Non approximability and non-continuity of the fall coloring graph problem

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Abstract

We investigate relations between interpolation properties and non-approximability aspects of some NP-complete graph coloring problems. The graph coloring we consider here is the *fall coloring*, i.e., a proper coloring of a graph G in which every colored vertex sees all the other colors in its neighborhood. Given a graph G, the optimization problem we focus on consists in determining the maximum cardinality of such a fall coloring (if at least one exists in this graph). The interpolation property of fall colorings says that for some graphs, there is no fall coloring for some cardinals between the minimum and the maximum cardinalities of fall colorings (we talk about not f-continuous graphs). We exhibit a family of graphs having at most two cardinalities of existing fall colorings and in which determining the maximal one leads to a NP-complete problem. By combining these interpolation and complexity properties, we show that the optimization problem about fall coloring can not be approximated with ratio less than $n^{1-\epsilon}$, for any $\epsilon > 0$, for any graph with n vertices, unless $P = NP$. Moreover, we give the complexity of problem of deciding whether a given graph is f-continuous and we prove that unless $P = NP$, there exists no polynomial-time approximation algorithm with approximation ratio bounded by a constant for problem of finding the maximum integer k for which a graph G has a fall k-coloring for f-continuous graphs. Finally, we study the properties of graphs having fall colorings and the f-spectrum of graphs (i.e., set of cardinal of existing fall colorings). We exhibit a class of graphs such that any integer set S (included in $\mathbb{N} \setminus \{0, 1\}$) can be associated with a graph having its f-spectrum equal to S.

Key words: Complexity, Graph Coloring

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1 Introduction

We investigate relations between interpolation properties and non approximability aspects of some NP-complete graph coloring problems. Given a graph G, we consider here a (k)-coloring of G, as a proper coloring with cardinality k [3]. It is well known that the problem of determining the minimum chromatic number $\chi(G)$ (i.e., the minimum possible value for k) cannot be approximated within a factor of $|V|^{1-\varepsilon}$ for any $\varepsilon > 0$ [14] unless $P = NP$. Moreover for some particular proper colorings, approximation results also exist. For instance, the NP-complete problem of determining the maximum cardinality k of an a-coloring, i.e., a (k)-coloring in which for any pair of colors there exists an edge connecting two vertices with these colors, can be approximated within a factor of $O(|V| \log\log |V| / \log |V|)$ (see [12]).

In this paper, we focus on the impact of interpolation properties of some graph colorings [9] on their approximation behavior. To illustrate this kind of property, let us consider b-colorings. A (k)b-coloring of a graph G is a (k)-coloring in which for any used color, there is at least one vertex called *colorful* vertex with this color having each other color in its neighborhood [10]. It is clear that any $(\chi(G))$-coloring of G is a b-coloring. Moreover, the problem of determining the maximum cardinality $b(G)$ of such a b-coloring is NP-complete [10]. In terms of interpolation property [9], it is easy to see that there exist some non b-continuous graphs G, i.e., there exist some integers k, $\chi(G) < k < b(G)$, such that there is no (k)b-coloring of G (see [10]). Moreover, in [2], for any nonempty integer set (included in $\mathbb{N} \setminus \{0, 1\}$), there exists a graph whose set of b-colorings cardinalities (called b-spectrum) is this given set. Up to our knowledge, there is no constant $\epsilon > 0$ for which the problem of determining $b(G)$, can be approximated within a factor of $120/113 - \epsilon$ in polynomial time, unless $P = NP$ [4]. Moreover, in [7], a distributed algorithm to compute a b-coloring is presented.

In this paper, we focus on a particular case of b-colorings called fall colorings [6], defined as proper colorings in which each vertex is a colorful vertex (i.e . We call fall chromatic number and fall achromatic number respectively the minimum and maximum cardinalities of a fall coloring of G which we denote by $\chi_f(G)$ and $\psi_f(G)$). Note that fall colorings do not exist in every graph and Dunbar et al. have shown that the problem of deciding if a given graph admits a fall coloring is NP-complete [6]. Moreover, in [5], some classes of graphs where the usual coloring problem is easy, such as chordal graphs are studied : the fall coloring can be computed in polynomial time for chordal graphs if there exists one.

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In this paper, we only consider graphs in which at least one fall coloring exists. As for b-colorings, it is easy to see that there exist some not f-continuous graphs (we give an example in Section 3). Answering a question of Dunbar et al. [6], we also show in this paper that given a graph G and an integer k, determining whether $\psi_f(G)$ is greater or equal to k is an NP-complete problem. The main question we focus on here is whether these "holes" in the cardinal set of fall colorings could negatively influence the approximation of $\psi_f(G)$. To answer this question, we define a class of graphs with n vertices the f-spectrum of which (i.e., set of cardinal of existing fall colorings) is either $\{2\}$ or $\{2, \alpha\}$, such that α is an integer less than n, the problem of deciding which of these two sets is the f-spectrum being NP-complete. This shows that there is no approximation algorithm with ratio less than $n^{1-\epsilon}$, for any $\epsilon > 0$ unless $P = NP$.

The proof of the main result is based on graphs in which the f-spectrum contains only $\chi_f(G)$ and $\psi_f(G)$. If all non f-continuous graphs would verify this property, we could think of a positive approximation result for all the other graphs. But actually we show that any integer set is the f-spectrum of a graph and that knowing if a graph is f-continuous or not is NP-complete.

The paper is organized as follows. In Section 2, we show that the maximization problem about fall coloring can not be approximated with ratio less than $n^{1-\epsilon}$, (for any $\epsilon > 0$), for any graph with n vertices, unless $P = NP$. Moreover, we prove that the problem of deciding whether a given graph is f-continuous is NP-complete and that unless $P = NP$, there exists no polynomial-time approximation algorithm with approximation ratio bounded by a constant for problem of finding the maximum integer k for which a graph G has a fall k-coloring for f-continuous graphs. In Section 3, we discuss on f-spectrum of graphs and in particular we show that for any integer set S included in $\mathbb{N} \setminus \{0, 1\}$, there exists a graph which cardinal set of fall colorings is S.

2 Approximation results

This section is devoted to determine the complexity of the optimization problem related to compute a fall coloring having a maximum number. Formally, the following problem is studied:

MAXIMUM FALL K-COLORING (MFKC)
Instance: Graph G having a fall coloring.
Solution: a fall α-coloring of G
Measure: α

Let $OPT(x)$ denote the optimal value for any arbitrary instance x of *MFKC* and let $B(x)$ be the solution found by an algorithm B. Consider a function $\lambda : \mathbb{Z}^+ \to \mathbb{R}^+$. We say that a polynomial-time algorithm B is a λ-approximation algorithm for *MFKC* iff for every instance x of *MFKC* of size n, $B(x)$ is in the range $[OPT(x)/\lambda(n), OPT(x)]$. We say that *MFKC* is approximated within a factor of λ if such an algorithm exists. The remaining of this section proves that $r-$approximating with $r < n^{1-\epsilon}$ for any $\epsilon > 0$, becomes computationally intractable.

