The Eulerian stretch of a digraph and the ending guarantee of a convergence routing*

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Abstract

In this paper, we focus on convergence packet routing techniques in a network, obtained from an Eulerian routing in the digraph modeling the target network. Given an Eulerian circuit $\mathcal{C}$ in a digraph G, we consider the maximal number $diamW_{\mathcal{C}}$ of arcs that a packet has to follow on $\mathcal{C}$ from its origin to its destination (we talk about the *ending guarantee* of the routing). We consider the *Eulerian diameter of* G as defined by $\mathcal{E}(G) = \min_{\mathcal{C} \in Eul(G)} diamW_{\mathcal{C}}$, where $Eul(G)$ is the set of all the Eulerian circuits in G. After giving a preliminary result about the complexity of finding $\mathcal{E}(G)$ for any digraph G, we give some lower and upper bounds of this parameter. We conclude by giving some families of digraphs having good Eulerian diameter.

Keywords : *network routing, ending routing guarantee, digraphs, Eulerian circuits, Eulerian diameter.*

1 Introduction

In this paper, we focus on some digraph parameters to evaluate the quality of a network, whoose the digraph is the topology, in terms of performances of specific packet routing algorithms. We consider packet routing strategies without intermediate storage of data packets (hereafter simply called packets) [1, 16], such as deflection routing [5, 6, 18] [1]. These techniques are known to clearly avoid deadlocks (packets in the network do not move) but livelocks could occur (packets move but never reach their destination), except for some cases of deflection routing in some classes of networks such as trees or triangulated graphs [8].

Thus, we want the techniques of routing to give performance guarantees about the life-time of a packet in the target network, similar to the ones defined in [7]. These techniques of routing

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[1] We especially focus on it in the RNRT project *ROM* dealing with all-optical telecommunication networks

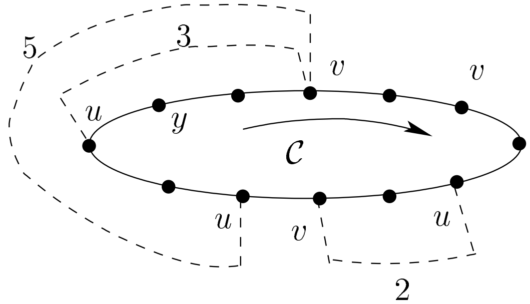


Figure 1: An example of Eulerian circuit $\mathcal{C}$ in a digraph G with $d_{\mathcal{C}}(u, v) = 5$.

without intermediate storage we consider, are such that any emitted packet reaches its destination within a finite maximal number of steps (i.e., no livelock). If such a number exists, we talk about the *ending guarantee* of the routing technique. For example, to obtain this ending guarantee with a deflection routing method, one solution is to use some priorities on the packets, depending on the time they have spent in the network [5, 7, 12, 19]. These techniques give various good ending guarantees in meshes under a *batch-routing model* (i.e., where each node is initially the origin of at most a fixed number k of packets and there is no new packet appearing in the node during time).

Another way to obtain an ending guarantee is to use a *convergence routing* technique [15, 16, 21]. Using such a routing strategy, packets are routed along a global sense of direction, which gives a ending guarantee. As proposed in [7, 15, 21], such a global sense of direction can be created by using decompositions of the target digraph (or of a covering sub-digraph of it) into circuits [2, 23]. In [7], Feige gives a technique, based on an Eulerian circuit in a sub-digraph, ensuring an ending guarantee equals to $O(n^{3/2})$ for any graph with a minimal number of edges. Here, as a particular case, we focus on the use of an *Eulerian routing*, i.e., a routing in which packets follow an Eulerian circuit in the digraph modeling the network.

Eulerian circuits are well known combinatorial structures in digraphs [4, 9]. The main strategies used to obtain Eulerian circuits in a digraph are given in [4, 11]. The most part of studies about these structures concerns the way to find pairwise compatible Eulerian circuits in a digraph [3, 13]. Here, we consider original combinatorial properties of an Eulerian circuit related to the quality of the ending guarantee of the related routing. Consider an Eulerian circuit $\mathcal{C}$ in a digraph G, where G represents the network. Each emitted packet follows $\mathcal{C}$ and, at each step, has priority on the next arc on this circuit. Then, a packet emitted by a node u and having destination node v will hopefully reach v. Let $d_{\mathcal{C}}(u, v)$ be the maximal number of arcs on $\mathcal{C}$ between one occurrence of the source vertex u and the first occurrence of vertex v encountered by following $\mathcal{C}$ from this occurrence of u. This parameter $d_{\mathcal{C}}(u, v)$ is the major parameter of this routing strategy. It represents the longest delay for packet delivery from vertex u to vertex v. In Figure 1, this maximal distance is 5, even if there exists a path of length 2 in $\mathcal{C}$ between u and v.

Using $\mathcal{C}$, any packet emitted in G reaches its destination in at most $diamW_{\mathcal{C}}$ steps, where

$$diamW_{\mathcal{C}} = \max_{u,v \in V(G)} d_{\mathcal{C}}(u, v).$$

This Eulerian routing technique is clearly interesting to obtain a good ending guarantee, but it gives poor average performance in comparison with classical deflection routing. Some authors have proposed to use it as a secure routing technique coupled to a deflection routing [2] and/or to use shortcuts [2, 14, 22], i.e., a packet can jump from an occurrence of a vertex to another one on the Eulerian circuit trying to go to a portion of the circuit where the relative distance to the destination is smaller than the distance it remains to do on the current part of the circuit. Note also that this routing technique can be implemented as a simple distributed algorithm in the network [2, 22].

