A MULTIVARIATE "INV" HOOK FORMULA FOR FORESTS

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To Dennis Stanton on his 60th birthday

ABSTRACT. Björner and Wachs provided two q-generalizations of Knuth's hook formula counting linear extensions of forests: one involving the major index statistic, and one involving the inversion number statistic. We prove a multivariate generalization of their inversion number result, motivated by specializations related to the modular invariant theory of finite general linear groups.

1. Introduction

This paper concerns formulas counting linear extensions of partial orders P on the set $\{1, 2, \ldots, n\}$ which are *forests*, in the sense that every element covers at most one other element. Recall that a permutation w is a *linear extension* of the poset P if the linear order $w_1 <_w \ldots <_w w_n$ has the property that $i <_P j$ implies $i <_w j$. Denote by $\mathcal{L}(P)$ the set of all linear extensions of P. Knuth observed the following.

Theorem. (Knuth [7, §5.1.4, Exer. 20]) *For any forest poset P on $\{1, 2, \ldots, n\}$, one has*

$$|\mathcal{L}(P)| = \frac{n!}{\prod_{i=1}^{n} h_i}$$

where $h_i := |P_{\geq i}|$ is the cardinality of the subtree $P_{\geq i}$ rooted at i.

Björner and Wachs [1] later gave two interesting q-generalizations of Knuth's result, both counting linear extensions according to certain statistics: the *inversion number* statistic inv, and the *major index* statistic maj. The following theorem rephrases a special case of the first of these results, relating to inv; see Remark 9.6 below for their second generalization.

Say that a forest poset P is *recursively labelled* if the label set on each subtree $P_{\geq i}$ forms an interval in the integers, that is, $P_{\geq i} = \{a, a+1, \ldots, b-1, b\}$ for some integers $a =: \min(P_{\geq i})$ and $b =: \max(P_{\geq i})$. Define the *inversion number* $\mathrm{inv}(P)$ to be the number of pairs $i <_{\mathbb{Z}} j$ for which $i >_P j$. For example, the following picture

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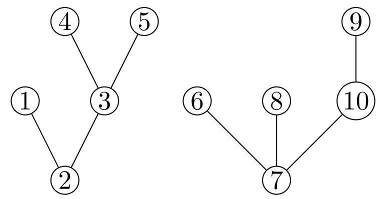
shows the Hasse diagram of a recursively labelled forest P on $\{1, 2, \ldots, 10\}$.



Here one has $P_{\geq 3} = \{3, 4, 5\}, P_{\geq 7} = \{6, 7, 8, 9, 10\}$, and

$$\mathrm{inv}(P) = 3 = |\{(1, 2), (6, 7), (9, 10)\}|.$$

Lastly, define the q-analogues

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1},$$
$$[n]!_q := [n]_q[n-1]_q[n-2]_q \cdots [2]_q[1]_q.$$

Theorem. (Björner and Wachs [1, Thm. 1.1])
Any recursively labelled forest P on $\{1, 2, \ldots, n\}$ has

$$\sum_{w \in \mathcal{L}(P)} q^{\mathrm{inv}(w)} = q^{\mathrm{inv}(P)} \frac{[n]!_q}{\prod_{i=1}^{n} [h_i]_q}. \tag{1.1}$$

Our goal is a multivariate generalization, Theorem 1.1 below. It is an identity within the field of rational functions $\mathbb{Q}(\mathbf{x}) := \mathbb{Q}(x_1, x_2, x_3, \ldots)$ in a sequence of indeterminates $x_1, x_2, x_3, \ldots$, related by a map F sending $x_i \mapsto x_{i+1}$ that we call the *Frobenius map*. We introduce the following multivariate analogues of the positive integers n and the factorial $n!$:

$$\begin{aligned} [1] :=& x_1 \\ [n] :=& [1] + F[1] + F^2[1] + \cdots + F^{n-1}[1] \\ =& x_1 + x_2 + \cdots + x_n \end{aligned} \tag{1.2}$$

$$\begin{aligned} [n]! :=& [n] \cdot F([n-1]) \cdot F^2([n-2]) \cdots F^{n-2}([2]) \cdot F^{n-1}([1]) \\ =& [n] \cdot F\left([n-1]!\right). \end{aligned} \tag{1.3}$$

For example,

$$[4]! = (x_1 + x_2 + x_3 + x_4)(x_2 + x_3 + x_4)(x_3 + x_4)x_4.$$

After defining in Section 5 a weight $\mathrm{wt}(w)$ lying in $\mathbb{Q}(\mathbf{x})$ for each permutation w, we prove in Section 7 the following main result.

Theorem 1.1. *Any recursively labelled forest P on $\{1, 2, \ldots, n\}$ has*

$$L(P) \;:=\; \sum_{w \in \mathcal{L}(P)} \mathrm{wt}(w) \;=\; \frac{[n]!}{\prod_{i=1}^{n} F^{\min(P_{\geq i})-1}[h_i]}.$$

Section 8 explains why Theorem 1.1 becomes (1.1) upon applying the following *q-specialization* map to both sides:

$$\begin{array}{rcl} \mathbb{Q}(x_1, x_2, \ldots) & \xrightarrow{\mathrm{sp}_q} & \mathbb{Q}(q) \\ x_i & \longmapsto & q^{i-1} - q^i\,. \end{array} \tag{1.4}$$

2. Invariant theory motivation

Aside from the Björner-Wachs *inv* formula, a second motivation for Theorem 1.1 stems from previous joint work in invariant theory with D. Stanton [11]. The reader interested mainly in Theorem 1.1 and its connection to the work of Björner and Wachs can safely skip this explanation of the invariant-theoretic connection.

There are two special cases of Theorem 1.1 that turn out be equivalent to results from [11], namely the cases where either

(a) P is a disjoint union of chains, each labelled by a contiguous interval of integers in increasing order [11, Theorem 8.6], or
(b) P is a *hook* poset [11, Eqn. (6.1) and (11.1)], having

$$1 >_P 2 >_P \cdots >_P m <_P m+1 <_P \cdots <_P n-1 <_P n\,.$$

The story from [11] begins with $G := GL_n(\mathbb{F}_q)$ acting by linear substitutions of variables on the polynomial algebra $S(q) := \mathbb{F}_q[x_1, \ldots, x_n]$. A well-known result of L.E. Dickson asserts that the G-invariant subalgebra $S(q)^G$ is again a polynomial algebra.

