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Trees, functional equations, and combinatorial Hopf algebras

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Abstract

One of the main virtues of trees is the representation of formal solutions of various functional equations which can be cast in the form of fixed point problems. Basic examples include differential equations and functional (Lagrange) inversion in power series rings. When analyzed in terms of combinatorial Hopf algebras, the simplest examples yield interesting algebraic identities or enumerative results.

1. Introduction

Let R be an associative algebra, and consider the functional equation for the power series $x \in R[[t]]$

$$x = a + B(x, x), \tag{1}$$

where $a \in R$ and $B(x, y)$ is a bilinear map with values in $R[[t]]$, such that the valuation of $B(x, y)$ is strictly greater than the sum of the valuations of x and y. Then, (1) has a unique solution

$$x = a + B(a, a) + B(B(a, a), a) + B(a, B(a, a)) + \cdots = \sum_{T \in \mathbf{CBT}} B_T(a), \tag{2}$$

where **CBT** is the set of (complete) binary trees, and for a tree T, $B_T(a)$ is the result of evaluating the expression formed by labeling by a the leaves of T and by B its internal nodes.

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Of course, the same can be done with m-ary trees, or more generally with *plane trees*. We are in particular interested in those counted by the little Schröder numbers, that is, plane trees without vertex of arity 2 [20, A001003], which solve equations of the form

$$x = a + \sum_{n \geq 2} F_n(x, x, \ldots, x) \tag{3}$$

each F_n being an n-linear operation.

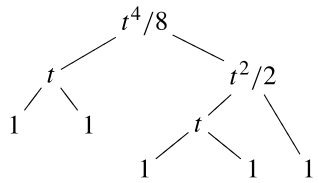
All this is well-known and rather trivial. However, the simplest example has still something to tell us. Consider the differential equation (for $x \in \mathbb{K}[[t]]$)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2, \quad x(0) = 1. \tag{4}$$

Its solution is obviously $x = (1-t)^{-1}$, but let us ignore this for the moment, and recast it as a fixed point problem

$$x = 1 + \int_0^t x^2(s)\mathrm{d}s = 1 + B(x, x), \tag{5}$$

where $B(x, y) = \int_0^t x(s)y(s)\mathrm{d}s$. Then, for a binary tree T with $n + 1$ leaves, $B_T(1)$ is the monomial obtained by putting 1 on each leaf and integrating at each internal node the product of the evaluations of its subtrees:

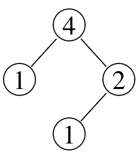


(6)

One can observe that

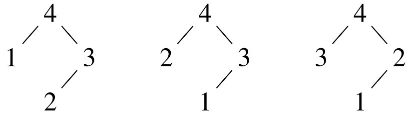
$$B_T(1) = c_{T'} \frac{t^n}{n!}, \tag{7}$$

where T' is the incomplete binary tree with n nodes obtained by removing the leaves of T, and $c_{T'}$ is the number of permutations $\sigma \in \mathfrak{S}_n$ whose decreasing tree has shape T'. Indeed, $c_{T'}$ is explicitly given by a *hook length formula* [11, Ex. 20 p. 70], which can be compared with the easily obtained closed form for $B_T(1)$. The *hook lengths* of T' are the number of nodes of all the subtrees



(8)

and $c_{T'}$ is $n!$ over the product of the hook lengths, here $4!/(4 \cdot 2 \cdot 1 \cdot 1) = 3$, the corresponding decreasing trees being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Our starting point will be the following question: Can one use this observation to derive the hook length formula for binary trees, and if yes, can we use the same method to obtain more interesting results?

For this, we have to lift our problem to the combinatorial Hopf algebra of Free quasi-symmetric functions **FQSym**. We can then derive in the same way the q-hook length formulas of Björner and Wachs [1,2]. The case of plane trees can be dealt with in the same way, the relevant Hopf algebra being **WQSym**, the Word Quasi-Symmetric invariants (or quasi-symmetric functions in noncommutative variables), and here the resulting formula is believed to be new. Finally, we give new proofs of some identities of Postnikov [18] and Du–Liu [6] by relating these to appropriate functional equations.

Notations. The symmetric group is denoted by $\mathfrak{S}_n$. The standardized $\mathrm{Std}(w)$ of a word w of length n is the permutation obtained by iteratively scanning w from left to right, and labeling $1, 2, \ldots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. All algebras are over a field $\mathbb{K}$ of characteristic 0.

