comes from the same idea as before: all permutations greater than a given permutation σ for the left weak order have LR-minima at the same position.

By the same argument as in the previous theorem, the maximal elements must be some w_I, where I has at most k ones. Now, it is clear that

$$w_I < w_J, \tag{166}$$

in the left weak order iff I can be obtained from J by gluing parts equal to 1 with their next part. So the compositions described in the statement are definitely maximal elements. And since all compositions with at most k ones can be obtained from these ones by the gluing process, this ends the proof. ∎

Here is a table of the number of maximal elements of $X_n^{(k)}$

(167)

$n\backslash k$	0	1	2	3	4	5	6	7	8
1	0	1							
2	1	1	1						
3	1	2	2	1					
4	2	3	3	3	1				
5	3	5	6	4	4	1			
6	5	9	9	10	5	5	1		
7	8	15	19	14	15	6	6	1	
8	13	27	31	34	20	21	7	7	1

Note that the first column is obviously given by Fibonacci numbers since these indeed count the number of compositions of n in parts at least 2. The other columns are not known to [34] and neither is the sequence of row sums.

But there exists a simple formula giving the number of maximal elements, coming directly from their characterization:

$$m_{n,k} := \binom{(n+k-1)/2}{k-1} + \sum_{\ell=0}^{\lfloor[\frac{n-k}{2}]\rfloor} \binom{\ell+k}{k}\binom{n-k-\ell-1}{\ell-1}, \tag{168}$$

with the convention that a binomial coefficient with entries not in the natural numbers is zero.

4.8. **Other bases of FQSym$^\sharp$.**

Conjecture 4.5. *Let $<'$ be the order on permutations defined by*

$$\sigma <' \tau \Longleftrightarrow \phi(\sigma) <_{\text{lex}} \phi(\tau). \tag{169}$$

Then the matrix of $\mathbf{S}^{\sigma\sharp}$ *of the* $\mathbf{S}$ *basis is triangular. Moreover, the diagonal values are* 1 *for the elements of* X_n *and* 0 *for* Y_n.

For example, here are the matrices for $n = 2$, 3, and 4 (Figure 1) where the zero entries have been represented by dots to enhance readability.

The permutations are ordered as follows:

$$[21,\ 12],\qquad [321,\ 312,\ 231,\ 123,\ 132,\ 213]. \tag{170}$$

$$\begin{aligned}&[4321,\ 4312,\ 4231,\ 4123,\ 4132,\ 4213,\ 3421,\ 3412,\\ &2341,\ 1234,\ 1243,\ 2314,\ 2431,\ 1423,\ 3241,\ 2134,\\ &3142,\ 1324,\ 1432,\ 2413,\ 2143,\ 3214,\ 1342,\ 3124].\end{aligned} \tag{171}$$

$$\begin{pmatrix} . & -1/2 \\ . & 1 \end{pmatrix} \qquad \begin{pmatrix} . & . & . & 1/3 & -1/3 & 2/3 \\ . & . & . & -1 & . & -1 \\ . & . & . & . & . & . \\ . & . & . & 1 & . & 1 \\ . & . & . & . & 1 & -1 \\ . & . & . & . & . & . \end{pmatrix} \tag{172}$$

5. Other combinatorial Hopf algebras

5.1. **The algebras PQSym and CQSym.** There is an internal product on **PQSym** extending that of $\mathbf{WQSym}^*$ [26]. The $\sharp$-transform is defined in **PQSym** (it contains **Sym** as a subalgebra), but $\mathbf{PQSym} * \mathbf{WQSym}^* \subseteq \mathbf{WQSym}^*$, so that $\mathbf{PQSym}^\sharp = \mathcal{W}^\sharp$, and we get nothing new.