For each composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of n, one associates two families of G-representations $V(q)$ over $\mathbb{F}_q$, described below. For both of these representations $V(q)$, the graded intertwiner spaces

$$M(q) := \mathrm{Hom}_{\mathbb{F}_q G}(V(q), S(q))$$

were shown in [11] to be *free* modules over $S(q)^G$, and explicit formulas were given for the degrees of their $S(q)^G$-basis elements, or equivalently for the *Hilbert series*

$$\mathrm{Hilb}_q(t) := \mathrm{Hilb}\left(M(q)/S(q)^G_+ M(q)\,,\ t \right).$$

These Hilbert series come from generating functions in $\mathbb{Q}(\mathbf{x})$ by applying the following (q,t)-*specialization map*

$$\begin{array}{ccc} \mathbb{Q}(x_1, x_2, \ldots) & \xrightarrow{\mathrm{sp}_{q,t}} & \mathbb{Q}(q) \\ x_i & \longmapsto & t^{q^{i-1}} - t^{q^i} \end{array} \tag{2.1}$$

which is less drastic than the specialization in (1.4).

The first family of G-representations $V(q)$ associated to α is the permutation module for G acting on α-flags of $\mathbb{F}_q$-subspaces

$$0 \subset V_{\alpha_1} \subset V_{\alpha_1+\alpha_2} \subset V_{\alpha_1+\alpha_2+\alpha_3} \subset \cdots \subset \mathbb{F}_q^n$$

where $\dim_{\mathbb{F}_q} V_i = i$. For this family one has $\mathrm{Hilb}_q(t) = \mathrm{sp}_{q,t} L(P)$ where the poset P is as described in case (a) above, when the chains have lengths $\alpha_1, \ldots, \alpha_\ell$.

The second family of G-representations $V(q)$ associated to α is the homology with $\mathbb{F}_q$-coefficients of the subcomplex of the *Tits building* generated by the faces indexed by α-flags. For this family one has $\mathrm{Hilb}_q(t) = \mathrm{sp}_{q,t} L(P)$ where the poset P is the *rim hook* poset P for α, having increasing chains of lengths $\alpha_1, \ldots, \alpha_\ell$, generalizing the $\alpha = (1^m, n-m)$ case described in (b) above.

In fact, for either of these classes of posets P associated to α, the more drastic q-specialization $\mathrm{sp}_q L(P)$ was shown to have two parallel representation-theoretic and invariant-theoretic interpretations. On one hand, $\mathrm{sp}_q L(P) = \dim_{\mathbb{F}_q} V(q)$. On the other hand, both classes of $\mathbb{F}_q G$-modules $V(q)$ have $(q = 1)$ analogous $\mathbb{Z}W$-module counterparts V where $W = \mathfrak{S}_n$ is the symmetric group. In particular, when one regards W acting on $S := \mathbb{Z}[x_1, \dots, x_n]$ by permuting the variables, so that S^W is the ring of symmetric polynomials, one finds that the graded intertwiner space

$$M := \mathrm{Hom}_{\mathbb{Z}W}(V, S)$$

turns out to be a free S^W-module, and that

$$\mathrm{Hilb}(M/S^W_+ M, q) = \mathrm{sp}_q\, L(P).$$

3. Binomial coefficient and Pascal recurrence

Definition 3.1. (cf. [11, (1.2)]) Define a multivariate analogue of a binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]! \cdot F^k([n-k]!)}.$$

It is an easy exercise in the definitions (1.3) to deduce the following analogue of the usual Pascal recurrence.

Proposition 3.2. (*cf.* [11, 1st equation in (4.2)])

$$\begin{bmatrix} n \\ k \end{bmatrix} = F \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \frac{F[k]!}{[k]!} \cdot F \begin{bmatrix} n-1 \\ k \end{bmatrix}. \quad \square$$

4. The weight of a subset

The Pascal recurrence leads to an interpretation of the binomial coefficient as a sum over certain partitions (cf. [11, (5.1)]). For our purpose, it is better to rephrase it as weight $\mathrm{wt}(S)$ defined for sets S of positive integers: a k-element set

$$S = \{i_1 > i_2 > \cdots > i_k\} \tag{4.1}$$

of positive integers, indexed in decreasing order, bijects with a partition λ whose Ferrers diagram fits inside a $k \times (n-k)$ rectangle:

$$\lambda(S) := (i_1, i_2, \dots, i_k) - (k, k-1, \dots, 2, 1)\,. \tag{4.2}$$

We thus re-encode the definition in [11, (5.1)] as follows.

Definition 4.1. For a k-element set S of positive integers indexed as in (4.1), define

$$\mathrm{wt}(S) := \frac{\prod_{j=1}^k F^{i_j - 1}[j]}{[k]!} = \prod_{j=1}^k \frac{F^{i_j-1}[j]}{F^{k-j}[j]}.$$

Example 4.2. For $k = 5$, the set $S = \{9, 7, 6, 4, 2\}$ has weight

$$\mathrm{wt}(S) = \frac{F^8[1] F^6[2] F^5[3] F^3[4] F^1[5]}{[5]!}.$$

Using the notation

$$S + 1 := \{i + 1 : i \in S\}$$

one can also define this weight recursively as follows:

$$\mathrm{wt}(S) := \begin{cases} 1 & \text{if } S = \emptyset \\ \frac{F[k]!}{[k]!} F\, \mathrm{wt}(\hat{S}) & \text{if } 1 \notin S \text{ and } S = \hat{S} + 1 \\ F\, \mathrm{wt}(\hat{S}) & \text{if } 1 \in S \text{ and } S = \{1\} \cup \left(\hat{S} + 1\right). \end{cases} \tag{4.3}$$

Proposition 4.3. (*cf.* [11, Theorem 5.3])

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_S \mathrm{wt}(S)$$

where the sum runs over all subsets S of cardinality k of $\{1, \ldots n\}$.

Proof. Induct on n with trivial base case $n = 0$. In the inductive step, the sum in the right hand side of the proposition decomposes as two subsums

$$\sum_{1 \in S} \mathrm{wt}(S) + \sum_{1 \notin S} \mathrm{wt}(S)$$

which correspond to the two terms in the Pascal recurrence, Proposition 3.2. Using the recursive definition (4.3) then completes the inductive step. □

5. The weight of a permutation via recursion

We wish to extend the definition of the weight $\mathrm{wt}(S)$ for a set S to a weight $\mathrm{wt}(w)$ for permutations w in $\mathfrak{S}_n$, defined recursively, following [11, §8].