In this paper, we only focus on the ending guarantee of an Eulerian routing and particularly on the *Eulerian diameter* of a digraph G defined as follows. Let $Eul(G)$ be the set of all the Eulerian circuits of G. The Eulerian diameter of G is defined by

$$\mathcal{E}(G) = \min_{\mathcal{C} \in Eul(G)} diamW_{\mathcal{C}}.$$

In fact, this parameter $diamW_{\mathcal{C}}$ is the best ending guarantee that can be obtained in G by using an Eulerian routing technique.

Our results : in the next section, we give some definitions and results about the NP-completeness of the problem of determining the Eulerian diameter of a digraph; we also provide some lower and upper bounds for the Eulerian diameter of a digraph. In Section 4, we show some families of digraphs having good Eulerian diameter. We conclude by giving some open problems and conjectures.

2 Definitions and preliminary results

2.1 Definitions

In this paper, we use the general digraph theory definitions of [4]. In particular, we say that that a digraph G is a *multi-digraph* if there are at least two occurrences of a same arc in $A(G)$; otherwise, we talk about simple digraph. Unless specified, we always consider simple digraphs in this paper. We deal here with Eulerian digraphs. Thus, for any vertex v of such a symmetric digraph G, the incoming degree $\delta^-(v)$ of v is equal to its outgoing degree $\delta^+(v)$. We denote by $\delta(v) = \delta^+(v) = \delta^-(v)$ the degree of vertex v, and by δ the minimum degree of G.

Let G be a digraph and $\mathcal{C}$ an Eulerian circuit in G. Consider a vertex v and an arc $\alpha = (u, y)$ of G. We denote by $tp_{\mathcal{C}}(\alpha, v)$ the length of the path on $\mathcal{C}$ beginning in u and ending in the first occurrence of v on $\mathcal{C}$, using α as first arc. In Figure 1, $tp_{\mathcal{C}}((u, y), v) = 3$.

We also define the parameter $\tau(\mathcal{C}, u)$ by:

$$\tau(\mathcal{C}, u) = \max_{\alpha=(u,y)\in A(G)} tp_{\mathcal{C}}(\alpha, u).$$

Thus, $\tau(\mathcal{C}, u)$ is the maximal distance on $\mathcal{C}$ between two consecutive occurrences of u on $\mathcal{C}$. In the example of Figure 1, $\tau(\mathcal{C}, u) = 7$.

We finally define the *stretch* $S_{\mathcal{C}}$ of $\mathcal{C}$ as $S_{\mathcal{C}} = \max_{u\in V(G)} \tau(\mathcal{C}, u)$.

2.2 Eulerian routing and NP-Completeness

Given a digraph G, the problem we focus on is to determine $\mathcal{E}(G)$. We first show that the parameters $diamW_{\mathcal{C}}$ and $S_{\mathcal{C}}$ are simply connected, which is interesting since $S_{\mathcal{C}}$ is simpler to use (and compute) than $diamW_{\mathcal{C}}$.

Lemma 1 *For each Eulerian circuit $\mathcal{C}$ in a digraph G, we have:*

$$diamW_{\mathcal{C}} = S_{\mathcal{C}} - 1. \qquad (1)$$

Proof: Let u be a vertex of G, and v another vertex such that $tp_{\mathcal{C}}((u,v),u)$ is maximal and equal to $S_{\mathcal{C}}$. Let w be the vertex such that the arc (v,w) follows (u,v) in $\mathcal{C}$ as shown in Figure 2 below. Thus, by definition, we have $tp_{\mathcal{C}}((v,w),u) = S_{\mathcal{C}} - 1$. Since, $diamW_{\mathcal{C}} = \max_{u,v \in V(G)} d_{\mathcal{C}}(u,v)$ and since $diamW_{\mathcal{C}} \geq d_{\mathcal{C}}(v,u) \geq tp_{\mathcal{C}}((v,w),u)$, we obtain :

$$S_{\mathcal{C}} - 1 \leq diamW_{\mathcal{C}}.$$

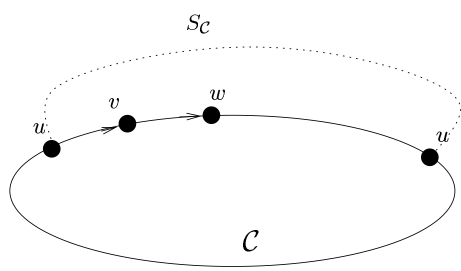


Figure 2: Lower bound of $diamW_{\mathcal{C}}$

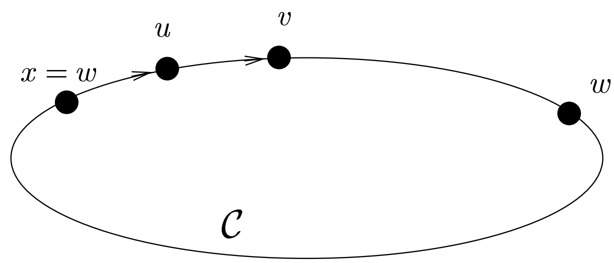


Figure 3: Upper bound of $diamW_{\mathcal{C}}$

In order to show the other part, we consider a triplet of vertices (u,v,w) that maximizes the definition of the worst Eulerian diameter, i.e., for which $tp_{\mathcal{C}}((u,v),w)$ is maximal as shown in Figure 3. Let x be the vertex just before the arc (u,v) in $\mathcal{C}$. Then $x = w$; otherwise $tp_{\mathcal{C}}((x,u),w) = tp_{\mathcal{C}}((u,v),w) + 1$, a contradiction with the hypothesis on (u,v,w). Consequently, we have

$$S_{\mathcal{C}} - 1 \geq diamW_{\mathcal{C}}.$$

This concludes the proof of this lemma. □

Note that Lemma 1 gives a simple way to linearly compute the value of the Eulerian diameter.