2. Free quasi-symmetric functions and hook length formulas

2.1. *A derivation of* **FQSym**

Recall from [5] that for an (infinite) totally ordered alphabet A, $\mathbf{FQSym}(A)$ is the subalgebra of $\mathbb{K}\langle A\rangle$ spanned by the polynomials

$$\mathbf{G}_\sigma(A) = \sum_{\mathrm{Std}(w)=\sigma} w \tag{10}$$

the sum of all words in A^n whose standardization is the permutation $\sigma \in \mathfrak{S}_n$. The multiplication rule is, for $\alpha \in \mathfrak{S}_k$ and $\beta \in \mathfrak{S}_l$,

$$\mathbf{G}_\alpha \mathbf{G}_\beta = \sum_{\substack{\gamma \in \mathfrak{S}_{k+l};\ \gamma = u\cdot v \\ \mathrm{Std}(u)=\alpha, \mathrm{Std}(v)=\beta}} \mathbf{G}_\gamma. \tag{11}$$

This sum has $\binom{k+l}{k}$ terms. Hence, the linear map

$$\phi : \mathbf{G}_\sigma \longmapsto \frac{t^n}{n!} \quad (\sigma \in \mathfrak{S}_n) \tag{12}$$

is a homomorphism of algebras $\mathbf{FQSym} \to \mathbb{K}[[t]]$. It is convenient to introduce the notation $\mathbf{F}_\sigma = \mathbf{G}_{\sigma^{-1}}$ and a scalar product satisfying $\langle \mathbf{F}_\sigma, \mathbf{G}_\tau\rangle = \delta_{\sigma,\tau}$. As a graded bialgebra, **FQSym** is self-dual, and its coproduct Δ satisfies $\langle FG, H\rangle = \langle F \otimes G, \Delta H\rangle$.

Let ∂ be the linear map defined by

$$\partial \mathbf{G}_\sigma = \mathbf{G}_{\sigma'}, \tag{13}$$

where σ' is the permutation whose word is obtained by erasing the letter n in $\sigma \in \mathfrak{S}_n$. Obviously,

$$\phi(\partial F) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(F) \tag{14}$$

for all $F \in \mathbf{FQSym}$, and moreover:

Proposition 2.1. *The map ∂ is a derivation of* **FQSym**. *It is the adjoint of the linear map $F \mapsto F \cdot \mathbf{F}_1$.*

Proof. By definition, $\langle \partial \mathbf{G}_\sigma, \mathbf{F}_\tau \rangle = \delta_{\sigma',\tau}$ is equal to 1 if σ occurs in $\tau \sqcup\!\sqcup\, n$ and to 0 otherwise. Hence,

$$\langle \partial \mathbf{G}_\sigma, \mathbf{F}_\tau \rangle = \langle \mathbf{G}_\sigma, \mathbf{F}_\tau \mathbf{F}_1 \rangle \tag{15}$$

whence the second part of the proposition. Now, $\mathbf{F}_1$ is a primitive element, so that ∂ is a derivation. ∎

The Leibniz relation

$$\partial(\mathbf{G}_\alpha \mathbf{G}_\beta) = \partial \mathbf{G}_\alpha \cdot \mathbf{G}_\beta + \mathbf{G}\alpha \cdot \partial \mathbf{G}_\beta \tag{16}$$

can be interpreted in terms of the *dendriform structure* of **FQSym**. Recall [13] that the product $\mathbf{G}_\alpha \mathbf{G}_\beta$ can be split into two parts (the dendriform operations)

$$\mathbf{G}_\alpha \mathbf{G}_\beta = \mathbf{G}_\alpha \prec \mathbf{G}_\beta + \mathbf{G}_\alpha \succ \mathbf{G}_\beta, \tag{17}$$

$$\mathbf{G}_\alpha \prec \mathbf{G}_\beta = \sum_{\substack{\gamma = u\cdot v,\ \max(u) > \max(v)\\ \mathrm{Std}(u)=\alpha, \mathrm{Std}(v)=\beta}} \mathbf{G}_\gamma, \tag{18}$$

$$\mathbf{G}_\alpha \succ \mathbf{G}_\beta = \sum_{\substack{\gamma = u\cdot v,\ \max(u) \le \max(v)\\ \mathrm{Std}(u)=\alpha, \mathrm{Std}(v)=\beta}} \mathbf{G}_\gamma. \tag{19}$$

Then,

$$\partial(\mathbf{G}_\alpha \prec \mathbf{G}_\beta) = \partial \mathbf{G}_\alpha \prec \mathbf{G}_\beta, \qquad \partial(\mathbf{G}_\alpha \succ \mathbf{G}_\beta) = \mathbf{G}_\alpha \succ \partial \mathbf{G}_\beta. \tag{20}$$

It will be convenient to consider the half-products as also defined on permutations, so that their sum is then the convolution $\alpha * \beta$.