We give now our main result concerning non-approximation of MFKC.

Theorem 1 *The optimization problem* MFKC *is not approximated within a factor of* $n^{1-\epsilon}$ *for any* $\epsilon > 0$ *where* n *is the number of vertices, unless* $P = NP$.

The remaining of this section is devoted to the proof of Theorem 1. First, in Section 2.1, we establish a polynomial time transformation from an instance of the NP-complete problem *NOT-ALL-EQUAL 3-SATISFIABILITY* (see [8,13]) to an instance G_t of MFKC. Then, in Section 2.2, we give the f-spectrum of graph G_t. Finally, we conclude the proof of our theorem in Section 2.3 and we give some extension results for particular graphs in Section 2.4.

2.1 Polynomial Transformation

The NP-complete problem *NOT-ALL-EQUAL 3-SATISFIABILITY (NAE-3-SAT)* is defined as follows [8,13]:

NOT-ALL-EQUAL 3-SATISFIABILITY (NAE-3-SAT)
Instance: Set U of variables, collection C of disjunctive clauses over U such that each clause $C_i \in C$ has $|C_i| = 3$.
Question: Is there a truth assignment for U such that each clause in C has at least one true literal and at least one false literal?

We give now a polynomial time transformation called $\mathcal{A}$ (see Table 1) which takes as input an instance $I = \langle U = \{u_1, \ldots, u_k\}, C = \{C_1, \ldots, C_p\}\rangle$ of *NAE-3-SAT* and an integer t with $t \geq 1$. It constructs a graph G_t, an instance of *MFKC*. Let $K'_{n,n}$ be a graph obtained from a complete bipartite graph $K_{n,n}$ by removing a perfect matching. Now, we will describe transformation $\mathcal{A}$.

Each clause C_j, for $j = 1, \ldots, p$, of I is represented only by one vertex c_j (see instruction 5a). Moreover, G_t contains a graph $K'_{2t,2t}$ such that every vertex c_j for $j = 1, \ldots, p$ has $2t - 1$ neighbors, in one of both bipartition class of graph $K'_{2t,2t}$. The goal of this construction is to consider that every vertex c_j for $j = 1, \ldots, p$ can easily have $2t - 1$ different colors in its neighborhood.

Fig. 1. An example of graph G_t such that instance I has $C_1 = (u_1, \overline{u_2}, u_k)$

Every variable u_i of I corresponds $4t + 1$ vertices (see instructions 4a, 4b and 4c): sets X_i and $\overline{X_i}$ contain t vertices which are adjacent to a vertex z_i (see intruction 4f). The goal of this construction is to consider that if z_u is colorful for a coloring with $2t+1$ colors then sets X_u and $\overline{X_u}$ must contain $2t$ distinct colors. For literal u (resp. $\overline{u}$), one vertex in X_u (resp. $\overline{X_u}$) is selected so that if literal u (resp. $\overline{u}$) appears in clause C_j, then this selected vertex is adjacent to vertex c_j in instruction 5c. Moreover other vertices are added so that vertices in $X_u \cup \overline{X_u}$ can be colorful for a coloring with $2t + 1$ colors (see instructions 4b, 4e and 4d). Finally, two disctint vertices a and b are added. Vertex a (resp. b) is adjacent to all vertices in Y_{k+1} (resp. X_{k+1}) and $Y_i \cup \overline{Y_i}$ in (resp. $X_i \cup \overline{X_i}$) for $i = 1 \dots k$. Algorithm $\mathcal{A}$ is described formally in Table 1.

In order to determine the running time of Algorithm $\mathcal{A}$, the total number of vertices of G_t will be computed according to the size of instance I of NAE-3-SAT $\langle U = \{u_1, \dots, u_k\}, C = \{C_1, \dots, C_p\}\rangle$. Every variable u of I corresponds

Algorithm $\mathcal{A}$.

Input: Instance I of NAE-3-SAT $\langle U = \{u_1, \ldots, u_k\}, C = \{C_1, \ldots, C_p\}\rangle$
an integer t

Output: Graph $G_t = (V_t, E_t)$

(1) $X_{k+1} \leftarrow \{x^1_{k+1}, x^2_{k+1}, \ldots, x^{2t}_{k+1}\}$, $Y_{k+1} \leftarrow \{y^1_{k+1}, y^2_{k+1}, \ldots, y^{2t}_{k+1}\}$,

(2) $V_t \leftarrow \{a, b\} \cup X_{k+1} \cup Y_{k+1}$

(3) $E_t \leftarrow \{(y^\ell_{k+1}, x^{\ell'}_{k+1}), (y^\ell_{k+1}, a), (x^\ell_{k+1}, b) : 1 \leq \ell \leq 2t \wedge 1 \leq \ell' \leq 2t \wedge \ell \neq \ell'\}$

(4) for each variable $u_i \in U$ do
  (a) $X_i \leftarrow \{x^1_i, x^2_i, \ldots, x^t_i\}$, $\overline{X_i} \leftarrow \{\overline{x^1_i}, \overline{x^2_i}, \ldots, \overline{x^t_i}\}$
  (b) $Y_i \leftarrow \{y^1_i, y^2_i, \ldots, y^t_i\}$, $\overline{Y_i} \leftarrow \{\overline{y^1_i}, \overline{y^2_i}, \ldots, \overline{y^t_i}\}$,
  (c) $V_t \leftarrow V_t \cup Y_i \cup \overline{Y_i} \cup X_i \cup \overline{X_i} \cup \{z_i\}$
  (d) $E_t \leftarrow E_t \cup \{(\overline{y^\ell_i}, \overline{x^{\ell'}_i}), (y^\ell_i, x^{\ell'}_i) : 1 \leq \ell \leq t \wedge 1 \leq \ell' \leq t\}$
  (e) $E_t \leftarrow E_t \cup \{(\overline{y^\ell_i}, x^{\ell'}_i), (y^\ell_i, \overline{x^{\ell'}_i}) : 1 \leq \ell \leq t \wedge 1 \leq \ell' \leq t \wedge \ell \neq \ell'\}$
  (f) $E_t \leftarrow E_t \cup \{(\overline{x^\ell_i}, z_i), (x^\ell_i, z_i), (\overline{x^\ell_i}, b), (x^\ell_i, b), (\overline{y^\ell_i}, a), (y^\ell_i, a) : 1 \leq \ell \leq t\}$

(5) For each clause $C_d \in C$ do,
  (a) $V_t \leftarrow V_t \cup \{c_d\}$
  (b) $E_t \leftarrow E_t \cup \{(c_d, x^\ell_{k+1}) : 2 \leq \ell \leq 2t \wedge \ell \neq t + 1\}$
  (c) for each literal u_α in C_d do $E_t \leftarrow E_t \cup \{(c_d, x^1_\alpha)\}$

(6) return G_t

Table 1

to $4t + 1$ vertices (see instructions 4a, 4b and 4c of Algorithm $\mathcal{A}$) and each clause of I is represented only by one vertex (see instruction 5a). Moreover, there are $4t+2$ additional vertices (see instructions 1 and 2). Since the number of vertices of G_t is equal to $4t(k+1)+k+p+2$, Algorithm $\mathcal{A}$ runs in polynomial time in term of t and of the size of instance I. The next section is devoted to the f-spectrum of G_t.