Given a digraph G and an integer k, the problem we now deal with is to know if there exists an Eulerian circuit $\mathcal{C}$ of G such that $S_{\mathcal{C}} \leq k$. From different results of Fleischner about graphs [9, Chapter IX], for any symmetric digraph G with n vertices and m arcs, $|Eul(G)| > 2^{m/2-n} \prod_{v \in V(G)} \left(\frac{\Delta^+(v)}{2} - 1\right)!$. Thus, it is not possible to solve problem VMS by computing the Eulerian diameter of all Eulerian circuits in G. Knowing whether this problem is NP-complete is still an open question. However, in this paper, we give an answer for the following problem for which we try to minimize the stretch only for a single vertex in the graph.

Problem Vertex_Min_Stretch (VMS)
Given : `a digraph` G, `a vertex` u `and an integer` k.
Question : `Does there exist an Eulerian circuit` $\mathcal{C}$ `of` G `such that` $\tau(\mathcal{C},u) \leq k$?

Theorem 1 *The problem* Vertex_Min_stretch *is NP-complete.*

Proof:
The problem VMS belongs to NP because we can verify in polynomial time that a given Eulerian circuit $\mathcal{C}$ satisfies the following property: $\tau(\mathcal{C},u) \leq k$. We will transform problem 3-Partition to a

restricted version of problem VMS. The problem 3-Partition (that is NP-complete in strong sense) is defined as follows:

Problem 3-Partition [10]

Given: a finite set S of $3m$ elements, a bound $B \in Z^+$, a weight $w(a) \in Z^+$, such that each $w(a)$ satisfies $B/4 < w(a) < B/2$ and such that $\sum_{a \in S} w(a) = mB$.

Question: Can S be partitioned into m disjoint sets S_1, $S_2, \ldots,$ S_m such that, for $1 \leq i \leq m$, $\sum_{a \in S_i} w(a) = B$?

Let a finite set S, a bound B, and a function w denote an arbitrary instance I of problem 3-Partition. We transform an instance of problem VMS which is composed by a graph $G = (V, A)$ with a distinct vertex u, and by an integer k from instance I. Let $\mathcal{A}$ be a transformation from instance I of problem 3-Partition to an instance of problem VMS which is defined by Figure 4 and illustrated in Figure 5.

1 $s := |S|$ and $\beta := max(s, mB) + 4$
2 Initialize graph G as follow : $V = \emptyset$ et $A = \emptyset$
3 Insert two distinct vertices u and v in V.
4 For each element a of S do
5   Insert an oriented symmetric path p_a of $\beta^2 * w(a)$ vertices in G.
6   Denote v_a and v'_a the extremities of path p_a.
7   Connect v to v_a with the arcs (v, v_a) and (v_a, v).
8 For $i = 1$ to m do
9   Insert new vertex x_i in G.
10   Insert arcs (x_i, u), (x_i, v), (u, x_i) and (v, x_i) in G.
11 $k := 2B\beta^2 + \beta^2 - 1$.
12 Return G, u and k.

Figure 4: Construction $\mathcal{A}$ from an instance I : a finite set S, a bound B, a function w.

From this construction, graph G can be split into 2 subgraphs:

- The first subgraph, denoted $\mathcal{P}$, is composed by the union of directed symmetric paths (see Instruction 5: it contains all paths p_a corresponding to all elements a of S).
- The second subgraph denoted $\mathcal{R}$ is graph G minus graph $\mathcal{P}$. This means that, it contains vertices u, v, x_i, $1 \leq i \leq m$, and v_a, for each element a of S.

To prove that construction $\mathcal{A}$ is polynomial, it is enough to count vertices of graph G.

- Graph $\mathcal{P}$ contains $\beta^2 * \sum_{a \in S} w(a) (= \beta^2 mB)$ vertices and $2\beta^2 * \sum_{a \in S} w(a)$ arcs.
- Graph $\mathcal{R}$ contains $2 + m + s$ vertices and $2(m + s)$ arcs.

Summing the vertices of the two subgraphs of G, we obtain that, graph G has $Bm\beta^2 + 2 + m$ vertices (vertices v_a, $a \in S$ belong to both subgraphs). The number of vertices of G is less than $B * max(3m, mB) + 2max(3m, mB)$. Also, we can conclude that the instance of problem VMS is constructed in polynomial time from an instance of problem 3-Partition by transformation $\mathcal{A}$. Finally, in order to prove that problem VMS is NP-complete we will show the following property:

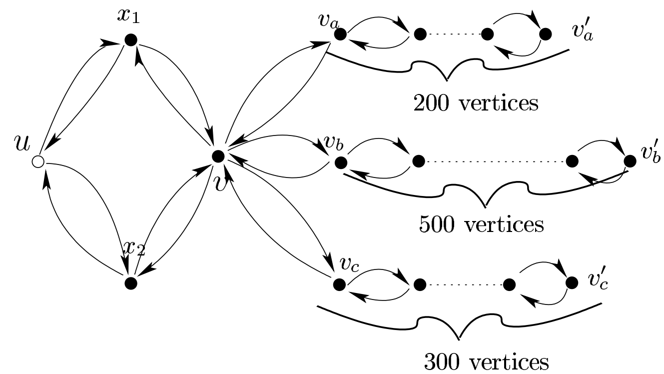


Figure 5: Example of construction $\mathcal{A}$: an instance of problem VMS is build from the following instance of problem 3-Partition: $m = 2$, $S = \{a, b, c\}$ and $w(a) = 2$, $w(b) = 5$ and $w(c) = 3$; we have here $\beta = 10$ and $k = 1099$.