2.2. *A differential equation in* **FQSym**

It follows from Proposition 2.1 that if we set $\mathbf{X} = (1 - \mathbf{G}_1)^{-1}$, we have

$$\partial \mathbf{X} = \mathbf{X}^2 \tag{21}$$

with $\mathbf{X}_0 = 1$ (constant term), and $\phi(\mathbf{X}) = (1-t)^{-1}$.

Note that thanks to the multiplication formula (11),

$$\mathbf{X} = \sum_\sigma \mathbf{G}_\sigma = \sum_{w \in A^*} w \tag{22}$$

is the sum of all permutations (interpreted as $\mathbf{G}$'s), that is, the sum of all words. If we can lift to **FQSym** the scalar bilinear map $B(x, y) = \int_0^t x(s)y(s)\mathrm{d}s$, it will also be interpretable as the sum of all complete binary trees.

2.3. The bilinear map

The required map is given by a simple operation, already introduced in [5], precisely with the aim of providing a better understanding of the Loday–Ronco algebra [13] of planar binary trees.

For $\alpha \in \mathfrak{S}_k$, $\beta \in \mathfrak{S}_l$, and $n = k + l$, set

$$\mathbf{B}(\mathbf{G}_\alpha, \mathbf{G}_\beta) = \sum_{\substack{\gamma = u(n+1)v \\ \mathrm{Std}(u)=\alpha, \mathrm{Std}(v)=\beta}} \mathbf{G}_\gamma. \tag{23}$$

Clearly,

$$\partial \mathbf{B}(\mathbf{G}_\alpha, \mathbf{G}_\beta) = \mathbf{G}_\alpha \mathbf{G}_\beta, \tag{24}$$

and our differential equation is now replaced by the fixed point problem

$$\mathbf{X} = 1 + \mathbf{B}(\mathbf{X}, \mathbf{X}). \tag{25}$$

Theorem 2.2. *In the binary tree solution (2) of (25),*

$$\mathbf{B}_T(1) = \sum_{\mathcal{T}(\sigma)=T} \mathbf{G}_\sigma, \tag{26}$$

where $\mathcal{T}(\sigma)$ denotes the shape of the decreasing tree of the permutation σ. In particular, $\mathbf{B}_T(1)$ coincides with $\mathbf{P}_T$, the natural basis of the Loday–Ronco algebra (in the notation of [10]).

Proof. By induction on the number n of internal nodes of T. For $n = 1$ the result is obvious, and if $n > 1$,

$$\mathbf{B}_T(1) = \mathbf{B}(\mathbf{B}_{T'}(1), \mathbf{B}_{T''}(1)),$$

where T' and T'' are the left and right subtrees of T. Hence, $\mathbf{B}_T(1)$ is the sum of the $\mathbf{G}_\sigma$ for $\sigma = \alpha n \beta$ such that $\mathbf{G}_{\mathrm{Std}(\alpha)}$ occurs in $\mathbf{B}_{T'}(1)$ and $\mathbf{G}_{\mathrm{Std}(\beta)}$ occurs in $\mathbf{B}_{T''}(1)$. Since we have assumed that (26) holds for T' and T'', this implies that it holds for T as well. ■

Corollary 2.3 (*The Hook Length Formula*). *The number of permutations whose decreasing tree has shape T is*

$$\frac{n!}{\prod_{v \in T} h_v}, \tag{27}$$

where for a vertex v of T, h_v is the number of nodes of the subtree with root v.

2.4. The q-hook length formula

Recall that under the q-specialization

$$A = \frac{1}{1-q} := \{\cdots < q^n < q^{n-1} < \cdots < q < 1\} \tag{28}$$

we have [12, (125)]

$$\mathbf{G}_\sigma\left(\frac{1}{1-q}\right) = \frac{q^{\mathrm{imaj}(\sigma)}}{(q)_n}, \tag{29}$$

where $\mathrm{imaj}(\sigma) = \mathrm{maj}(\sigma^{-1})$, $\mathrm{maj}(\sigma)$ is the classical major index (sum of the descents) of $\sigma \in \mathfrak{S}_n$ and $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$.