Figure 1 gives an example for the partial graph resulting from algorithm $\mathcal{A}$.

2.2 The f-spectrum of G_t

Now, we will give the f-spectrum of G_t. This f-spectrum depends on the instance I of *NAE-3-SAT* and on an integer t.

Proposition 1 *Let $I = \langle U, C \rangle$ be an instance of* NAE-3-SAT. *Let G_t be the output of algorithm $\mathcal{A}$ having I and an arbitrary integer t as input.*

(1) The f-spectrum of G_t is $\{2, 2t+1\}$ if set U of variables has a truth assignment with desired properties
(2) The f-spectrum of G_t is $\{2\}$ otherwise

Now, we will prove Proposition 1. First, it is easy to see that graph G_t is a bipartite graph because all edges have only one vertex in $\cup_{i=1}^{k} \left(X_i \cup \overline{X_i} \right) \cup X_{k+1} \cup \{a\}$. So, we deduce that:

Lemma 1 *Graph G_t has a fall 2-coloring.*

Moreover, graph G_t satisfies the following lemma:

Lemma 2 *Graph G_t does not have any fall j-coloring for $j = 3, \ldots, 2t$.*

Proof. We show Lemma 2 by contradiction. Assume that G_t has a fall j-coloring with $2 < j < 2t+1$.

Without loss of generality, we assume that the color of vertex z_1 is j. The neighborhood of z_1 is $X_1 \cup \overline{X_1}$. Since z_1 is colorful, for each color c in $\{1, \ldots, j-1\}$ there exists at least one vertex in $X_1 \cup \overline{X_1}$ colored with c. In addition, vertex b must be colored with j, because it is adjacent to all the vertices of $X_1 \cup \overline{X_1}$. Since the vertices in $\cup_{i=1}^{k+1} X_i \cup_{i=1}^{k} \overline{X_i}$ are adjacent to vertex b, their color is not j.

Moreover, the neighborhood of c_1 is included in $\cup_{i=1}^{k+1} X_i \cup_{i=1}^{k} \overline{X_i}$. Since that all neighboors of c_1 are not colored with color j, c_1 should be colored with color j.

Since $j - 1 < 2t$ and since $|X_1 \cup \overline{X_1}| = 2t$, at least two vertices in $X_1 \cup \overline{X_1}$ are colored with the same color, say color α. This implies that color α cannot be assigned to any vertex of $Y_1 \cup \overline{Y_1}$, because each vertex of $Y_1 \cup \overline{Y_1}$ has $2t - 1$ neighbors in $X_1 \cup \overline{X_1}$. So, every vertex of $Y_1 \cup \overline{Y_1}$ has at least one neighbor in $X_1 \cup \overline{X_1}$ colored with α.

Since there are at least $j - 1$ distinct colors in $X_1 \cup \overline{X_1}$, there exists at least one vertex u of $X_1 \cup \overline{X_1}$ colored with α' such that $\alpha' \neq \alpha$. Consequently vertex u cannot be colorful. Indeed, vertex u has not any neighbor of color α. This leads to contradiction with the definition of fall coloring. This concludes the proof of Lemma 2. $\square$

According to Lemmas 1 and 2, it remains to prove that the set U of variables has a truth assignment with desired properties if and only if G_t has a fall $(2t+1)$-coloring.

Lemma 3 *If set U of variables has a truth assignment with desired properties, then G_t has a fall $2t+1$-coloring.*

Proof. Assume first that I has a satisfying truth assignment $f : U \to \{T, F\}$. Color vertices $\{a, b, z_i, c_\ell : 1 \leq i \leq k \wedge 1 \leq \ell \leq p\}$ with color $2t+1$. Next, for $i = 1, \ldots, k$, if $f(u_i) = T$, color vertices $\{x_i^\ell, y_i^\ell, \overline{x_i^\ell}, \overline{y_i^\ell}\}$, for $\ell = 1, \ldots, t$ with colors $\{\ell, \ell+t, \ell+t, \ell\}$ respectively, otherwise with colors $\{\ell+t, \ell, \ell, \ell+t\}$ respectively. Finally, color vertices x_{k+1}^ℓ and y_{k+1}^ℓ with color ℓ for $\ell = 1, \ldots, 2t$. Clearly this coloring is a proper coloring. Moreover, by construction, it is easy to see that every vertex not in $\{c_i : 1 \leq i \leq p\}$ is colorful. Now, it remains to check that all the vertices in $\{c_i : 1 \leq i \leq p\}$ are colorful. Let C_j be a clause in C. Vertex c_j is adjacent to vertices in $\{x_{k+1}^\ell, x_{k+1}^{\ell+t} : 1 < \ell \leq t\}$ and so it is adjacent to vertices of colors ℓ, $1 < \ell \leq t$ and to vertices of colors ℓ, $t+1 < \ell \leq 2t$. Since f is a satisfying truth assignment such that every clause C_j has at least one true literal and at least one false literal, vertex c_i is adjacent to vertices of colors 1, $t+1$ and vertex c_j is colorful. So this coloring is a fall coloring. This concludes the proof of Lemma 3. $\square$

Conversely, we prove the following lemma:

Lemma 4 *If G_t has a fall $2t+1$-coloring. Then, the set U of variables has a truth assignment with desired properties.*

Proof. Assume that graph G_t has a fall $(2t+1)$-coloring. Without loss of generality, we assume that the color of vertex b is $2t+1$. Therefore no vertex of any X_i or $\overline{X_i}$ has color $2t+1$.

Let i be an integer between 1 and k. Since z_i is a colorful vertex and since the neighborhood of z_i is $X_i \cup \overline{X_i}$, this set contains $2t$ distinct colors and it does not contain the color of z_i. Since vertex b is adjacent to $X_i \cup \overline{X_i}$, the color of b is the same as that of z_i. So, for $i = 1, \ldots, k$, the color of z_i is $2t+1$. Since, the vertices of Y_{k+1} are of degree $2t$, the neighborhood of any vertex y of Y_{k+1} must have distinct colors, otherwise y would not be colorful. Now, if two vertices of X_{k+1} have the same color then there exists at least one vertex of Y_{k+1} with two neighbors of the same color. So the vertices of X_{k+1} have $2t$ distinct colors in X_{k+1}. With regard to vertex a, its color is $2t+1$ using the same previous properties.