Property 1 *There exists a partition of S into m subsets S_1, $S_2, \dots,$ S_m such that S_i, $1 \le i \le m$ satisfies $\sum_{a \in S_i} w(a) = B$, if and only if there exists an Eulerian circuit $\mathcal{C}$ of G such that $\tau(\mathcal{C}, u) \le k$.*

First, assume that S can be split into m sets S_1, $S_2, \dots, S_m$ such that $\sum_{a \in S_i} w(a) = B$, $1 \le i \le m$. From this partition, we will build an Eulerian circuit $\mathcal{C}$ such that $\tau(\mathcal{C}, u) \le k$. Eulerian circuit $\mathcal{C}$ is defined as the union of m circuits $\mathcal{C}_i$, $1 \le i \le m$.

Any subset S_i, $1 \le i \le m$, is represented by circuit $\mathcal{C}_i$. We assume that $S_i = \{a_1, \dots, a_k\}$. The circuit $\mathcal{C}_i$ is defined as follows : $\{u, x_i, v\} \cup_{i=1}^{k} \{v, v_a, \overrightarrow{p_{a_i}}, v'_a, \overleftarrow{p_{a_i}}, v_a\} \cup \{v, x_i, u\}$ where:

- $\overrightarrow{p_a}$ represents the direct path to v_a from v'_a.
- $\overleftarrow{p_a}$ represents the direct path to v'_a from v_a.

We can notice that the length of the circuit $\mathcal{C}_i$ is equal to $2\beta^2 \sum_{a \in S_i} w(a) + 2|S_i| + 4$. Its length is equal to $2B\beta^2 + 2|S_i| + 4$.

As we have $\beta > s + 1 \ge |S_i| + 1$, and $\beta > 4$, we obtain $\beta^2 - 1 \ge 2|S_i| + 4$. We can deduce that the length of the circuit $\mathcal{C}_i$ is less or equal to k. We conclude that Eulerian circuit $\mathcal{C}$ already defined is such that $\tau(\mathcal{C}, u) \le k$.

Conversely, we assume that there exists an Eulerian circuit $\mathcal{C}$ of G such that $\tau(\mathcal{C}, u) \le k$. We will prove that there exists a partition of S into m subsets S_1, $S_2, \dots,$ S_m such that S_i, $1 \le i \le m$, satisfies $\sum_{a \in S_i} w(a) = B$. We use the circuit C to build a partition of S.

By definition of G, vertex u appears m times in circuit $\mathcal{C}$. So, circuit $\mathcal{C}$ is split into m circuits denoted $\mathcal{C}_i$, $1 \le i \le m$, such that u appears once in each of these circuits. Before building this partition, we can notice that

Remark 1: If vertex v_a belongs to $\mathcal{C}_i$, then every vertex of p_a belongs to $\mathcal{C}_i$.

This remark is due to the fact that path p_a is only connected to the rest of graph G by only two arcs (v_a, v) and $(v,\ v_a)$. Now, we build the partition of finite set S. S is split into m subsets S_i, $i = 1, \dots, m$ as follows: an element a of S belongs to S_i if and only if vertex v_a belongs to circuit $\mathcal{C}_i$. It is easy to notice that all the sets S_i, $1 \le i \le m$, form a partition of S. Afterwards, we will compute $\sum_{a \in S_i} w(a)$ for each subset S_i, $1 \le i \le m$. To do this, we focus on the length of each $\mathcal{C}_i$. Each $\mathcal{C}_i$ is composed of path p_a , $a \in S_i$ plus γ arcs belonging to subgraph $\mathcal{R}$. So the length of this circuit is equal to $2\beta^2 \sum_{a \in S_i} w(a) + \gamma$ and is less than k. By definition of the parameter k, we can deduce the following equations:

$$k \geq 2\beta^2 \sum_{a \in S_i} w(a) + \gamma \tag{2}$$

$$2B\beta^2 + \beta^2 - 1 \geq 2\beta^2 \sum_{a \in S_i} w(a) + \gamma \tag{3}$$

As γ can be less than the number of arc of $\mathcal{R}$, we have $2(s+m) \geq \gamma$. Moreover, as $\beta > 4$, we obtain $\beta^2 - 1 \geq 4 * \beta \geq \gamma$. From Equation 3, we have

$$B\beta^2 \geq \beta^2 \sum_{a \in S_i} w(a).$$

$$B \geq \sum_{a \in S_i} w(a).$$

So we can deduce that for each $i = 1, \ldots, m$, we have $B \geq \sum_{a \in S_i} w(a)$. So, finite set S can be split in m subsets S_1, S_2,..., S_m such that for each $i = 1, \ldots, m$, we have $\sum_{a \in S_i} w(a) = B$. We have proven Property 1 and Theorem 1 holds.

□

3 Some bounds on the Eulerian diameter of a digraph

In this section, we prove the following theorem that gives simple bounds on the stretch of any graph.

Theorem 2 *Let G be a digraph with minimal degree δ, n vertices and m arcs. Then*

$$\frac{m}{\delta} - 1 \leq \mathcal{E}(G) \leq m - 2\delta - 3. \tag{4}$$

If G is a δ-regular digraph, with $\delta > 3$ and n vertices, then

$$n + 1 \leq \mathcal{E}(G). \tag{5}$$

Note that δ can not be equal to 1 because we only consider the symmetric digraph (for any vertex v, we have $\delta^+(v) = \delta^-(v)$). It is also easy to show that, if $\delta = 2$, then $\mathcal{E}(G) \geq n$ and that this bound is tight. The upper bound given in this proposition is based on a trivial impossibility. In fact, we conjecture that there is no digraph with degree $\delta \geq 2$ such that $\mathcal{E}(G) = m - 2\delta - 3$.