Hence, the map

$$\phi_q(\mathbf{G}_\sigma) = \frac{q^{\mathrm{imaj}(\sigma)}t^n}{[n]_q!} = (t(1-q))^n\mathbf{G}_\sigma\left(\frac{1}{1-q}\right) \tag{30}$$

is a homomorphism of algebras. The image of (25) under ϕ_q reads

$$x = 1 + B_q(x, x), \tag{31}$$

where the bilinear map is now a q-integral

$$B_q(f, g) = \int_0^t \mathrm{d}_q s f(s) g(qs), \tag{32}$$

and the q-integral is defined by

$$\int_0^t s^n \mathrm{d}_q s = \frac{t^{n+1}}{[n+1]_q}. \tag{33}$$

To show this, we have to compute $\phi_q(\mathbf{B}(\mathbf{G}_\alpha, \mathbf{G}_\beta))$.

Lemma 2.4. *Let $\alpha \in \mathfrak{S}_k$, $\beta \in \mathfrak{S}_l$. The inverse major index is distributed over the half-products according to*

$$\sum_{\gamma\in\alpha\succ\beta} q^{\mathrm{imaj}(\gamma)} = q^{\mathrm{imaj}(\alpha)+\mathrm{imaj}(\beta)} \begin{bmatrix} k+l-1 \\ l-1 \end{bmatrix}_q, \tag{34}$$

and

$$\sum_{\gamma\in\alpha\prec\beta} q^{\mathrm{imaj}(\gamma)} = q^{\mathrm{imaj}(\alpha)+\mathrm{imaj}(\beta)+l} \begin{bmatrix} k+l-1 \\ l \end{bmatrix}_q. \tag{35}$$

Proof. Straightforward by induction on $n = k + l$. ■

From this, one deduces immediately

$$\sum_{\substack{\gamma=u\cdot(n+1)\cdot v \\ \mathrm{Std}(u)=\alpha,\mathrm{Std}(v)=\beta}} q^{\mathrm{imaj}(\gamma)} = q^{\mathrm{imaj}(\alpha)+\mathrm{imaj}(\beta)+l} \begin{bmatrix} k+l \\ k \end{bmatrix}_q, \tag{36}$$

which in turn implies the following:

Lemma 2.5. *If $f(t) = \phi_q(F)$ and $g(t) = \phi_q(G)$, then*

$$\phi_q(\mathbf{B}(F, G)) = \int_0^t \mathrm{d}_q s f(s) g(qs). \tag{37}$$

Corollary 2.6 (*The q-Hook Length Formula of [1]*). *The inverse major index polynomial of the set of permutations whose decreasing tree has shape T is*

$$\sum_{\mathcal{T}(\sigma)=T} q^{\operatorname{imaj}(\sigma)} = [n]_q! \prod_{v \in T} \frac{q^{\delta_v}}{[h_v]_q}, \tag{38}$$

where δ_v is the number of nodes in the right subtree of v.

2.5. Another approach

It has been observed in [5] that **FQSym** had a natural q-deformation, obtained by replacing the ordinary shuffle $\sqcup\!\sqcup$ by the q-shuffle $\sqcup\!\sqcup_q$ in the product formula for the basis $\mathbf{F}_\sigma$. That is, $\mathbf{FQSym}_q$ is the algebra with basis $\mathbf{F}_\sigma = \mathbf{G}_{\sigma^{-1}}$ and product rule

$$\mathbf{F}_\alpha \mathbf{F}_\beta = \sum_\gamma (\gamma | \alpha \sqcup\!\sqcup_q \beta[k]) \mathbf{F}_\gamma = \sum_\gamma (\gamma | \alpha \sqcup\!\sqcup \beta[k]) q^{l(\gamma)-l(\beta)-l(\alpha)} \mathbf{F}_\gamma, \tag{39}$$

where $(\gamma | f)$ means the coefficient of γ in f, k is the length of α and $\beta[k] = (\beta_1 + k) \cdots (\beta_l + k)$, (the shifted word), $l(\sigma)$ being the number of inversions of σ.

Then, the map $\phi_q : \mathbf{FQSym}_q \to \mathbb{K}[[t]]$ defined by

$$\phi_q(\mathbf{G}_\sigma) = \frac{t^n}{[n]_q!} \tag{40}$$

is a homomorphism of algebras.