Definition 5.1. [11, Definition 8.1] Given $w = (w_1, w_2, \ldots, w_n)$ in $W := \mathfrak{S}_n$, let $k := w_1 - 1$, so that $0 \leq k \leq n-1$ and $w_1 = k+1$. Regarding w as a shuffle of its restrictions to the alphabets $[1, k]$ and $[k+1, n]$, one can factor it uniquely

$$w = u \cdot a \cdot b \tag{5.1}$$

with u a minimum-length coset representative of uW_J for the *parabolic* or *Young subgroup*

$$\begin{aligned} W_J &:= \mathfrak{S}_{[1,k]} \times \mathfrak{S}_{[k+1,n]} \\ &\cong \mathfrak{S}_k \times \mathfrak{S}_{n-k} \end{aligned}$$

and where a, b lie in $\in \mathfrak{S}_{[1,k]}, \mathfrak{S}_{[k+1,n]}$, respectively.

Since u is a shuffle of the increasing sequences $(1, 2, \ldots, k), (k+1, k+2, \ldots, n)$, it can be encoded via the set

$$S(u) := \{u^{-1}(k) > u^{-1}(k-1) > \cdots > u^{-1}(2) > u^{-1}(1)\}. \tag{5.2}$$

Since $w_1 = k+1$ implies $b(k+1) = k+1$, the permutation b in $\mathfrak{S}_{[k+1,n]}$ actually lies in the subgroup $\mathfrak{S}_{[k+2,n]}$ that fixes $k+1$, isomorphic to $\mathfrak{S}_{n-k-2}$. Denote by $\hat{b}$ the corresponding element of $\mathfrak{S}_{n-k-2}$.

Now define $\mathrm{wt}(w)$ recursively by saying that the identity element e in $\mathfrak{S}_0$ has $\mathrm{wt}(e) := 1$, and otherwise

$$\mathrm{wt}(w) := \mathrm{wt}(S(u)) \cdot \mathrm{wt}(a) \cdot F^{k+1}(\mathrm{wt}(\hat{b})). \tag{5.3}$$

Note that since $k = w_1 - 1$, the integer 1 is never in $S(u)$. Therefore writing $S(u) = \hat{S}(u) + 1$ for a k-element subset of $\{1, 2, \ldots, n-1\}$, one can use (4.3) to rewrite (5.3) as

$$\mathrm{wt}(w) := \frac{F[k]!}{[k]!} F\left(\mathrm{wt}(\hat{S}(u))\right) \cdot \mathrm{wt}(a) \cdot F^{k+1}(\mathrm{wt}(\hat{b})). \tag{5.4}$$

Example 5.2. For $n = 9$, consider within $\mathfrak{S}_9$ the permutation

$$\begin{aligned} w &= \begin{pmatrix} 1 & \underline{2} & 3 & \underline{4} & 5 & \underline{6} & \underline{7} & 8 & \underline{9} \\ 6 & \mathbf{2} & 9 & \mathbf{1} & 7 & \mathbf{5} & \mathbf{3} & 8 & \mathbf{4} \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 1 & \underline{2} & 3 & \underline{4} & 5 & \underline{6} & \underline{7} & 8 & \underline{9} \\ 6 & \mathbf{1} & 7 & \mathbf{2} & 8 & \mathbf{3} & \mathbf{4} & 9 & \mathbf{5} \end{pmatrix}}_{u} \cdot \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix}}_{a} \cdot \underbrace{\begin{pmatrix} 6 & 7 & 8 & 9 \\ 6 & 9 & 7 & 8 \end{pmatrix}}_{b} \end{aligned}$$

One has $k = w_1 - 1 = 6 - 1 = 5$ here, and note that $b(6) = 6$, with

$$\hat{b} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Since the values $\{1, 2, 3, 4, 5(= k)\}$ occur in positions $S(u) = \{9, 7, 6, 4, 2\}$ of u or w, one has that

$$\begin{aligned} \mathrm{wt}(w) &= \mathrm{wt}(\{9, 7, 6, 4, 2\}) \cdot \mathrm{wt}(a) \cdot F^6\, \mathrm{wt}(\hat{b}) \\ &= \frac{F^8[1]F^6[2]F^5[3]F^3[4]F[5]}{[5]!} \cdot \mathrm{wt}(a) \cdot F^6\, \mathrm{wt}(\hat{b}). \end{aligned}$$

Finishing the recursive computation, one finds

$$\mathrm{wt}(a) = \mathrm{wt}(21534) = \frac{(x_4 + x_5)x_2x_5}{(x_3 + x_4)x_1x_4}, \qquad \mathrm{wt}(b) = \mathrm{wt}(312) = \frac{(x_2 + x_3)x_3}{(x_1 + x_2)x_2},$$

$$\mathrm{wt}(\{9, 7, 6, 4, 2\}) = \frac{x_9(x_7 + x_8)(x_6 + x_7 + x_8)(x_4 + x_5 + x_6 + x_7)(x_2 + \cdots + x_6)}{x_5(x_4 + x_5)(x_3 + x_4 + x_5)(x_2 + x_3 + x_4 + x_5)(x_1 + \cdots + x_5)},$$

and therefore

$$\mathrm{wt}(w) = \frac{x_2x_9^2(x_8 + x_9)(x_6 + x_7 + x_8)(x_4 + x_5 + x_6 + x_7)(x_2 + \cdots + x_6)}{x_1x_4x_8(x_3 + x_4)(x_3 + x_4 + x_5)(x_2 + x_3 + x_4 + x_5)(x_1 + \cdots + x_5)}.$$

Example 5.3. Here are the values of $\mathrm{wt}(w)$ for w in $\mathfrak{S}_3$:

w	$\mathrm{wt}(w)$
123	1
132	$\frac{F^2[1]}{F[1]} = \frac{x_3}{x_2}$
213	$\frac{F[1]}{[1]} = \frac{x_2}{x_1}$
231	$\frac{F^2[1]}{[1]} = \frac{x_3}{x_1}$
312	$\frac{F[2]!}{[2]!} = \frac{F[2]F^2[1]}{[2]F[1]} = \frac{(x_2+x_3)x_3}{(x_1+x_2)x_2}$
321	$\frac{F[2]!}{[2]!}\frac{F[1]}{[1]} = \frac{F[2]F^2[1]}{[2][1]} = \frac{(x_2+x_3)x_3}{(x_1+x_2)x_1}$

Four out of these six permutations w in $\mathfrak{S}_3$, namely all except for $\{213, 231\}$, are themselves recursively labelled forests when regarded as linear orders. For these four one can check that the value of $\mathrm{wt}(w)$ given in the table agrees with the product formula predicted by Theorem 1.1.