To prove Equation 5, we use the technical result given in Lemma 2, based on the following definition.

Definition 1 *Let G and $\mathcal{C}$ be respectively a digraph and an Eulerian circuit of G. The* min-stretch *of $\mathcal{C}$, denoted by $\alpha_{\mathcal{C}}(G)$, is the smallest distance between two occurrences of a same vertex of G in $\mathcal{C}$.*

$$\alpha_{\mathcal{C}}(G) = \min_{(u,v) \in A(G)} tp_{\mathcal{C}}((u,v), u).$$

Let G be a digraph and $\mathcal{C}$ an Eulerian circuit of G. Consider $V(G) = \{0, 1, \ldots, n-1\}$. By Definition 1, $\mathcal{C}$ contains the following pattern (up to a permutation of the vertices' labels):

$$\ldots \quad 0 \quad 1 \quad 2 \quad \ldots \quad (\alpha_{\mathcal{C}}(G)-1) \quad 0 \quad \ldots$$

Lemma 2 *Let G and $\mathcal{C}$ be respectively a digraph of order n and an Eulerian circuit of G with min-stretch smaller than $n-1$ ($\alpha_{\mathcal{C}}(G) < n-1$). Then we have:*

$$S_{\mathcal{C}} \geq n+2.$$

Proof: Wlog, we can assume that the property of min-stretch is obtained for vertex 0. Then, $\mathcal{C}$ contains the following pattern:

$$a_1 \quad a_2 \quad \ldots \quad a_k \quad 0 \quad 1 \quad 2 \quad \ldots \quad (\alpha_{\mathcal{C}}(G)-1) \quad 0 \quad b_1 \quad b_2 \quad b_3 \quad \ldots$$

where k is the smallest integer such that the sequence $\mathcal{S} = a_1 \ldots a_k$, contains at least one occurrence of each vertex in $\{\alpha_{\mathcal{C}}(G), \ldots, n-1\}$ (note that, since G is a simple digraph, $k \geq 2$). Then, $k \geq n - \alpha_{\mathcal{C}}(G)$. Wlog, let $a_1 = n-1$. So, $tp_{\mathcal{C}}((n-1, a_2), n-1)$ is minimal if $b_1 = n-1$ and then, the stretch of $(n-1)$ is such that $\tau(\mathcal{C}, n-1) \geq k + \alpha_{\mathcal{C}}(G) + 1$. We consider now two cases.

- If $k > n - \alpha_{\mathcal{C}}(G)$ and $b_1 = n-1$, then $\tau(\mathcal{C}, n-1) > n - \alpha_{\mathcal{C}}(G) + \alpha_{\mathcal{C}}(G) + 1$, i.e., $\tau(\mathcal{C}, n-1) \geq n+2$.
- If $k = n - \alpha_{\mathcal{C}}(G)$ and $b_1 = n-1$, then all the a_i, $1 \leq i \leq k$, are different and greater than $\alpha_{\mathcal{C}}(G)$. Wlog, let $a_2 = n-2$. So, b_2 can not be vertex $n-2$ because circuit $\mathcal{C}$ already contains arc $n-1, n-2$. Thus, $tp_{\mathcal{C}}((n-2, a_3), n-2)$ is minimal if $b_1 = n-1$ and $b_3 = n-2$, i.e., $\tau(\mathcal{C}, n-2) \geq n+2$.

Thus, in all cases, $S_{\mathcal{C}} \geq n+2$. □

Remark 2: Let consider $\mathcal{C}$ an Eulerian circuit of a given digraph G. Assume that $\mathcal{C}$ has stretch β. Then, by Lemma 2, $\mathcal{C}$ clearly satisfies the following constraints:

min-stretch constraint: the distance on $\mathcal{C}$ between two occurrences of the same vertex is at least $\alpha_{\mathcal{C}}(G)$;

stretch constraint: the distance on $\mathcal{C}$ between two occurrences of the same vertex is less or equal to β.

Proof of Theorem 2. By Lemma 1, in all this proof we focus on the stretch $S_{\mathcal{C}}$ of an Eulerian circuit in a digraph G.

Proof of Equation 4. Let u be a vertex of degree δ in G. Thus, $S_{\mathcal{C}}(u) \geq \frac{m}{\delta}$, since the occurrences of u in the Eulerian circuit divide this circuit into δ parts. We have:

$$\mathcal{E}(G) = \min_{\mathcal{C} \in Eul(G)} diamW_{\mathcal{C}} \text{ by definition, and } \tfrac{m}{\delta} \leq S_{\mathcal{C}} = diamW_{\mathcal{C}}(G) + 1 \text{ by Lemma 1.}$$

Let consider a special Eulerian circuit that places all the occurrences of vertex u (of minimal degree δ) as shown in Figure 6. In this case, $S_{\mathcal{C}}(u) = m - 2(\delta - 1)$. This corresponds to the largest possible part since between two occurrences of the same vertex in the circuit there must have a least one vertex.

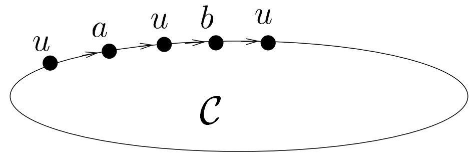


Figure 6: Eulerian circuit having large $S_{\mathcal{C}}$ with $\delta = 3$.

Proof of Equation 5. Assume now that $\delta > 3$ and that $\mathcal{E}(G) = n$. Then, there exists an Eulerian circuit $\mathcal{C}$ with stretch $S_{\mathcal{C}} = n+1$. Using Lemma 2 and Definition 1, we know that $n-1 \leq \alpha_{\mathcal{C}}(G) \leq n+1$.