Similarly, the Catalan algebra **CQSym** [28], we have

$$\mathbf{CQSym} * \mathbf{Sym} \subseteq \mathbf{Sym}, \tag{173}$$

so that

$$\mathbf{CQSym}^\sharp = \mathbf{Sym}^\sharp. \tag{174}$$

5.2. **The algebra of planar binary trees PBT.** The Loday-Ronco algebra of planar binary tree is not stable by the $\sharp$-transform. Since **PBT** is the subalgebra of **FQSym** generated by the S^σ where σ avoids the pattern 132 (see [13]), we have, for example :

$$S^{213\sharp} = S^{123} - S^{132} - S^{312} + \frac{2}{3}S^{321} \notin \mathbf{PBT}. \tag{175}$$

However, $\mathbf{PBT}^\sharp$ is a well-defined Hopf subalgebra of **FQSym**.

Conjecture 5.1. *The algebra* $\mathbf{PBT}^\sharp$ *is free over the set* $\mathbf{P}^\sharp_T$*, where* T *runs over trees with at least two nodes, and such that the right subtree of the root is empty.*

In particular, the conjecture implies that the dimension of the homogeneous components $\mathbf{PBT}^\sharp_n$ are given by the Fine numbers [8, 34], sequence A000957:

$$1,\ 0\,1\,2\,6\,18\,57\,186,\ 622\ldots \tag{176}$$

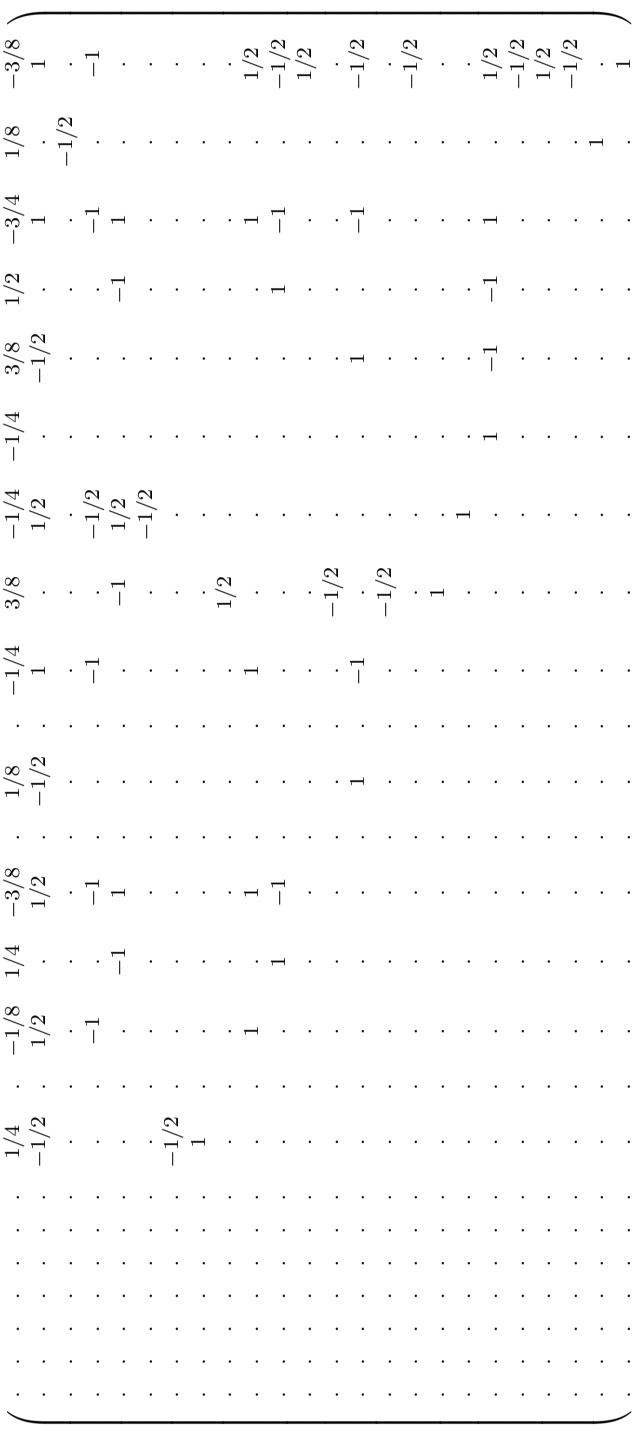

FIGURE 1. The matrix of $\sharp$ in the $\mathbf{S}$ basis of $\mathbf{FQSym}$

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