On the other hand, the two exceptions $\{213, 231\}$ comprise $\mathcal{L}(P)$ for the recursively labelled forest poset $1 >_P 2 <_P 3$. One then checks from the values in the table that

$$L(P) = \mathrm{wt}(213) + \mathrm{wt}(231) = \frac{x_2}{x_1} + \frac{x_3}{x_1} = \frac{x_2 + x_3}{x_1} = \frac{F[2]}{[1]}$$

which again agrees with the prediction of Theorem 1.1, namely

$$\frac{[3]!}{F^{\min(P_{\geq}1)-1}[h_1] \cdot F^{\min(P_{\geq}2)-1}[h_2] \cdot F^{\min(P_{\geq}3)-1}[h_3]} = \frac{[3]F[2]F^2[1]}{F^0[1]F^0[3]F^2[1]} = \frac{F[2]}{[1]}.$$

For later use in Section 8, we explain how $\mathrm{wt}(w)$ behaves under the specialization map sp_q from (1.4) which sends $x_i = F^i[1]$ to $q^{i-1} - q^i$. Note that

$$\mathrm{sp}_q\, F^i[n] = \mathrm{sp}_q(x_i + x_{i+1} + \cdots + x_{i+n-1}) = q^{i-1} - q^{i+n-1}$$

so that

$$\mathrm{sp}_q \frac{F^a[n]}{F^b[m]} = q^{a-b} \frac{1-q^n}{1-q^m}. \tag{5.5}$$

In particular, when $m = n$ one has

$$\mathrm{sp}_q \frac{F^a[n]}{F^b[n]} = q^{a-b}, \tag{5.6}$$

and hence for a k-subset $S = \{i_1 > i_2 > \cdots > i_k\}$,

$$\mathrm{sp}_q\, \mathrm{wt}(S) = \mathrm{sp}_q \prod_{j=1}^{k} \frac{F^{i_j-1}[j]}{F^{k-j}[j]} = q^{\sum_{j=1}^{k}(i_j-(k-j)-1)}. \tag{5.7}$$

Corollary 5.4. *For any permutation w in $\mathfrak{S}_n$ one has* $\mathrm{sp}_q\, \mathrm{wt}(w) = q^{\mathrm{inv}(w)}$.

Proof. Induct on n, with $n = 0$ as a trivial base case. In the inductive step, if $w_1 = k+1$ and $w = u \cdot a \cdot b$ is the parabolic factorization from (5.1), then

$$\mathrm{inv}(w) = \mathrm{inv}(u) + \mathrm{inv}(a) + \mathrm{inv}(b). \tag{5.8}$$

Note that $\mathrm{inv}(b) = \mathrm{inv}(\hat{b})$. Also note that in (5.2), if one has $S(u) = \{i_1 > \cdots > i_k\}$, then $\mathrm{inv}(u) = \sum_{j=1}^{k} (i_j - (k-j) - 1)$ so that (5.7) implies

$$q^{\mathrm{inv}(u)} = \mathrm{sp}_q\, \mathrm{wt}(S(u)). \tag{5.9}$$

Since by definition one has

$$\mathrm{wt}(w) = \mathrm{wt}(S(u)) \cdot \mathrm{wt}(a) \cdot F^{k+1}\, \mathrm{wt}(\hat{b})$$

the assertion of the corollary follows from (5.8), (5.9), together with the inductive hypothesis applied to a and $\hat{b}$. □

6. The weight of a permutation, via a search tree

The goal of this section is to encode the recursive nature of the definition of the weight $\mathrm{wt}(w)$ for a permutation w in a standard combinatorial data structure, an increasing binary search tree. Once this tree is computed, one no longer needs recursion to define $\mathrm{wt}(w)$.

Definition 6.1. (cf. Stanley [10, §1.3]) For any word $w = w_1 \dots w_m$ without repetition, define recursively its *increasing binary tree* $\mathrm{T}(w)$ as follows:

- if w is empty (i.e. $m = 0$), then $\mathrm{T}(w)$ is the empty binary tree;
- else denote by k the index of the smallest letter of w. Then $\mathrm{T}(w)$ is the binary tree whose root is labelled w_k, whose left subtree is $\mathrm{T}(w_1 \dots w_{k-1})$ and whose right subtree is $\mathrm{T}(w_{k+1} \dots w_m)$.

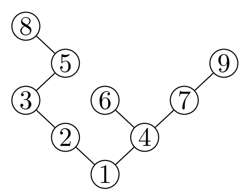
Now for a given permutation w, consider the tree $\mathrm{T}(w^{-1})$. For each pair of labeled nodes (α, β) such that α occurs in the left subtree rooted at β, define a *numerator* polynomial $N(\alpha, \beta)$ and *denominator* polynomial $D(\alpha, \beta)$ by

$$\begin{aligned} D(\alpha,\beta) &:= x_{w(\beta)-1} + \cdots + x_{w(\beta)-\ell} &&= F^{w(\beta)-\ell-1}[\ell] \\ N(\alpha,\beta) &:= F^{r+1}(D(\alpha,\beta)) &&= F^{w(\beta)+r-\ell}[\ell] \end{aligned}$$

where $\ell := \ell(\alpha, \beta)$ (resp. $r := r(\alpha, \beta)$) is the number of nodes in the left (resp. right) subtree of β whose label is larger or equal (resp. smaller) than α. Note that since α is in the left subtree of β, one always has $\ell \geq 1$.

Example 6.2. For example, consider the permutation $w = 541736829$. Its inverse is $w^{-1} = 385216479$. The corresponding increasing tree $\mathrm{T}(w^{-1})$ is therefore

$\mathrm{T}(w^{-1}) =$



and the relevant pairs (α, β) and polynomials $N(\alpha, \beta), D(\alpha, \beta)$ are as follows:

α	β	$w(\beta)$	ℓ	r	$N(\alpha,\beta)$ $:= F^{r+1}D(\alpha,\beta)$	$D(\alpha,\beta)$ $:= F^{w(\beta)-\ell-1}([\ell])$
2	1	5	4	0	$x_5 + x_4 + x_3 + x_2$	$x_4 + x_3 + x_2 + x_1$
3	1	5	3	0	$x_5 + x_4 + x_3$	$x_4 + x_3 + x_2$
5	1	5	2	1	$x_6 + x_5$	$x_4 + x_3$
8	1	5	1	3	x_8	x_4
3	2	4	3	0	$x_4 + x_3 + x_2$	$x_3 + x_2 + x_1$
5	2	4	2	0	$x_4 + x_3$	$x_3 + x_2$
8	2	4	1	0	x_4	x_3
8	5	3	1	0	x_3	x_2
6	4	7	1	0	x_7	x_6

Proposition 6.3. *For any permutation w, the weight of w equals*

$$\mathrm{wt}(w) = \prod_{(\alpha,\beta)} \frac{N(\alpha,\beta)}{D(\alpha,\beta)}\,, \tag{6.1}$$

where the product is over (α, β) with α in the left subtree of $\mathrm{T}(w^{-1})$ rooted at β.