Moreover, it is easy to see that $\alpha_{\mathcal{C}}(G) < n+1$ since G is a simple digraph. If $\alpha_{\mathcal{C}}(G) = n$, then, w.l.o.g., $\mathcal{C} = \underline{\mathbf{0}\ \mathbf{1}}\ \mathbf{2}\ \ldots\ (\mathbf{n-1})\ \mathbf{0}\ a_1\ \ldots$. Due to the min-stretch constraint, a_1 must be equal to 1, that leads to a contradiction since G is a simple digraph. Consequently, we must have $\alpha_{\mathcal{C}}(G) = n-1$ and then, $\mathcal{C}$ contains the following pattern:

$$b \quad a \quad \mathbf{0} \quad \mathbf{1} \quad \mathbf{2} \quad \ldots \quad (\mathbf{n-2}) \quad \mathbf{0} \quad c \quad \ldots$$

where a, b and c are vertices in $\{0, \ldots, n-1\}$.

Since $S_{\mathcal{C}} = n+1$, then a and c are equal to $n-1$ (a can not equal to $n-2$ because G is a simple graph). Thus, $\mathcal{C}$ contains the pattern:

$$(\mathbf{n-1}) \quad \mathbf{0} \quad \mathbf{1} \quad \mathbf{2} \quad \ldots \quad (\mathbf{n-2}) \quad \mathbf{0} \quad (\mathbf{n-1}) \quad a_1 \quad a_2 \quad a_3 \quad \ldots$$

Since $S_{\mathcal{C}} = n+1$, we have to consider two possible cases for a_1 and a_2.

If $a_1 = 2$ and $a_2 = 1$ then it is easy to see that the first $n+1$ elements and the last $n+1$ elements follow the same pattern, up to a permutation of $V(G)$:

$$\overline{(n-1) \quad 0 \quad \underline{1 \quad 2 \quad \ldots \quad (n-2) \quad 0 \quad (n-1)}} \quad \underline{2 \quad 1}$$

If $a_1 = 1$, $a_2 = 3$ and $a_3 = 2$ then once again, the first $n+1$ elements and the last $n+1$ elements follow the same pattern, up to a permutation of $V(G)$:

$$\overline{(n-1) \quad 0 \quad 1 \quad \underline{2 \quad 3 \quad \ldots \quad (n-2) \quad 0 \quad (n-1)}} \quad \underline{1 \quad 3 \quad 2}$$

Consider two consecutive arcs (u_{i-1}, v) and (u_i, v') in $\mathcal{C}$, where u_{i-1} and u_i are two occurrences u_i of a same u vertex of G. The *interval of* u_i in $\mathcal{C}$ denoted $IS(u_i)$ is $tp_{\mathcal{C}}((u,v),u) - 1$, i.e., the number j of vertices in $\mathcal{C}$ between u_{i-1} and u_i. In $\mathcal{C}$, the notation $u_i^{(j)}$ indicates that $IS(u_i) = j$.

We are interested in the possible sequences of intervals of consecutive vertices in the Eulerian circuit. From the two possible patterns of $\mathcal{C}$ given before, the sequence of intervals in the Eulerian circuit can be described from two elementary patterns:

$$\begin{array}{llll} \mathcal{P}_1: & (n-1) & (n+1) & \\ \mathcal{P}_2: & n & (n-1) & (n+1) \end{array}$$

The two following claims study the sequence of consecutive intervals of a given vertex in $\mathcal{C}$. First note that such a sequence is composed with values $n-1$, n, and $n+1$. Since a vertex of G occurs δ times in $\mathcal{C}$, there exists the same number of intervals of size $n-1$ as intervals of size $n+1$.

Claim 1 *Let G be a digraph of degree $\delta > 3$ and order n, and $\mathcal{C}$ an Eulerian circuit of G. The sequence of intervals of the consecutive occurrences of a same vertex in $\mathcal{C}$ does not contain the patterns $(n-1, n+1)$ and $(n+1, n-1)$.*

• Let us simply show the property for pattern $(n+1, n-1)$. The other one is obtained by reversing the following arguments. In this case, $\mathcal{C}$ has the following form:

$$
\begin{array}{llllllll}
\mathbf{(n-1)} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \ldots & \mathbf{(n-3)} & \mathbf{(n-2)} & \mathbf{0} \\
\mathbf{(n-1)}^{(n+1)} & & & & a_1 & a_2 & \ldots & a_{n-2} \\
\mathbf{(n-1)}^{(n-1)} & & & & & & & a_n
\end{array}
$$

Using the min-stretch constraint, a_n can only be equal to 0 or a_1. However, any of these two values can't be used since they are the two previous neighbours of this occurrence of $n-1$, a contradiction with the definition of G as a digraph. This ends the proof of the claim. •

In the following claim, we give another interval impossibility.

Claim 2 *Let G be a digraph of degree $\delta > 3$ and order n, and $\mathcal{C}$ an Eulerian circuit of G. Let k be an integer. The sequence of intervals of consecutive occurrences of a same vertex in $\mathcal{C}$ does not contain the pattern $(n, (n-1)^k, n)$, where $(n-1)^k$ is the sequence made of k times $(n-1)$.*