Let C_j be a clause in C. Since the neighborhood of c_j is contained in set $\bigcup_{i=1}^{k+1} X_i \cup \bigcup_{i=1}^{k} \overline{X_i}$, and since this fall $(2t+1)$-coloring is proper, c_j must have color $2t+1$. Moreover, for any clause C_j, vertex c_j is adjacent to $\{x_{k+1}^\ell : 1 < \ell \leq 2t \wedge \ell \neq t+1\}$. Since vertices in $\{x_{k+1}^\ell : 1 < \ell \leq 2t \wedge \ell \neq t+1\}$ have $2t-2$ distinct colors, we assume without loss of generality that the two missing colors are 1 and $t+1$. We define a function $f : U \to \{T, F\}$ by setting $f(u_i) = T$ if vertex x_i^1 is colored with color 1, otherwise $f(u_i) = F$. Since the coloring is a fall $(2t+1)$-coloring, each vertex c_i is adjacent to at least one vertex of color 1 and at least one vertex of color $t+1$. This function f is a satisfying truth assignment with desired properties for *NAE-3-SAT*, which concludes the proof of Lemma 4. □

Lemmas 1, 2, 3 and 4 imply Proposition 1.

2.3 Non-approximation proof

This section is devoted to prove Theorem 1. From Proposition 1 we first prove the following proposition:

Proposition 2 *The problem of deciding, for a graph $G = (E, V)$ having a fall 2-coloring and an integer t, whether its f-spectrum is $\{2, 2t+1\}$ is NP-complete.*

By setting $t = 1$, we can deduce that the problem of deciding, for a graph $G = (E, V)$ having a fall 2-coloring, whether G has a fall 3-coloring is NP-complete.

Moreover, using the gap technique ([1], p. 100), by Propositions 1 and 2, we prove that no polynomial-time t-approximation algorithm for *MFKC* may exist unless $P = NP$. Now, we will compare t to n (number of vertices of graph G_t). In Section 2.1, n is computed : $n = 4t(k+1) + k + p + 2$. Choose arbitrarily a sufficiently small constant $\epsilon > 0$. We consider $t = \lceil (4(k+p))^{1/\epsilon - 1} \rceil$. By definition, we have $n < 4t(k+1)$ and $(4(k+p))^{1/\epsilon-1} \leq t$. Thus, we can deduce that $n \leq t^{1/(1-\epsilon)}$. Then $t \leq n^{1-\epsilon}$ and problem *MFKC* is not approximated within a factor of $n^{1-\epsilon}$ for any $\epsilon > 0$ in polynomial time. This concludes the proof of Theorem 1.

We have shown that the problem of deciding, for a graph $G = (E, V)$ having a fall 2-coloring and an integer t, whether its f-spectrum is $\{2, 2t+1\}$ is NP-complete. In this last case, the "hole" between 2 and $2t+1$ leads to the non-approximability of MFKC. But these graphs have a very particular f-spectrum and once could ask if except them, it is possible to obtain better approximation results. We give some first negative answers in the next section.

2.4 Complexity of f-continous graphs.

One could ask if similar negative results presented previously occur if only f-continuous graphs are considered. This is an open question, but we can give the following first result about the difficulty of deciding whether a graph is f-continuous or not. In order to prove it, we first give a simple lemma about combining the f-spectrums of two graphs. The *join* graph (see [11]) of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets is the graph $G = (V, E)$ defined by $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup R$, where $R = \{(v_1, v_2) : v_1 \in V_1, v_2 \in V_2\}$. From the f-spectrum of two graphs G_1 and G_2, the f-spectrum of the join graph of G_1 and G_2 can be computed.

Lemma 5 *Let G_1 and G_2 be two graphs. The f-spectrum of the join of G_1 and G_2 is $\{i + j : i \in S_f(G_1), j \in S_f(G_2)\}$*

Proof. The edges added between the vertex sets of G_1 and G_2 prohibit colors from appearing in both sets. Thus the proper colorings of the join consist of proper colorings of G_1 and G_2 using disjoint sets of colors. $\square$

Let us prove the following proposition.

Proposition 3 *The problem of deciding for a graph $G = (E, V)$ having a fall coloring whether G is f-continuous, is NP-complete.*

Proof. This problem is in NP: given a graph G, for each integer k between 2 and n (where n is the number of vertices of G), a non-deterministic polynomial time algorithm can determine whether there exists a fall k-coloring of G. We prove now that this problem is NP-hard. We transform *NAE-3-SAT* to this problem. The transformation takes as input an instance $I = \langle U = \{u_1, \ldots, u_k\}, C = \{C_1, \ldots, C_j\}\rangle$. First, G_1 is computed by polynomial time algorithm $\mathcal{A}$ described in Table 1 taken as input I and $t = 1$. Afterward, graph G is constructed from the join of the graph G_1 and the graph H_3 that is the hypercube graph of dimension 3. This transformation runs in polynomial time since Algorithm $\mathcal{A}$ is a polynomial time algorithm and since the join of G_1 and H_3 can be performed in polynomial time. To complete the proof, we show that this transformation is indeed a reduction: graph G is f-continuous if and only if U has a truth assignment with desired properties.

It is easy to compute the f-spectrum of H_3: the f-spectrum of H_3 is the set $\{2, 4\}$. Moreover, by Proposition 2, we know that the f-spectrum of G_1 is $\{2, 3\}$ if and only if U has a truth assignment with desired properties. From Lemma 5, we deduce that the f-spectrum of graph G is the set $\{4, 5, 6, 7\}$ if and only if U has a truth assignment with desired properties, otherwise it is the set $\{4, 6\}$. So G is f-continuous if and only if U has a truth assignment with desired properties. $\square$

Proposition 3 immediately leads that there is no constant $\epsilon > 0$ for which the optimization problem *MFKC* for f-continuous graphs can be approximated within a factor of $3/2 - \epsilon$. In fact, we can generalize this remark: unless $P = NP$, there exists no polynomial-time approximation algorithm with approximation ratio bounded by a constant for problem *MFKC* for f-continuous graphs. In other worlds, we have:

Theorem 2 *Unless $P = NP$, Problem* MFKC *for f-continuous graphs does not belong to APX.*

Proof. As previously, *NAE-3-SAT* is considered. The transformation $\mathcal{B}$ takes as input an instance $I = \langle U = \{u_1, \ldots, u_k\}, C = \{C_1, \ldots, C_j\}\rangle$ and an integer ℓ. This transformation constructs ℓ graphs $B_1, \ldots, B_\ell$ as follows:

- B_1 is computed by polynomial time algorithm $\mathcal{A}$ described in Table 1 taken as input I and $t = 1$.
- for any $i = 1, \ldots, \ell - 1$, B_{i+1} is constructed from B_i and graph D_i which is the output of algorithm $\mathcal{A}$ described in Table 1 taken as input I and $t_i = \frac{2^i+1}{3}$ if i is odd otherwise $t_i = \frac{2^i+2}{3}$. B_{i+1} is the join of B_i and D_i.