One has now

$$\phi_q(\partial F) = D_q \phi_q(F), \tag{41}$$

where D_q is the q-derivative

$$D_q f(t) = \frac{f(qt) - f(t)}{qt - t}. \tag{42}$$

In $\mathbf{FQSym}_q$, ∂ is not anymore a derivation, but satisfies

$$\partial(FG) = \partial F(A) \cdot G(qA) + F(A) \cdot \partial G(A) \tag{43}$$

so that the noncommutative functional equation is now

$$\partial \mathbf{X}(A) = \mathbf{X}(A)\mathbf{X}(qA), \qquad \mathbf{X}_0 = 1 \tag{44}$$

and its one-variable projection under ϕ_q is

$$D_q x(t) = x(t)x(qt), \qquad x(0) = 1. \tag{45}$$

This is equivalent to

$$x = 1 + B_q(x, x), \tag{46}$$

where we have again

$$B_q(x, y) = \int_0^t \mathrm{d}_q s\, x(s) y(qs). \tag{47}$$

Theorem 2.7 (*q-Hook Length Formula for Inversions [2]*). *The inversion polynomial of the set of permutations having a decreasing tree of shape T is given by the same hook length formula as for the inverse major index,*

$$\sum_{\mathcal{T}(\sigma)=T} q^{l(\sigma)} = [n]_q! \prod_{v\in T} \frac{q^{\delta_v}}{[h_v]_q}. \tag{48}$$

In particular imaj *and l are equidistributed on these sets.*

This is a refinement of a classical result of Foata and Schützenberger [7].

3. Word quasi-symmetric functions and plane trees

To interpret (3), we need to work in **WQSym**, the algebra of Word Quasi-Symmetric functions, which contains an algebra of plane trees (the free dendriform trialgebra on one generator [14]) in the same way as **FQSym** contains an algebra of binary trees [16].

The basis elements $\mathbf{M}_u$ of **WQSym** are labeled by packed words u, or if one prefers, surjections $[n] \to [k]$, set compositions, or faces of the permutohedron [3]. These objects are counted by the ordered Bell numbers [20, A000262]. There is a canonical way to associate a plane tree with such an object [16], and the sums over the fibers of this map span a Hopf subalgebra of **WQSym**. Hence, we need to define on **WQSym** an analogue of our derivation ∂ of **FQSym**.

Recall that a word w over the alphabet of positive integers is said to be *packed* if the set of letters occurring in w is an initial interval $[a_1, a_k]$ of the alphabet A. The *packed word* $u = \text{pack}(w)$ associated with a word $w \in A^*$ is obtained by the following process. If $b_1 < b_2 < \cdots < b_k$ are the letters occurring in w, u is the image of w by the semigroup homomorphism $b_i \mapsto a_i$. For example, $\text{pack}(34364) = 12132$. A word u is said to be *packed* if $\text{pack}(u) = u$. To such a word is associated a polynomial $\mathbf{M}_u$, defined as the sum of all words w such that $\text{pack}(w) = u$.

The product on **WQSym** is given by

$$\mathbf{M}_{u'}\mathbf{M}_{u''} = \sum_{u\in u'\star u''} \mathbf{M}_u, \tag{49}$$

where the *convolution* $u' \star u''$ of two packed words is defined as

$$u' \star u'' = \sum_{v,w;u=v\cdot w\in \text{PW},\text{pack}(v)=u',\text{pack}(w)=u''} u. \tag{50}$$

For example,

$$\mathbf{M}_{11}\mathbf{M}_{21} = \mathbf{M}_{1121} + \mathbf{M}_{1132} + \mathbf{M}_{2221} + \mathbf{M}_{2231} + \mathbf{M}_{3321}. \tag{51}$$

The coproduct can be defined by the usual trick of noncommutative symmetric functions, considering the alphabet A as an ordered sum of two mutually commuting alphabets $A'\hat{+}A''$. First, by direct inspection, one finds that

$$\mathbf{M}_u(A'\hat{+}A'') = \sum_{0\le k\le \max(u)} \mathbf{M}_{(u|_{[1,k]})}(A')\mathbf{M}_{\text{pack}(u|_{[k+1,\max(u)]})}(A''), \tag{52}$$

where $u|_B$ denote the subword obtained by restricting u to the subset B of the alphabet.

For a packed word u, let u' be the word obtained from u by erasing all the occurrences of the maximal letter $m = \max(u)$, e.g., $(5211354)' = 21134$. Now, define a linear map δ by

$$\delta \mathbf{M}_u = \mathbf{M}_{u'}. \tag{53}$$

This is not anymore a derivation, but rather a finite difference operator: indeed, it follows from (52) that

$$\delta \mathbf{M}_u(A) = \mathbf{M}_u(A \hat{+} 1) - \mathbf{M}_u(A), \tag{54}$$

where $A\hat{+}1$ is the ordered sum of A and $\{1\}$ (the scalar 1, so that $\mathbf{M}_u(1) = 1$ if u is of the form $11 \cdots 1$, and is 0 otherwise). Alternatively, δ is the adjoint of the right multiplication by $\sum_{n\geq 1} \mathbf{M}^*_{1^n}$, where $\mathbf{M}^*_u$ is the dual basis of $\mathbf{M}_u$.