• Assume first that $k > 0$. Wlog, consider that the intervals patterns$(n, (n-1)^k, n)$ occurs for vertex 0. Then, $\mathcal{C}$ is of form

$$
\begin{array}{llllll}
\mathbf{0} & a_{1,k+2} & a_{2,k+2} & \ldots & a_{n-2,k+2} & a_{n-1,k+2} \\
\mathbf{0}^{(\mathbf{n})} & a_{1,k+1}^{(n-1)} & a_{2,k+1}^{(n+1)} & \ldots & a_{n-2,k+1} & \\
\mathbf{0}^{(\mathbf{n-1})} & a_{1,k}^{(n+1)} & a_{2,k}^{(b_k)} & \ldots & a_{n-2,k} & \\
\ldots & & & & & \\
\mathbf{0}^{(\mathbf{n-1})} & a_{1,2}^{(n+1)} & a_{2,2}^{(b_2)} & \ldots & a_{n-2,2} & \\
\mathbf{0}^{(\mathbf{n-1})} & a_{1,1}^{(n+1)} & a_{2,1}^{(b_1)} & \ldots & a_{n-2,1} & a_{n-1,1} \\
\mathbf{0}^{(\mathbf{n})} & a_{1,0}^{(n-1)} & a_{2,0}^{(n+1)} & \ldots & &
\end{array}
$$

where all the $a_{i,j}$ are vertices of G and the vertex following $a_{n-1,k+2}$ in the Eulerian circuit is 0 (on the next line). Remind that $a^{(j)}$ indicates that the previous occurrence of vertex a is j elements before this occurrence.

The intervals are obtained from patterns $\mathcal{P}_1$ and $\mathcal{P}_2$. Wlog, consider $a_{1,j} = 1 + j$, $0 \leq j \leq k+1$. Then, using the interval of $a_{2,0}$, this latter vertex is 2. We also have $a_{2,1} = 1$. Thus, the Eulerian circuit has the form given in the left part of Figure 7.

Let consider the interval b_1. Using patterns $\mathcal{P}_1$ and $\mathcal{P}_2$, b_1 can only be equal to n or $n-1$. If $b_1 = n$ then vertex $a_{1,2}$ would be 1 and the Eulerian circuit would use twice the arc $(0,1)$. Thus, $b_1 = n-1$ and $a_{2,2} = 1$. Using a simple recurrence argument, all the b_i's are equal to $n-1$ and the form of the Eulerian circuit is given in the right part of Figure 7.

Using the interval of $a_{2,k+1}$, we can conclude that $a_{1,k+2} = 1$ and the arc $(0,1)$ is used twice in the Eulerian circuit, that leads to a contradiction.

For $k = 0$, just remark that the same scheme is directly applicable and leads to the same contradiction. This ends the proof of the claim. •

$\mathbf{0}$	$a_{1,k+2}$	$\mathbf{k+2}$	$\dots$	$a_{n-2,k+2}$	$\mathbf{k+1}$
$\mathbf{0}$	$\mathbf{k+2}$	$a_{2,k+1}^{(n+1)}$	$\dots$	$\mathbf{k}$	
$\mathbf{0}$	$\mathbf{k+1}$	$a_{2,k}^{(b_k)}$	$\dots$	$\mathbf{k-1}$	
$\dots$					
$\mathbf{0}$	$\mathbf{3}$	$a_{2,2}^{(b_2)}$	$\dots$	$a_{n-2,2}$	
$\mathbf{0}$	$\mathbf{2}$	$\mathbf{1}^{(b_1)}$	$\dots$	$a_{n-2,1}$	$a_{n-1,1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{2}$	$\dots$		

$\mathbf{0}$	$a_{1,k+2}$	$\mathbf{k+2}$	$\dots$	$a_{n-2,k+2}$	$\mathbf{k+1}$
$\mathbf{0}$	$\mathbf{k+2}$	$\mathbf{1}^{(n+1)}$	$\dots$	$\mathbf{k}$	
$\mathbf{0}$	$\mathbf{k+1}$	$\mathbf{1}^{(n-1)}$	$\dots$	$\mathbf{k-1}$	
$\dots$					
$\mathbf{0}$	$\mathbf{3}$	$\mathbf{1}^{(n-1)}$	$\dots$	$a_{n-2,2}$	
$\mathbf{0}$	$\mathbf{2}$	$\mathbf{1}^{(n-1)}$	$\dots$	$a_{n-2,1}$	$a_{n-1,1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{2}$	$\dots$		

Figure 7: Two resolution steps of the construction of an Eulerian circuit of stretch $n+1$. The min-stretch indications are left only when necessary.

As a consequence of these two previous claims, let consider $\mathcal{C}$ an Eulerian circuit in G of stretch $n+1$. For symbol 0, let consider the sequence of intervals. This must have length δ and must contain the same number of $n-1$ as $n+1$. Using Claim 1, $n-1$ cannot follow, or be followed by, $n+1$. Thus, any sub-sequence of $n-1$ must be delimited by two n. Using Claim 2, this latter condition cannot be satisfied. Consequently, the sequence is only composed by n, that is impossible again using Claim 2 with $k=0$. Then, we can conclude that $\mathcal{C}$ cannot exists.

Thus, $S_C \geq n+2$, and then by Lemma 1, $\mathcal{E}(G) \geq n+1$.

□

4 Some digraphs having good Eulerian diameters

In this section, we give families of digraphs having good Eulerian diameter considering Theorem 2. These digraphs will be induced from the construction of good Eulerian circuits for the complete digraph K_n^+. To do this, we first give the construction of a particular matrix we use in the following.

4.1 A useful matrix construction

Definition 2 *Let n be even. We denote by B_n the (n,n) matrix defined by*

$$\begin{aligned} B_n(0,j) &= j/2 && && \text{if } j \text{ is even,} \\ &= -\tfrac{j+1}{2} && \bmod n && \text{if } j \text{ is odd,} \\ B_n(i,j) &= B_n(0,j)+i && \bmod n && 0<i<n, 0\leq j<n. \end{aligned}$$

Each line i of the matrix is denoted by Π_i and we note $B_n = (\Pi_0, \dots, \Pi_{n-1})$. An example of such a matrix is given in Figure 8.