Proof. Induct on n, with trivial base cases $n = 0, 1$. In the inductive step, let L and R be the left and right subtrees of the root of $\mathrm{T}(w^{-1})$. Define a, u and $\hat{b}$ as in Definition 5.1. Then

$$\mathrm{wt}(w) := \mathrm{wt}(S(u)) \cdot \mathrm{wt}(a) \cdot F^{k+1}(\mathrm{wt}(\hat{b})). \tag{6.2}$$

Assume (6.1) holds for $w := a$ or $w := \hat{b}$; we wish to prove it holds for $w = u \cdot a \cdot b$.

The tree $\mathrm{T}(a^{-1})$ is obtained from L by renumbering the labels to $\{1, \dots, k\}$ keeping their relative order. Let (α, β) be two nodes of L and (α', β') their renumbering in $\mathrm{T}(a^{-1})$. It should be clear that

$$\begin{aligned} r(\alpha,\beta) &= r(\alpha',\beta')\,,\\ \ell(\alpha,\beta) &= \ell(\alpha',\beta')\,,\\ w(\beta) &= a(\beta')\,. \end{aligned}$$

As a consequence

$$\mathrm{wt}(a) = \prod_{(\alpha',\beta')} \frac{N(\alpha',\beta')}{D(\alpha',\beta')} = \prod_{\substack{(\alpha,\beta)\\ \alpha,\beta\in L}} \frac{N(\alpha,\beta)}{D(\alpha,\beta)}\,. \tag{6.3}$$

Similarly, the values of r and ℓ also agree in $\mathrm{T}(\hat{b}^{-1})$ and R, but the difference is that for two corresponding nodes $\beta \in R$ and $\beta' \in \mathrm{T}(\hat{b}^{-1})$, one has $w(\beta) = \hat{b}(\beta') + k + 1$. It follows that

$$F^{k+1}(\mathrm{wt}(\hat{b})) = \prod_{(\alpha',\beta')} F^{k+1}\left(\frac{N(\alpha',\beta')}{D(\alpha',\beta')}\right) = \prod_{\substack{(\alpha,\beta)\\ \alpha,\beta\in R}} \frac{N(\alpha,\beta)}{D(\alpha,\beta)}\,. \tag{6.4}$$

It therefore remains to show that $\mathrm{wt}(S(u))$ is exactly the product over pairs (α, β) with $\alpha = 1$. Ordering decreasingly the labels $\{\alpha_1 > \cdots > \alpha_k\}$ of L which are also the elements of $S(u)$, one sees that

$$\begin{aligned} \ell(\alpha_j, 1) &= j\,,\\ r(\alpha_j, 1) &= \alpha_j - 1 - (k-j)\,. \end{aligned}$$

Since $w(1) = k + 1$, one has

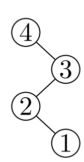
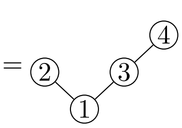
$$\begin{aligned} D(\alpha_j, 1) &= F^{k-j}[j]\,,\\ N(\alpha_j, 1) &= F^{\alpha_j-1}[j]\,. \end{aligned}$$

Therefore

$$\prod_{\alpha\in L}\left(\frac{N(\alpha,1)}{D(\alpha,1)}\right) = \prod_{j=1}^{f} \frac{F^{\alpha_j-1}[j]}{F^{k-j}[j]} = \frac{\prod_{j=1}^{k} F^{\alpha_j-1}[j]}{[k]!} = \mathrm{wt}(S(u))\,. \tag{6.5}$$

This proves that (6.1) holds for $w = u \cdot a \cdot b$. □

Example 6.4. Continuing Example 6.2, one sees that $a = 4132$ so that $a^{-1} = 2431$ and $b = 57689$ so that $\hat{b} = 2134$ and $\hat{b}^{-1} = 2134$. As a consequence:

$\mathrm{T}(a^{-1}) =$  and $\mathrm{T}(\hat{b}^{-1}) =$ 

This gives a different way to view the assertion $\mathrm{sp}_q\, \mathrm{wt}(w) = q^{\mathrm{inv}(w)}$. of Corollary 5.4.

Second proof of Corollary 5.4. Rephrasing Proposition 6.1 as

$$\mathrm{wt}(w) = \prod_{(\alpha,\beta)} \frac{F^{r(\alpha,\beta)+1} D(\alpha,\beta)}{D(\alpha,\beta)}$$

and bearing in mind (5.6), it suffices to check that

$$\mathrm{inv}(w) = \sum_{(\alpha,\beta)} (r(\alpha,\beta)+1) \ . \tag{6.6}$$

Let $(i<j)$ be an inversion of w, meaning that $w_j < w_i$. Looking at w^{-1}, this means that j occurs to the left of i in the word $w^{-1} = (w^{-1}(1), w^{-1}(2), \ldots, w^{-1}(n))$. There are two possibilities:

- For all r such that $w_j < r < w_i$ one has $i < w^{-1}(r)$.
  In other words, in w^{-1} all letters between j and i are bigger than i. By the construction of the tree $T = \mathrm{T}(w^{-1})$, this implies that j lies in the left subtree of i.
- There exists an r such that $w_j < r < w_i$ and $w^{-1}(r) < i$.
  In other words, one can find a letter smaller than i lying between j and i in w^{-1}. Let k be the minimal such letter:

$$k := \min\{w^{-1}(r) \ \mid \ w_j < r < w_i\}. \tag{6.7}$$

  By the construction of $T = \mathrm{T}(w^{-1})$, the letter k is the label of the only node m of T such that j and i are in the left and right subtrees of m. Therefore this i counts for 1 in $r(\alpha,\beta)$ where $\alpha := j$ and $\beta := k$.

As a consequence, fixing α, the sum $\sum_\beta (r(\alpha,\beta)+1)$ is exactly the number of $i<\alpha$ such that $w_i > w_\alpha$. This proves (6.6). □

7. Proof of Theorem 1.1

For a recursively labelled forest P on $\{1,2,\ldots,n\}$, we wish to prove equality of the two rational functions

$$\begin{aligned} L(P) &:= \sum_{w\in\mathcal{L}(P)} \mathrm{wt}(w)\,, \\ H(P) &:= \frac{[n]!}{\prod_{i=1}^n F^{\min(P_{\geq i})-1}[h_i]}\,. \end{aligned} \tag{7.1}$$

Proceed by induction on the following quantity: the sum of n and the number of incomparable pairs i,j in P. In the base case where this quantity is zero, in particular $n=0$, and the result is trivial. In the inductive step, there are two cases.