This implies that δ satisfies

$$\delta(FG) = (\delta F)G + (\delta F)(\delta G) + F(\delta G), \tag{55}$$

but this formula can be refined in terms of the tridendriform structure of **WQSym** [16]. Indeed, it is known that $\mathbf{WQSym}^+$ is a sub-dendriform trialgebra of $\mathbb{K}\langle A\rangle^+$, the partial products being given by

$$\mathbf{M}_{w'} \prec \mathbf{M}_{w''} = \sum_{w=u\cdot v\in w'\star w'', |u|=|w'|; \max(v)<\max(u)} \mathbf{M}_w, \tag{56}$$

$$\mathbf{M}_{w'} \circ \mathbf{M}_{w''} = \sum_{w=u\cdot v\in w'\star w'', |u|=|w'|; \max(v)=\max(u)} \mathbf{M}_w, \tag{57}$$

$$\mathbf{M}_{w'} \succ \mathbf{M}_{w''} = \sum_{w=u\cdot v\in w'\star w'', |u|=|w'|; \max(v)>\max(u)} \mathbf{M}_w r, \tag{58}$$

and it follows from the multiplication rule (49) that

$$\delta(F \prec G) = (\delta F)G, \qquad \delta(F \circ G) = (\delta F)(\delta G), \qquad \delta(F \succ G) = F(\delta G). \tag{59}$$

Now, let

$$\mathbf{X} = (1 - q\mathbf{M}_1)^{-1} = \sum_u q^{|u|} \mathbf{M}_u = \sum_w q^{|w|} w. \tag{60}$$

It follows from (55) that

$$\delta \mathbf{X} = q\mathbf{X}^2(1 - q\mathbf{X})^{-1} = \sum_{n\geq 2} q^{n-1}\mathbf{X}^n. \tag{61}$$

For packed words $u_1, \ldots, u_k$, define

$$\mathbf{F}_k(\mathbf{M}_{u_1}, \ldots, \mathbf{M}_{u_k}) = \sum \mathbf{M}_w, \tag{62}$$

where the sums runs over packed words w such that

$$w = w_1 m w_2 m \cdots m w_k, \qquad \mathrm{pack}(w_i) = u_i, \qquad m = \max(w_1, \ldots, w_k) + 1. \tag{63}$$

For example,

$$\mathbf{F}_2(\mathbf{M}_{11}, \mathbf{M}_{21}) = \mathbf{M}_{11321} + \mathbf{M}_{11432} + \mathbf{M}_{22321} + \mathbf{M}_{22431} + \mathbf{M}_{33421}. \tag{64}$$

Then, obviously,

$$\mathbf{X} = 1 + \sum_{n\geq 2} q^{n-1}\mathbf{F}_n(\mathbf{X}\ldots,\mathbf{X}) \tag{65}$$

which does indeed give back (61), since

$$\delta\mathbf{F}_k(\mathbf{M}_{u_1},\ldots,\mathbf{M}_{u_k}) = \mathbf{M}_{u_1}\cdots\mathbf{M}_{u_k}. \tag{66}$$

It follows from (49) that the linear map $\psi :$ **WQSym** $\to \mathbb{K}[[t]]$ defined by

$$\psi(\mathbf{M}_u) = \binom{t}{\max(u)} \tag{67}$$

is a homomorphism of algebras. Moreover, it maps δ over the finite difference operator

$$\psi(\delta F) = \Delta\psi(F), \tag{68}$$

where $\Delta f(t) = f(t+1) - f(t)$. Hence, the images of (61) and (65) by ψ are

$$\Delta x = \sum_{n\geq 2} q^{n-1}x^n \tag{69}$$

$$x = 1 + \sum_{n\geq 2} q^{n-1}F_n(x,x,\ldots,x), \tag{70}$$

where

$$F_n(x_1,\ldots,x_n) = \Sigma_0^t x_1(s)x_2(s)\cdots x_n(s)\delta s, \tag{71}$$

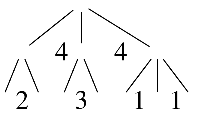
the discrete integral being defined by

$$\Sigma_0^t f(s)\delta s = \sum_{i=0}^{t-1} f(i). \tag{72}$$

The realization of the free dendriform trialgebra given in [16] involves the following construction. With any word w of length n, associate a plane tree $\mathcal{T}(w)$ with $n+1$ leaves, as follows: if $m = \max(w)$ and if w has exactly $k-1$ occurrences of m, write

$$w = v_1 m v_2 \cdots v_{k-1} m v_k, \tag{73}$$

where the v_i may be empty. Then, $\mathcal{T}(w)$ is the tree obtained by grafting the subtrees $\mathcal{T}(v_1), \mathcal{T}(v_2), \ldots, \mathcal{T}(v_k)$ (in this order) on a common root, with the initial condition $\mathcal{T}(\epsilon) = \emptyset$ for the empty word. For example, the tree associated with 243411 is