We denote the number of vertices of G by $|V(G)|$. We focus on theses three following properties of graph B_ℓ with $\ell \geq 0$:

Property 1: $|V(B_\ell)| \leq \ell(3k + p + 4) + 4(k + 1)\frac{2^\ell}{3}$

Property 2: If U has a truth assignment with desired properties, then the f-spectrum of B_ℓ is $\{i : 2\ell \leq i \leq 2\ell + \alpha_\ell\}$ where $\alpha_\ell = \begin{cases} \frac{2^{\ell+1}-1}{3} & \text{if } \ell \text{ odd} \\ 2(\frac{2^\ell-1}{3}) & \text{otherwise} \end{cases}$.

Property 3: If U has not a truth assignment with desired properties, then graph B_ℓ has only one fall 2ℓ-coloring.

First, we prove Property 1. By construction, we can notice that B_1 has $5k+p+6$ vertices (recall that in Section 2.1 the number of vertices of B_1 is computed). We count the vertices of graph $B_{\ell+1}$. Since $B_{\ell+1}$ is the join of B_ℓ and D_ℓ, the number of vertices of $B_{\ell+1}$ is equal to the sum of those of D_ℓ and B_ℓ: $|V(B_{\ell+1})| = |V(B_\ell)| + |V(D_\ell)|$. Note that by construction, D_ℓ has $4t_\ell(k+1) + k + p + 2$ vertices. By computation, for any $\ell \geq 1$, we have

$$\begin{aligned}
|V(B_\ell)| &= |V(B_1)| + \textstyle\sum_{j=1}^{\ell-1} |V(D_j)| \\
&= 5k + p + 6 + \textstyle\sum_{j=1}^{\ell-1}(4t_j(k + 1) + k + p + 2) \\
&= 4k + 4 + \ell(k + p + 2) + 4(k + 1)\textstyle\sum_{j=1}^{\ell-1} t_j \\
&\leq 4k + 4 + \ell(k + p + 2) + 4(k + 1)(\frac{2^\ell}{3} + \frac{\ell}{2}) \\
&\leq \ell(k + p + 2) + 4(k + 1)(\frac{2^\ell}{3} + \frac{\ell}{2}) \\
&\leq \ell(3k + p + 4) + 4(k + 1)\frac{2^\ell}{3}
\end{aligned}$$

So, we can state that Property 1 is true and it remains to prove Properties 2 and 3.

Second, we prove Properties 2 and 3 by induction on ℓ. First, we check if the two properties are satisfied for the base case where $\ell = 1$. By construction, B_1 is a bipartite graph and hence B_1 has a fall 2-coloring. By Proposition 2, we know that the f-spectrum of B_1 is $\{2, 3\}$ if and only if U has a truth assignment with desired properties. Since $\alpha_1 = 1$, if U has a truth assignment with desired properties, then the f-spectrum of B_ℓ is $\{2, 2 + \alpha_1\}$. So the two properties are satisfied for $\ell = 1$.

Now we have to show that if Properties 2 and 3 are satisfied for ℓ, then they also hold when $\ell + 1$. Assume the following properties are true for ℓ.

Assume that set U of variables has a truth assignment with desired properties.

By Proposition 1, the f-spectrum of graph D_ℓ is $\{2, 2t_\ell + 1\}$. From induction hypothesis, the f-spectrum of B_ℓ is $\{i : 2\ell \leq i \leq 2\ell + \alpha_\ell\}$ because U has a truth assignment with desired properties.

Recall that $B_{\ell+1}$ is the join of B_ℓ and D_ℓ. From Lemma 5, we deduce that the f-spectrum of graph $B_{\ell+1}$ is the set $\{i : 2\ell + 2 \leq i \leq 2\ell + \alpha_\ell + 2\} \cup \{i : 2\ell + 2t_\ell + 1 \leq i \leq 2\ell + \alpha_\ell + 2t_\ell + 1\}$. By computation, we have

- $2\ell + 2 + \alpha_{\ell+1} = 2\ell + \alpha_\ell + 2t_\ell + 1$ because

$$\begin{aligned}
\text{if } \ell \text{ is odd then,} \quad 2\ell + \alpha_\ell + 2t_\ell + 1 &= 2\ell + 2 - 1 + \tfrac{2^{\ell+1}-1}{3} + 2 \times \tfrac{2^\ell+1}{3} \\
&= 2\ell + 2 + \tfrac{2^{\ell+2}+1}{3} - 1 \\
&= 2\ell + 2 + 2(\tfrac{2^{\ell+1}-1}{3}) \\
&= 2\ell + 2 + \alpha_{\ell+1} \\
\text{if } \ell \text{ is even then,} \quad 2\ell + \alpha_\ell + 2t_\ell + 1 &= 2\ell + 2 - 1 + 2(\tfrac{2^\ell-1}{3}) + 2 \times \tfrac{2^\ell+2}{3} \\
&= 2\ell + 2 + \tfrac{2^{\ell+2}+1}{3} \\
&= 2\ell + 2 + \alpha_{\ell+1}
\end{aligned}$$

- $2\ell + 2t_\ell + 1 \leq 2\ell + \alpha_\ell + 2$ because

$$\begin{aligned}
\text{if } \ell \text{ is odd then,} \quad 2\ell + 2t_\ell + 1 &= 2\ell + 2 \times \tfrac{2^\ell+1}{3} + 1 \\
&= 2\ell + 2 + \tfrac{2^{\ell+1}-1}{3} \\
&= 2\ell + 2 + \alpha_\ell \\
\text{if } \ell \text{ is even then,} \quad 2\ell + 2t_\ell + 1 &= 2\ell + 1 + 2 \times \tfrac{2^\ell+2}{3} \\
&= 2\ell + 2 + \tfrac{2^{\ell+1}+1}{3} \\
&\leq 2\ell + 2 + \alpha_\ell
\end{aligned}$$

So the f-spectrum of $B_{\ell+1}$ is $\{i : 2(\ell+1) \leq i \leq 2(\ell+1)+\alpha_{\ell+1}\}$ and the second property holds.