Case 1. There exist two elements i,j having subtrees $P_{\geq i}, P_{\geq j}$ labelled by two contiguous intervals of integers, say

$$\begin{aligned} P_{\geq i} &= [r+1, r+s]\,, \\ P_{\geq j} &= [r+s+1, r+s+t]\,. \end{aligned}$$

In this case, form the poset $P_{i<j}$ by taking the transitive closure of P and the extra relation $i<j$. Defining $P_{j<i}$ similarly, one has the disjoint decomposition

$$\mathcal{L}(P) = \mathcal{L}(P_{i<j}) \sqcup \mathcal{L}(P_{j<i})$$

since any w in $\mathcal{L}(P)$ either has $i <_w j$ or $j <_w i$. Therefore

$$L(P) = L(P_{i<j}) + L(P_{j<i})\,,$$

and hence it remains to show

$$H(P) = H(P_{i<j}) + H(P_{j<i})\,. \tag{7.2}$$

Because $P, P_{i<j}, P_{j<i}$ share the same size n, and share the same label sets on their subtrees $P_{\geq k}$ for $k \neq i, j$, the desired equality (7.2) is equivalent to checking

$$\frac{1}{F^r[s] \cdot F^{r+s}[t]} = \frac{1}{F^r[s+t] \cdot F^{r+s}[t]} + \frac{1}{F^r[s] \cdot F^r[s+t]}.$$

Over a common denominator, this amounts to checking

$$F^r[s+t] = F^r[s] + F^{r+s}[t],$$

which is immediated from the definition (1.2) of $[n]$.

Case 2. There are no such pairs of elements i, j as in Case 1.

This means that P is a *recursively labelled binary tree*, meaning that it has a minimum element, say $k+1$, and every element i in P is covered by at most one element $j <_{\mathbb{Z}} i$ and at most one element $j >_{\mathbb{Z}} i$. In particular, this means that the poset P_1 obtained by restricting P to the values $[1, k]$ is again a recursively labelled binary tree. Similarly the restriction of P to the values $[k+2, n]$ is obtained from some recursively labelled binary tree P_2 on values $[1, n-k-1]$ by adding $k+1$ to all of its vertex labels; denote this restriction $F^{k+1}(P_2)$.

One then calculates that

$$\begin{aligned} H(P) &= \frac{[n]!}{\prod_{i=1}^n F^{\min(P_{\geq i})-1}[h_i]} \\ &= \frac{F[n-1]!}{\prod_{i \neq k+1} F^{\min(P_{\geq i})-1}[h_i]} \\ &= \frac{F[k]!}{[k]!} \cdot F \begin{bmatrix} n-1 \\ k \end{bmatrix} \cdot \frac{[k]!}{\prod_{i=1}^k F^{\min(P_{\geq i})-1}[h_i]} \cdot \frac{F^{k+1}[n-1-k]!}{\prod_{i=k+2}^n F^{\min(P_{\geq i})-1}[h_i]} \\ &= \frac{F[k]!}{[k]!} \cdot F \begin{bmatrix} n-1 \\ k \end{bmatrix} \cdot H(P_1) \cdot F^{k+1} H(P_2). \end{aligned}$$

It remains to show that $L(P)$ satisfies the same recurrence. Note that each w in $\mathcal{L}(P)$ has $w_1 = k+1$, because $k+1$ is the minimum element of P. Furthermore, when one decomposes $w = u \cdot a \cdot b$ as in the parabolic factorization (5.1) used to define $\mathrm{wt}(w)$, one finds that $a, \hat{b}$ lie in $\mathcal{L}(P_1), \mathcal{L}(P_2)$, respectively. Conversely, any such triple $(u, a, \hat{b})$, where u is a shuffle of the sequences $(1, 2, \ldots, k), (k+1, k+2, \ldots, n)$ having $u(1) = k+1$, gives rise to an element $w = u \cdot a \cdot b$ of $\mathcal{L}(P)$. Thus

$$\begin{aligned} L(P) &= \sum_{(u,a,\hat{b})} \mathrm{wt}(S(u))\, \mathrm{wt}(a) F^{k+1}\, \mathrm{wt}(\hat{b}) \\ &= \frac{F[k]!}{[k]!} \left(\sum_{\substack{k-\text{subsets } \hat{S} \\ \text{of } \{1,2,\ldots,n-1\}}} F\left(\mathrm{wt}(\hat{S})\right) \right) \left(\sum_a \mathrm{wt}(a) \right) \left(\sum_{\hat{b}} F^{k+1}\, \mathrm{wt}(\hat{b}) \right) \\ &= \frac{F[k]!}{[k]!} F \begin{bmatrix} n-1 \\ k \end{bmatrix} \cdot L(P_1) \cdot F^{k+1} L(P_2) \end{aligned}$$

using (5.4) and (4.3).

Thus in both cases, $L(P)$ and $H(P)$ satisfy the same recurrence, concluding the proof of Theorem 1.1.

8. Specializing to the formula of Björner and Wachs

It is now easy to deduce Björner and Wachs' identity (1.1) as the q-specialization of Theorem 1.1: one has from Corollary 5.4 that

$$\mathrm{sp}_q\, L(P) = \sum_{w \in \mathcal{L}(P)} q^{\mathrm{inv}(w)} ,$$

while the right side $H(P)$ of Theorem 1.1 has q-specialization

$$\begin{aligned} \mathrm{sp}_q\, H(P) &= \mathrm{sp}_q \prod_{i=1}^{n} \frac{F^{i-1}[n-i+1]}{F^{\min(P_{\geq i})-1}[h_i]} \\ &= q^{\sum_{i=1}^{n}(i-\min(P_{\geq i}))} \prod_{i=1}^{n} \frac{1-q^{n-i+1}}{1-q^{h_i}} \qquad \text{using (5.5)} \\ &= q^{\mathrm{inv}(P)} \frac{[n]!_q}{\prod_{i=1}^{n}[h_i]} \end{aligned}$$

where the last equality used the following fact: since P is a recursively labelled forest, for each i, the quantity $i - \min(P_{\geq i})$ counts the contribution to $\mathrm{inv}(P)$ coming from the pairs (i,j) where j lies in $P_{\geq i}$.