(74)

From the previous considerations, one can now deduce a formula for the number of packed words yielding a given plane tree, which can be regarded as another generalization of the hook length formula for binary trees:

Theorem 3.1. *If a term $F_T(1)$ in the plane tree solution has the decomposition*

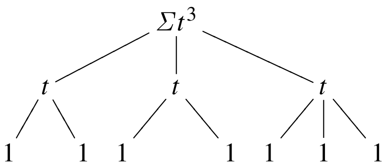
$$F_T(1) = \sum_k c_k \binom{t}{k} \tag{75}$$

then, c_k is the number of packed words u with maximal letter k such that $\mathcal{T}(u) = T$.

Proof. A straightforward induction, from (63) and (73). ■

For example, the following tree

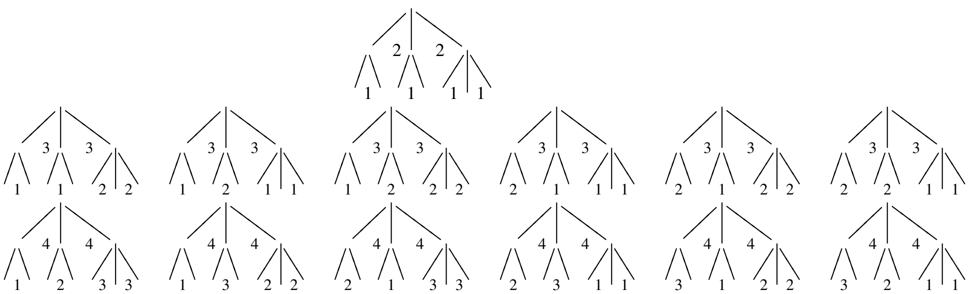
$$\mathbf{F}_3(\mathbf{F}_2(1,1), \mathbf{F}_2(1,1), \mathbf{F}_3(1,1,1)) = \tag{76}$$



gives

$$\Sigma_0^t s^3 \delta s = 6\binom{t}{4} + 6\binom{t}{3} + \binom{t}{2} \tag{77}$$

so that there are $6 + 6 + 1 = 13$ packed words whose plane trees have this shape:



4. Functional equations associated with some generalizations of the hook length formula

4.1. Postnikov's identity and Eisenstein's exponential series

Postnikov [18] has obtained the following identity

$$(n+1)^{n-1} = \frac{n!}{2^n} \sum_{T \in \mathbf{BT}_n} \prod_{v \in T} \left(1 + \frac{1}{h_v}\right), \tag{78}$$

where $\mathbf{BT}_n$ is the set of (incomplete) binary trees with n nodes. Combinatorial proofs are given in [4,19], and generalizations occur in [6,8].

Let $g(t)$ be the exponential generating function of the lhs of (78), that is,

$$g(t) = \sum_{n \geq 0} (n+1)^{n-1} \frac{t^n}{n!}. \tag{79}$$

This is a famous power series, known as Eisenstein's generalized exponential (see, e.g., [17] or [9, Chap. 5]). It satisfies the functional equation

$$g(t) = e^{tg(t)}. \tag{80}$$

Hence, $x = g(t)$ is a solution of the differential equation

$$x' = x^2 + txx' = x^2 + t\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x^2}{2}\right), \tag{81}$$

and integrating by parts, we obtain the fixed point equation

$$x = 1 + t\frac{x^2}{2} + \frac{1}{2}\int_0^t x^2(s)\mathrm{d}s = 1 + B(x, x) \tag{82}$$

with

$$B(x, y) = t\frac{xy}{2} + \frac{1}{2}\int_0^t x(s)y(s)\mathrm{d}s. \tag{83}$$

From this, one derives that

$$B_T(1) = \frac{1}{2^n}\prod_{v\in T}\left(1 + \frac{1}{h_v}\right)t^n, \tag{84}$$

since, by induction, if T has T_1 (resp. T_2) as left (resp. right) subtree with n_1 nodes (resp. n_2 nodes), then