Conversely, assume that set U has not a truth assignment with desired properties. By Proposition 1, and from induction hypothesis, the f-spectrum of D_ℓ and of B_ℓ are $\{2\}$ and $\{2\ell\}$ respectively. From Lemma 5, we deduce that the f-spectrum of graph $B_{\ell+1}$ is the set $\{2(\ell+1)\}$. So the third property holds.

From Properties 2 and 3, we can deduce that for any ℓ, B_ℓ is a f-continuous graph.

Let r be an arbitrary constant with $r > 3/2$. Let ℓ the smallest integer such that $r \leq \frac{2^\ell+1}{3\ell}$ and such that $\ell > 2$. By definition, we have $\frac{2^{\ell-1}+1}{3(\ell-1)} < r \leq \frac{2^\ell+1}{3\ell}$ and we can deduce that $\ell < (r+1)$ and $2^\ell < 6\ell r < 6r(r+1)$.

Assume that there exists a polynomial-time r-approximation algorithm $\mathcal{D}$ for this problem. Let $I = \langle U = \{u_1, \ldots, u_k\}, C = \{C_1, \ldots, C_j\}\rangle$ be an instance of *NAE-3-SAT*. Let G be a graph, the output of algorithm $\mathcal{B}$ taken as input I and ℓ. We can note that G has at most $\ell(3k+p+4)+4(k+1)\frac{2^\ell}{3}$ vertices (Property 1). So G has at most $O(r^2(k+p))$ vertices and algorithm $\mathcal{B}$ constructs graph G in polynomial time in size of instance I since r is a constant.

If the answer to problem *NAE-3-SAT* for instance I is postive, the r-approximation algorithm $\mathcal{D}$ returns a solution has a measure at least $\frac{2\ell+\alpha_\ell}{r}$. Note that:

$$\frac{2\ell+\alpha_\ell}{r} \geq \frac{2\ell+\frac{2^{\ell+1}-2}{3}}{r} \geq \frac{(6\ell+2^{\ell+1}-2)3\ell}{3(2^\ell+1)} \geq 2\ell + \frac{(6\ell-4)\ell}{(2^\ell+1)} > 2\ell$$

So, if the answer to problem *NAE-3-SAT* for instance I is postive, the r-approximation algorithm $\mathcal{D}$ returns a solution has a measure at least $2\ell+1$. Otherwise, it returns a solution has a measure at most 2ℓ. So due to Property (2), we could use it to decide whether the answer to problem *NAE-3-SAT* for instance I is postive or not in polynomial time. Thus, unless $P = NP$ there is no r- approximation polynomial algorithm for Problem *MFKC* for any contant $r \geq 1$. So, this completes the proof of this theorem. □

3 First results about f-spectrum

This section is devoted to study the f-spectrum of some particular graphs. First, we answer to an open question of Dunbar et al. [6].

Proposition 4 *For any integer ℓ, there is a non f-continuous graph G, such that $\chi_f(G) - \chi(G) > \ell$.*

Proof. We consider the graph G obtained from the complete bipartite graph $K_{\ell+4,\ell+4} = (U \cup V, E)$ by removing a perfect matching and by adding an edge (u_1, u_2), between two vertices $u_1, u_2 \in U$. Clearly $\chi(G) = 3$. Moreover, it is easy to see that G has a fall $(\ell+4)$-coloring. To show that $\chi_f(G) = \ell + 4$, it remains to prove that G does not have fall k-colorings for $3 \geq k < \ell + 4$. Assume that G has such a fall k-coloring with $k < |V|$. As $k < |V|$, at least two vertices of V are colored with the same color, say color c. Since the neighborhood of any two vertices of V is the set U, color c cannot be assigned to any vertex of U. This means that every vertex $v \in V$ must be colored with color c (if it were not the case, then v would not be colorful). This implies that no vertex $u \in U \setminus \{u_1, u_2\}$ is colorful, since all its neighbors are colored with the same color c. Therefore $\chi_f(G) = \ell + 4$ and $\chi_f(G) - \chi(G) > \ell$. □

Now, we will extend this result. The remainder of this section is devoted to prove that there exists a graph G such that $S_f(G) = S$ for any finite nonempty set $S \subset (\mathbb{N} \setminus \{0, 1\})$. Let us first define the *categorical product* of two graphs. For graphs G and H, the *categorical product* of G and H is the graph $G \times H$ with vertices $\{(u, v) | u \in G, v \in H\}$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times H$ if and only if u_1 is adjacent to u_2 in G and v_1 is adjacent to v_2 in H. Figure 2 shows the categorical product $K_2 \times K_4$. In [6] Dunbar et al. have shown that:

Fig. 2. The categorical product $K_2 \times K_4$

Theorem 3 *The f-spectrum of the graph $K_{n_1} \times K_{n_2}$ where $n_1 \geq 2$ and $n_2 \geq 2$ is the set $\{n_1, n_2\}$.*

Note, that this theorem does not generalize to categorical products of three or more complete graphs. For example graph $K_2 \times K_3 \times K_4$ have fall $2-, 3-$ and $4-$colorings but have also a fall $6-$coloring (see [6]).

Theorem 4 *For any finite nonempty set $S \subset (\mathbb{N} \setminus \{0, 1\})$, there exists a graph G such that $S_f(G) = S$.*

Let $S = \{n_0, n_1, n_2, \ldots, n_p\}$ such that $n_{i-1} < n_i$ for all $i \in [1, \ldots, p]$. We deal with particular cases. If $S = \{n\}$, then the complete graph K_n has the required property. If $S = \{n_1, n_2\}$, then by Theorem 3, S is the f-spectrum of the graph $K_{n_1} \times K_{n_2}$. Now, we will consider the general case where $p \geq 2$.

First, the case where $n_0 = 2$ is considered.

Let $S = \{n_0, n_1, n_2, \ldots, n_p\}$ such that $2 \leq p$ and $n_0 = 2 < n_1 < n_2 < \ldots < n_p$. We initialize G_S with graph $K_2 \times K_{n_1} \times \ldots \times K_{n_p}$. We add to it a new vertex v and connect v to the nodes of the subset B defined by : $B = \{(1, i_1, \ldots, i_1) : 1 \leq i_1 \leq n_1 - 1\} \cup_{\ell=2}^{p} \{(n_0 - 1, n_1 - 1, n_2 - 1, \ldots, n_{\ell-1} - 1, \beta, \ldots, \beta) : n_{\ell-1} \leq \beta \leq n_\ell - 1\}$. Figure 3 gives a partial construction of graph G_S for $S = \{2, 4, 6\}$, and vertex v is adjacent to nodes in $\{(1,1,1), (1,2,2), (1,3,3), (1,3,4), (1,3,5)\}$.