9. Algebra morphisms

Theorem 1.1 has an interesting rephrasing in terms of a $\mathbb{Q}$-linear map from the ring of *free quasisymmetric functions* $\mathcal{FQS}ym$ (or *Malvenuto-Reutenauer algebra*) into a certain target ring. We define these objects here.

Definition 9.1. Recall from [9] that the algebra $\mathcal{FQS}ym$ has $\mathbb{Q}$-basis elements

$$\left\{ \mathbf{F}_w : w \in \bigsqcup_{n \geq 0} \mathfrak{S}_n \right\},$$

with multiplication defined $\mathbb{Q}$-bilinearly as follows: for a, b lying in $\mathfrak{S}_k, \mathfrak{S}_\ell$ one has

$$\mathbf{F}_a \cdot \mathbf{F}_b := \sum_w \mathbf{F}_w$$

where w runs through all *shuffles* of the words

$$a = (a_1, \ldots, a_k), \text{ and}$$
$$F^k(b) := (b_1 + k, \ldots, b_\ell + k) .$$

One of the original motivations for introducing the ring $\mathcal{FQS}ym$ is the following. Define for each poset P the element

$$\mathbf{F}_P := \sum_{w \in \mathcal{L}(P)} \mathbf{F}_w \tag{9.1}$$

in $\mathcal{FQS}ym$. Then for two posets P, Q on elements $\{1, 2, \ldots, k\}, \{1, 2, \ldots, \ell\}$, respectively, one has in $\mathcal{FQS}ym$ that

$$\mathbf{F}_P \cdot \mathbf{F}_Q = \mathbf{F}_{P \sqcup F^k(Q)} , \tag{9.2}$$

where $P \sqcup F^k(Q)$ denotes the poset on $\{1, 2, \ldots, k+\ell\}$ which is the disjoint union of P with the poset $F^k(Q)$ on $\{k+1, k+2, \ldots, k+\ell\}$ obtained by adding k to each label in Q.

Definition 9.2. Let the semigroup $\mathbb{N} = \{1, F, F^2, \ldots\}$ act on the rational functions $\mathbb{Q}(\mathbf{x}) = \mathbb{Q}(x_1, x_2, \ldots)$ via the Frobenius map as before: $F(x_i) = x_{i+1}$. Then define the *skew semigroup algebra* $\mathbb{Q}(\mathbf{x})\#\mathbb{N}$ to be the free $\mathbb{Q}(\mathbf{x})$-module on basis $\{1, u, u^2, \ldots\}$, with multiplication defined $\mathbb{Q}$-linearly by

$$f(\mathbf{x})u^k \cdot g(\mathbf{x})u^\ell = \left(f(\mathbf{x})F^k(g(\mathbf{x}))\right) u^{k+\ell}\,.$$

One of our motivations for introducing $\mathbb{Q}(\mathbf{x})\#\mathbb{N}$ is that, in addition to its $\mathbb{Q}(\mathbf{x})$-basis $\{1, u, u^2, \ldots\}$, it also has a $\mathbb{Q}(\mathbf{x})$-basis of *divided powers* $\{1, u^{(1)}, u^{(2)}, \ldots\}$, where

$$u^{(n)} := \frac{1}{[n]!}u^n,$$

and this basis has our binomial coefficients as multiplicative structure constants:

$$u^{(k)} \cdot u^{(\ell)} = \begin{bmatrix} k+\ell \\ k \end{bmatrix} u^{(k+\ell)}\,. \tag{9.3}$$

Definition 9.3. Define the $\mathbb{Q}$-linear map

$$\begin{array}{rcl} \mathcal{FQS}ym & \stackrel{\phi_{\mathrm{inv}}}{\longrightarrow} & \mathbb{Q}(\mathbf{x})\#\mathbb{N} \\ \mathbf{F}_w & \longmapsto & \frac{\mathrm{wt}(w)}{[n]!}u^n = \mathrm{wt}(w) \cdot u^{(n)} \end{array}$$

for w in $\mathfrak{S}_n$. Note that

$$\phi_{\mathrm{inv}}(\mathbf{F}_P) = L(P) \cdot u^{(n)}\,. \tag{9.4}$$

This $\mathbb{Q}$-linear map ϕ_{inv} turns out *not* to be an algebra morphism. E.g., one can check via explicit computations that

$$\begin{aligned} \phi_{\mathrm{inv}}(\mathbf{F}_1 \cdot \mathbf{F}_{213}) &= \phi_{\mathrm{inv}}(\mathbf{F}_{1324}) + \phi_{\mathrm{inv}}(\mathbf{F}_{3124}) + \phi_{\mathrm{inv}}(\mathbf{F}_{3214}) + \phi_{\mathrm{inv}}(\mathbf{F}_{3241}) \\ &\neq \phi_{\mathrm{inv}}(\mathbf{F}_1) \cdot \phi_{\mathrm{inv}}(\mathbf{F}_{213})\,. \end{aligned}$$

However, the import of Theorem 1.1 is that ϕ_{inv} *becomes* an algebra morphism when restricted to an appropriate subalgebra of $\mathcal{FQS}ym$.

Definition 9.4. Recall from [8] that the *Loday-Ronco algebra of binary trees* $\mathcal{PBT}$ can be defined as the subalgebra of $\mathcal{FQS}ym$ spanned by all $\{\mathbf{F}_P\}$ as P runs through all recursively labelled forests.

Proposition 9.5. *When restricted from $\mathcal{FQS}ym$ to $\mathcal{PBT}$, the map ϕ_{inv} becomes an algebra homomorphism $\mathcal{PBT} \stackrel{\phi_{\mathrm{inv}}}{\longrightarrow} \mathbb{Q}(\mathbf{x})\#\mathbb{N}$.*

Proof. It is easy to check that the product formula $H(P)$ defined in (7.1) for a recursively labelled forest P satisfies

$$H(P \sqcup F^k Q) = \begin{bmatrix} k+\ell \\ k \end{bmatrix} H(P) \cdot F^k H(Q)\,. \tag{9.5}$$

Hence for recursively labelled forests P, Q of sizes k, ℓ, one has

$$\begin{array}{rll} \phi_{\mathrm{inv}}(\mathbf{F}_P \cdot \mathbf{F}_Q) & = \phi_{\mathrm{inv}}\left(\mathbf{F}_{P \sqcup F^k Q}\right) & \text{by (9.2)} \\ & = L(P \sqcup F^k Q) \cdot u^{(k+\ell)} & \text{by (9.4)} \\ & = H(P \sqcup F^k Q) \cdot u^{(k+\ell)} & \text{by Theorem 1.1} \\ & = \begin{bmatrix} k+\ell \\ k \end{bmatrix} H(P) \cdot F^k H(Q) \cdot u^{(k+\ell)} & \text{by (9.5)} \\ & = H(P)u^{(k)} \cdot H(Q)u^{(\ell)} & \text{by (9.3)} \\ & = L(P)u^{(k)} \cdot L(Q)u^{(\ell)} & \text{by Theorem 1.1} \\ & = \phi_{\mathrm{inv}}(\mathbf{F}_P) \cdot \phi_{\mathrm{inv}}(\mathbf{F}_Q) & \text{by (9.4)}\,. \end{array}$$