$$\begin{aligned} B_T(1) &= B(B_{T_1}(1), B_{T_2}(1)) \\ &= \frac{1}{2^{n_1}}\prod_{v\in T_1}\left(1 + \frac{1}{h_v}\right)\frac{1}{2^{n_2}}\prod_{v\in T_2}\left(1 + \frac{1}{h_v}\right)\left(\frac{1}{2}t^{n_1+n_2+1} + \frac{1}{2}\frac{t^{n_1+n_2+1}}{n_1+n_2+1}\right) \\ &= \frac{1}{2^{n_1+n_2+1}}\prod_{v\in T}\left(1 + \frac{1}{h_v}\right)t^n, \end{aligned} \tag{85}$$

which explains (78). Note in particular that both terms of $B(x, y)$ contribute to one term (either 1 or $1/h_v$) for each node.

4.2. Du–Liu identities

Lascoux proposed a one-parameter generalization of (78):

$$\sum_T\prod_v\left(\alpha + \frac{1}{h_v}\right) = \frac{1}{(n+1)!}\prod_{i=0}^{n-1}((n+1+i)\alpha + n + 1 - i) \tag{86}$$

which has been proved by Du and Liu [6], who reformulated it as

$$\sum_T\prod_v\frac{(h_v+1)\alpha + 1 - h_v}{2h_v} = \frac{1}{n+1}\binom{(n+1)\alpha}{n} \tag{87}$$

and obtained the further generalization

$$\sum_T \prod_v \frac{(mh_v+1)\alpha+1-h_v}{(m+1)h_v} = \frac{1}{mn+1}\binom{(mn+1)\alpha}{n}, \tag{88}$$

where now, T runs over plane $(m+1)$-ary trees.

These identities can also be obtained from the tree solution of a functional equation. Let $x = f(t)$ be the ordinary generating function of the rhs of (88), that is,

$$f(t) = \sum_{n\geq 0} \binom{(mn+1)\alpha}{n} \frac{t^n}{mn+1}. \tag{89}$$

It follows from the Lagrange inversion formula (see, e.g., [15, p. 35 ex. 25]) that x is solution of the fixed point equation

$$x = (1+tx^m)^\alpha. \tag{90}$$

Taking derivatives, we obtain the differential equation

$$x' = \alpha x^{m+1} + (\alpha m - 1)t \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x^{m+1}}{m+1}\right) \tag{91}$$

and integrating by parts, we arrive at

$$x = 1 + \frac{\alpha m - 1}{m+1} t x^{m+1} + \frac{\alpha+1}{m+1}\int_0^t x^{m+1}(s)\mathrm{d}s \equiv 1 + F_{m+1}(x, x, \ldots, x). \tag{92}$$

As in the Postnikov identity, the $(m+1)$-ary tree expansion of the solution associates with each tree T the lhs of (88), where both terms of F_{m+1} contribute to one term (either with coefficient 1 or $1/h_v$) for each node.

5. Concluding remarks

The original hook length formula for Young tableaux can be interpreted as giving the image of a Schur function by the ring homomorphism $f \mapsto f(\mathbb{E})$ defined on the power sums

$$p_n \mapsto p_n(\mathbb{E}) = \begin{cases} 1 & n = 1 \\ 0 & n > 1. \end{cases} \tag{93}$$

There are generalizations giving the images by the morphisms

$$\begin{cases} p_n \mapsto p_n\left(\dfrac{1}{1-q}\right) = \dfrac{1}{1-q^n}, \\ p_n \mapsto p_n(\alpha) = \alpha, \\ p_n \mapsto p_n\left(\dfrac{1-t}{1-q}\right) = \dfrac{1-t^n}{1-q^n}, \end{cases} \tag{94}$$

the last one giving back the first one for $t = 0$ and the second one for $t = q^\alpha$ and $q \to 1$.

The theory of noncommutative symmetric functions allows one to define analogs of these specializations for quasi-symmetric functions [12], and therefore also for those combinatorial Hopf algebras H which admit homomorphisms $H \to QSym$. This is the case of **PBT** and **WQSym**, and Corollary 2.6 and Theorem 3.1 can be interpreted as evaluation of $\mathbf{P}_T(1/(1-q))$

and $\mathbf{M}_T(\alpha)$ respectively. It will be shown in a forthcoming paper that it is in fact possible to evaluate both $\mathbf{P}_T$ and $\mathbf{M}_T$ on $(1-t)/(1-q)$ defined in the right way, and to get (q,t)-hook length formulas for binary and plane trees.

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