In fact, for every integer k in an arbitrarily specified set $S = \{n_0, n_1, \ldots, n_p\}$ of positive integers, the categorical product G of complete graphs K_{n_i} for every $n_i \in S$, has fall n_i-colorings, for every $n_i \in S$, but graph G has also other fall k-coloring for some $k \notin S$. And the rule of vertex v is to prevent from some undesirable fall colorings in G_S.

Fig. 3. Partial construction of graph G for $S = \{2, 4, 6\}$. The black nodes represent v and its neighborhood. We give only the edges between the vertex $(1, 4, 1)$ and its neighborhood

By construction, we have the following remark:

Remark 1 *Two vertices $x = (x_0, x_1, \ldots, x_p)$ and $y = (y_0, y_1, \ldots, y_p)$ from $V(G_S) \setminus v$ are adjacent if and only if $x_i \neq y_i$ for any i, $0 \leq i \leq p$.*

Let us prove that G_S satisfies the following Lemma:

Lemma 6 *For any i, $0 \leq i \leq p$, G_S has a fall n_i-coloring.*

Proof. G_S is a bipartite graph because all edges of G_S have exactly one extremity in vertex set $\{(1, x_1, x_2, \ldots, x_p) | 1 \leq x_i \leq n_i, 1 \leq i \leq p\}$. So G_S has a fall 2-coloring.

Given an integer i such that $1 \leq i \leq p$, a fall n_i-coloring π of G_S is constructed as follows. The color of vertex v is n_i. For $x_i = 1, \ldots, n_i$, vertices in $\{(x_0, x_1, \ldots, x_i, \ldots, x_p) : 1 \leq x_j \leq n_j, 0 \leq j \leq p \wedge j \neq i\}$ have the color x_i.

We check first that coloring π is proper. If two vertices $x = (x_0, x_1, \dots, x_p)$ and $y = (y_0, y_1, \dots, y_p)$ in set $V(G_S) \setminus \{v\}$ are of the same color then $x_i = y_i$, therefore x and y are not adjacent (see Remark 1). Moreover, by construction, vertex v is not adjacent to $x = (x_0, x_1, \dots, x_p)$ if $x_i = n_i$. So π is proper. Furthermore, the vertices of $V(G_S) \setminus \{v\}$ are colorful because each vertex $x = (x_0, x_1, \dots, x_\ell, \dots, x_p)$ of color x_ℓ is adjacent to vertex $x' = (x'_0, x'_1, \dots, x'_j, \dots, x'_p)$, with $x'_i \neq x_i$, of color x_j with $j \neq \ell$. Since v is adjacent to the vertices $(1, j_1, j_1, \dots, j_1), (1, n_1 - 1, j_2, \dots, j_2) \dots (1, n_1 - 1, \dots n_{i-1} - 1, j_i, \dots, j_i)$ where $1 \leq j_1 \leq n_1 - 1$ and $n_{h-1} - 1 \leq j_h \leq n_h - 1$ with $2 \leq h \leq i$, vertex v is colorful. So coloring π is a fall n_i-coloring of G. This concludes the proof of Lemma 6. $\square$

It remains to prove that if G_S has a fall k-coloring, then $k \in S$. For given $x_0, x_1, \dots x_i$, where $1 \leq x_j \leq n_j$ and $0 \leq j \leq i \leq p - 1$, let $A_{i+1}(x_0, x_1, \dots, x_i)$ be the subset of $V(G_S) \setminus \{v\}$ defined by :

$$\{(x_0, x_1, \dots, x_i, x_{i+1}, \dots, x_p) : 1 \leq x_r \leq n_r, with\ i + 1 \leq r \leq p\}.$$

Recall that two vertices $u = (u_0, u_1, , \dots, u_p)$ and $w = (w_0, w_1, , \dots, w_p)$ are ajacent in G_S if and only if $\forall i, 1 \leq i \leq p, u_i \neq w_i$. By Remark 1, set $A_{i+1}(x_0, x_1, \dots, x_i)$ is an independent set.

Let u_1 and u_2 be two vertices of set $A_p(x_0, x_1, \dots, x_{p-1})$. We have $\Gamma(u_1) \cup \Gamma(u_2) = \Gamma(A_p(x_0, x_1, \dots, x_{p-1}))$, where $\Gamma(v)$ and $\Gamma(A_p(x_0, x_1, \dots, x_{p-1}))$ denote respectively the neighborhood of vertex v and set $A_p(x_0, x_1, \dots, x_{p-1})$.

Lemma 7 *For any fall k-coloring, for any r, $1 \leq r \leq p-1$ such that $k < n_{r+1}$, of graph G_S, all the vertices of $A_{r+1}(x_0, x_1, \dots, x_r)$ have the same color.*

Proof. Let us begin with $r = p - 1$. Suppose that G has a fall k-coloring with $k < n_p$. We consider two cases: $x_0 = 2$ and $x_0 = 1$.

Case 1: $x_0 = 2$, this implies that v and $A_p(2, x_1, \dots, x_{p-1})$ are in the same class of the bipartition. As $|A_p(2, x_1, \dots, x_{p-1})| = n_p > k$, at least two vertices of $A_p(2, x_1, \dots, x_{p-1})$ have the same color denoted by c. Furthermore, the neighborhood of set $A_p(x_0, x_1, \dots, x_{p-1})$ and the union of the neighbors of any two elements of $A_p(x_0, x_1, \dots, x_{p-1})$ are equal. So color c will not appear on the neighborhood of set $A_p(x_0, x_1, \dots, x_{p-1})$. Since all the vertices of G must be colorful, all the vertices of $A_p(x_0, x_1, \dots, x_{p-1})$ have to be colored with c.

Case 2: $x_0 = 1$. Assume that v has a color c. Vertex $x = (1, x_1, \dots, x_{p-1}, n_p)$ of $A_p(1, x_1, \dots, x_{p-1})$ is not adjacent to v. Since vertex x must be colorful, then for every color $c' \neq c$, x is adjacent to a vertex $y = (2, y_1, \dots, y_p)$ colored with c'. By case 1, if $y = (2, y_1, \dots, y_p)$ is colored with c' then all the vertices of set $A_p(2, y_1, \dots, y_{p-1})$ are colored with c'. This implies that every vertex of set $A_p(1, x_1, \dots, x_{p-1})$ have color c' on its neighborhood and this fact holds for every color $c' \neq c$. This means that all the vertices of $A_p(1, x_1, \ldots, x_{p-1})$ are of color c.