□

Remark 9.6. This twisted semigroup algebra $\mathbb{Q}(\mathbf{x})\#\mathbb{N}$ also appears implicitly in the theory of P-partitions, as the target of a *different* map $\phi_{\mathrm{maj}} : \mathcal{FQS}ym \to \mathbb{Q}\#\mathbb{N}$, which *is* an algebra morphism. This is related to a recent multivariate generalization of Björner and Wachs' *other* "maj" q-hook formula for forests. We describe both connections briefly here.

For a poset P on $\{1, 2, \ldots, n\}$, a *P-partition* (see [10, §4.5 and 7.19]) is a weakly order-reversing function $f : P \to \mathbb{N}$ (meaning $i <_P j$ implies $f(i) \geq f(j)$) which is strictly decreasing along *descent* covering relations: whenever j covers i in P and $i >_{\mathbb{Z}} j$ then $f(i) > f(j)$. Define their generating function $\gamma(P, \mathbf{x}) := \sum_f \mathbf{x}^f$ where here f runs over all P-partitions, and $\mathbf{x}^f := x_1^{f(1)} \cdots x_n^{f(n)}$. The relevant algebra morphism is defined $\mathbb{Q}$-bilinearly as follows:

$$\begin{array}{rcl} \mathcal{FQS}ym & \xrightarrow{\phi_{\mathrm{maj}}} & \mathbb{Q}\#\mathbb{N} \\ \mathbf{F}_w & \longmapsto & \gamma(w, \mathbf{x}) \cdot u^n \,. \end{array}$$

The main proposition on P-partitions [10, Theorem 4.54] asserts that

$$\gamma(P, \mathbf{x}) = \sum_{w \in \mathcal{L}(P)} \gamma(w, \mathbf{x}) \tag{9.6}$$

or equivalently,

$$\phi_{\mathrm{maj}}(\mathbf{F}_P) = \gamma(P, \mathbf{x}) u^n \,.$$

This then shows that ϕ_{maj} is an algebra morphism, since for any posets P, Q on $[1, k]$ and $[1, \ell]$, one has

$$\begin{aligned} \phi_{\mathrm{maj}}(\mathbf{F}_P \cdot \mathbf{F}_Q) &= \phi_{\mathrm{maj}}(\mathbf{F}_{P \sqcup F^k(Q)}) \\ &= \gamma(P \sqcup F^k(Q), \mathbf{x}) u^{k+\ell} \\ &= \gamma(P, \mathbf{x}) \cdot F^k(\gamma(Q, \mathbf{x})) u^{k+\ell} \\ &= \gamma(P, \mathbf{x}) u^k \cdot \gamma(Q, \mathbf{x}) u^\ell \\ &= \phi_{\mathrm{maj}}(\mathbf{F}_P) \cdot \phi_{\mathrm{maj}}(\mathbf{F}_Q) \,. \end{aligned}$$

The Björner-Wachs *maj* formula arises when P is a *dual forest*, that is, every element i in P is covered by at most one other element j; say that i is a *descent* of P if in addition $i >_{\mathbb{Z}} j$. Let $\mathrm{Des}(P)$ denote the set of descents of P, and $\mathrm{maj}(P) := \sum_{i \in \mathrm{Des}(P)} i$. In particular, permutations $w = (w_1, \ldots, w_n)$ considered as linear orders are dual forests, and for them one has $\mathrm{maj}(w) = \sum_{i : w_i > w_{i+1}} i$. For any dual forest P, note that the subtree rooted at i is $P_{\leq i}$, and again denote its cardinality by h_i. The Björner-Wachs *maj* formula asserts the following.

Theorem. ([1, Theorem 1.2]) *Any dual forest P on $\{1, 2, \ldots, n\}$ has*

$$\sum_{w \in \mathcal{L}(P)} q^{\mathrm{maj}(w)} = q^{\mathrm{maj}(P)} \frac{[n]!_q}{\prod_{i=1}^n [h_i]_q} \,. \tag{9.7}$$

The following generalization was observed recently in [2]:

Theorem. *For any dual forest P on $\{1,2,\ldots,n\}$, one has*

$$\gamma(P,\mathbf{x}) := \frac{\prod_{i\in \mathrm{Des}(P)} \mathbf{x}_{P_{\leq i}}}{\prod_{i=1}^{n}(1-\mathbf{x}_{P_{\leq i}})} \tag{9.8}$$

where $\mathbf{x}_S := \prod_{j\in S} x_j$*, so that* (9.6) *becomes*

$$\sum_{w\in\mathcal{L}(P)} \left(\frac{\prod_{i\in \mathrm{Des}(w)} x_{w_1}\cdots x_{w_i}}{\prod_{i=1}^{n}(1-x_{w_1}\cdots x_{w_i})} \right) = \frac{\prod_{i\in \mathrm{Des}(P)} x_{P_{\leq i}}}{\prod_{i=1}^{n}(1-x_{P_{\leq i}})}. \tag{9.9}$$

The Björner-Wachs maj formula is immediate upon specializing $x_i = q$ in (9.9):

$$\sum_{w\in\mathcal{L}(P)} \frac{q^{\mathrm{maj}(w)}}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{q^{\mathrm{maj}(P)}}{\prod_{i=1}^{n}(1-q^{h_i})}.$$

Remark 9.7. The maps $\phi_{\mathrm{inv}}, \phi_{\mathrm{maj}} : \mathcal{FQS}ym \to \mathbb{Q}(\mathbf{x})\#\mathbb{N}$ are reminiscent of the formalism of *moulds* discussed by Chapoton, Hivert, Novelli and Thibon [3], but we have not yet found a deeper connection.

One might also hope that the (q,t)-specializations $\mathrm{sp}_{q,t}\, L(P)$ for recursively labelled binary trees P can be given a representation-theoretic interpretation, similar to the discussion in Section 2, but related to q-analogues of the indecomposable projective modules for the algebras whose existence is conjectured by Hivert, Novelli and Thibon in [4, §5.2]. At the moment this is purely speculative.

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