Suppose now that this lemma holds for r such that $i+1 \leq r \leq p-1$, and let us prove it for $r = i$. We assume that G has a fall k-coloring with $k < n_{i+1}$. We have

$$A_{i+1}(x_0, x_1, \ldots, x_i) = \bigcup_{1 \leq x_{i+1} \leq n_{i+1}} A_{i+2}(x_0, x_1, \ldots, x_{i+1}).$$

Since $k < n_{i+1} < n_{i+2}$, by hypothesis, for any x_{i+1}, $1 \leq x_{i+1} \leq n_{i+1}$ all the vertices in set $A_{i+2}(x_0, x_1, \ldots, x_{i+1})$ have the same color. We consider two cases: $x_0 = 2$ and $x_0 = 1$.

Case a: $x_0 = 2$. Since $k < n_{i+1}$, all the vertices of at least two subsets $A_{i+2}(2, x_1, \ldots, x^1_{i+1})$ and $A_{i+2}(2, x_1, \ldots, x^2_{i+1})$ of set $A_{i+1}(2, x_1, \ldots, x_i)$ must have the same color, say color c. Furthermore, we consider a neighbour y of a vertex z in $A_{i+1}(2, x_1, \ldots, x_i)$. We will prove that $y = (1, y_1, \ldots, y_p)$ is also a neighbour of at most one vertex in $A_{i+2}(2, x_1, \ldots, x^1_{i+1}) \cup A_{i+2}(2, x_1, \ldots, x^2_{i+1})$. Since y is a neighbour of z, by Remark 1, we have $x_\ell \neq y_\ell$ for any ℓ, $0 \leq \ell \leq i$. Assume that $x^1_{i+1} = y_{i+1}$. So, it implies that $x^2_{i+1} \neq y_{i+1}$. We consider the vertex $w = (1, w_1, \ldots, w_p)$ in $A_{i+2}(2, x_1, \ldots, x^2_{i+1})$ such that

- $w_\ell = x_\ell$ for any ℓ, $0 \leq \ell \leq i$.
- $w_{i+1} = x^2_{i+1}$
- $w_\ell = (y_\ell + 1 \mod n_\ell) + 1$ for any ℓ, $i+2 \leq \ell \leq p$.

By construction, we have $w_\ell \neq y_\ell$ for any ℓ, $0 \leq \ell \leq p$. So y is a neighbour of w and y is included in the neighborhood of set $A_{i+1}(2, x_1, \ldots, x_i)$. Note that y is not of color c. We can apply the argument for the case where $x^1_{i+1} \neq y_{i+1}$. So, the neighborhood of set $A_{i+1}(2, x_1, \ldots, x_i)$ is included in the neighborhood of subset $A_{i+2}(2, x_1, \ldots, x^1_{i+1}) \cup A_{i+2}(2, x_1, \ldots, x^2_{i+1})$. This implies that color c will not appear in the neighborhood of $A_{i+1}(2, x_1, \ldots, x_i)$. So, in order to be colorful, vertices of $A_{i+1}(2, x_1, \ldots, x_i)$ must have color c.

Case b: $x_0 = 1$. Vertex $x = (1, x_1, \ldots, x_{p-1}, n_p)$ in set $A_{i+1}(1, x_1, \ldots, x_i)$ is not adjacent to v. Vertex x is colorful, say for color c. So for any color $c' \neq c$, x is adjacent to at least one vertex $y = (2, y_1, \ldots y_i, \ldots, y_p)$ colored with c'. By Case a, all the vertices of set $A_{i+1}(2, y_1, \ldots, y_i)$ have the same color. But every vertex in $A_{i+1}(x_0, x_1, \ldots, x_i)$ has at least one neighbor in $A_{i+1}(2, y_1, \ldots, y_i)$. Hence color c' cannot be assigned to any vertex of set $A_{i+1}(1, x_1, \ldots, x_i)$, and this for every $c' \neq c$. This implies that all the vertices of $A_{i+1}(x_0, x_1, \ldots, x_i)$ are of color c.

So the lemma holds for $r = i$. This concludes the proof of Lemma 7. $\square$

Now, we can determine the f-spectrum of graph G_S:

Lemma 8 $S_f(G_S) = S$

Proof. By Lemma 6, we know that $S \subseteq S_f(G_S)$ Now, we will prove that, if G_S has a fall k-coloring with $n_i \leq k < n_{i+1}$ and $0 \leq i \leq p-1$, then $k = n_i$. We assume that such fall k-coloring exists. Vertex v has some neighbors in exactly $n_i - 1$ distinct sets $A_{i+1}(x_0, x_1, \ldots, x_i)$. Indeed, vertex v has some neighbors in the sets $A_{i+1}(1, j_1, \ldots, j_1), \ldots, A_{i+1}(1, n_1 - 1, \ldots, n_{h-1} - 1, j_h, \ldots j_h) \ldots, A_{i+1}(1, n_1 - 1, \ldots, n_{i-1} - 1, j_i)$, with $n_{h-1} - 1 \leq j_h \leq n_h - 1$ and $1 \leq h \leq i$. On the other hand, by Lemma 7, for every $x_1, x_2, \ldots, x_i$ with $x_0 = 1, 2$, $1 \leq x_j \leq n_j$ and $1 \leq j \leq i$, all the vertices of $A_{i+1}(x_0, x_1, \ldots, x_i)$ have the same color. This implies that vertex v has at most $n_i - 1$ colors on its neighborhood. It follows from this argument that v is colorful if and only if $k = n_i$. Finally is it clear that G_S does not have any fall k-coloring with $k > n_p$, because the degree of vertex v is $n_p - 1$. $\square$

So, we have obtained a construction of graph G_S such that $S_f(G_S) = S$ and $\min S = 2$. Now, we will consider the case where S with $\min S > 2$. Let $S = \{n_0, n_1, n_2, \ldots, n_p\}$ such that $n_{i-1} < n_i$ for all $i \in [1, \ldots, p]$. We construct graph G such that its f-spectrum is $\{2, n_1 - n_0 + 2, n_2 - n_0 + 2, \ldots, n_p - n_0 + 2\}$. By Lemma 5, the f-spectrum of the join of G and $K_{n_0 - 2}$ is S. Hence, for each set of integers S included in $\mathbb{N} \setminus \{0, 1\}$, there exists a graph G having the set S as a f-spectrum.

4 Conclusion

The main result of this paper given in Theorem 1 shows that problem MFKC can not be approximated within a factor of $n^{1-\epsilon}$ for any $\epsilon > 0$. As far as we know, this is the first result giving a relation between interpolation properties of a coloring (i.e., there exist not f-continuous graphs) and the non-approximability of its maximum colors cardinality. This result is directly deduced from the fact that there is a "hole" between this (possible) maximal number and before the last one. We also answer some open question from Hedetniemi et al concerning f-continuity. As we say in the introduction, an open question is to know if a similar result about non-approximability can be obtained for the b-coloring.

Acknowlegment

We would like to thank one anonymous referee for many helpful comments, and for suggestions about new statements to improve some results.

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