This matrix has many similarities with the one shown by Tillson in [20]. Indeed, it is built by the same way; Tillson's one has a different constant for the definition of $B_n(0,j)$, for even j. Many properties of B_n can be directly shown from the ones given by Tillson.

$$
\begin{array}{lcllllll}
\Pi_0 & = & 0 & 5 & 1 & 4 & 2 & 3 \\
\Pi_1 & = & 1 & 0 & 2 & 5 & 3 & 4 \\
\Pi_2 & = & 2 & 1 & 3 & 0 & 4 & 5 \\
\Pi_3 & = & 3 & 2 & 4 & 1 & 5 & 0 \\
\Pi_4 & = & 4 & 3 & 5 & 2 & 0 & 1 \\
\Pi_5 & = & 5 & 4 & 0 & 3 & 1 & 2
\end{array}
$$

Figure 8: B_6 matrix.

4.1.1 Properties of B_n

In the following, we give some basic properties of the B_n matrix. The two first lemmas show that B_n is a row complete Latin square (see [20] for definition). Lemma 5 gives some symmetry property of this matrix and Lemma 6 shows some basic properties between two consecutive lines.

Remark 3: First note that we have the following property on this matrix:

$$\forall\, 0 \leq i, j < n \quad B_n((i+1) \bmod n, j) = 1 + B_n(i, j).$$

Lemma 3 *Each row and column of B_n is a permutation of $0, \ldots, n-1$.*

Proof: It is clear that any column is a permutation. Let us show that the first row is also a permutation.

Let j and j' two distinct integers ($0 \leq j < n$) such that $B_n(0, j)$ and $B_n(0, j')$ are equal. Assume first that they are both even. Then, we have the following equalities:

$$
\begin{aligned}
B_n(0, j) = B_n(0, j') \quad &\Longrightarrow \quad \frac{j}{2} = \frac{j'}{2} \\
&\Longrightarrow \quad j = j' \pmod{2n} \quad \textit{Impossible}
\end{aligned}
$$

If they are both odd, we obtain the same contradiction. If j is even and j' odd, we have:

$$
\begin{aligned}
B_n(0, j) = B_n(0, j') \quad &\Longrightarrow \quad \frac{j}{2} = -\frac{j'+1}{2} \mod n \\
&\Longrightarrow \quad j + j' = -1 \pmod{2n}.
\end{aligned}
$$

Since, $0 \leq j, j' < n$, we have

$$B_n(0, j) = B_n(0, j') \quad \Longrightarrow \quad j + j' = 2n - 1 \ ,$$

a contradiction.

All the symbols in the first row are different, and thus by Remark 3 each row represents a permutation of $0, \ldots, n-1$. □

Lemma 4 *For all ordered pair (u, v) of distinct integers, there exist unique integers i and j, $0 \leq i < n$, $0 \leq j < n-1$, such that:*

$$u = B_n(i, j) \text{ and } v = B_n(i, j+1).$$

The easy proof of this lemma is left to the reader.

Lemma 5 *For any integers n, i and j, we have*

$$B_n(i,j) = B_n(n/2 + j \bmod n, n-1-j).$$

Proof: By using Remark 3, we need only to prove this lemma for $i=0$. Let $n = 2p$ and assume first that j is even (i.e., $j = 2j'$).

$$\begin{array}{rclclclcl} B_n(p, 2p-1-2j') & = & p + B_n(0, 2p-1-2j') & = & p - \frac{(2p-2j'-1)+1}{2} & = & j' & = & B_n(0, 2j'). \\ B_n(p, n-1-j) & = & B_n(0,j). \end{array}$$

Next, consider $j = 2j'+1$

$$\begin{array}{rclclcl} B_n(p, 2p-1-2j'-1) & = & p + \frac{2p-2j'-2}{2} & = & 2p - j' - 1 & = & -(j'+1). \\ B_n(0, 2j'+1) & = & -\frac{(2j'+1)+1}{2} & = & -(j'+1). \end{array}$$

Thus, for any case, we have shown the desired property. □

Lemma 6 *For i, j and k such that $B_n(i,j) = B_n(i+1 \bmod n, k)$. We have:*

$$k - j \leq 2.$$

Proof: By using Remark 3, we need only to prove this lemma for $i = 0$. This proof is divided into four simple cases. We only develop the first one and leave the exact calculus to the interested reader.

Let $j = 2p+1 < n-2$ be odd. Then, we have the following equalities (all the sums are done modulo n):

$$B(1, j+2) \;=\; 1 + B(0, j+2) \;=\; 1 + \left(-\frac{(j+2)+1}{2}\right) \;=\; 1 - \frac{j+3}{2} \;=\; -\frac{j+1}{2} \;=\; B(0,j).$$

For $j = n-1$, a similar calculus shows that $B_n(0, n-1) = B_n(1, n-2)$.

When $0 < j = 2p < n$, we have $B_n(0,j) = B_n(1, j-2)$.

Finally, $B_n(0,0) = B_n(1,1)$.

Thus, we have shown that when we have a symbol in a given row, in the next one it is at most two columns after. □

4.2 Case of symmetric digraphs

Many networks are based on full-duplex communication links [17]. In general, these networks are modeled by symmetric digraphs. In this section, we exhibit some symmetric digraphs, of any degree and any size, with small Eulerian diameters (note that the degree of a symmetric digraph is always even). All this section consists in proving the following main result.

Theorem 3 *Let n, d and p be integers such that $0 < d \leq p$, and $p = \lfloor \frac{n-1}{2} \rfloor$. There exists a symmetric digraph G of degree $2d$ such that*

$$\mathcal{E}(G) \leq n + 2\left\lceil \frac{p}{d} \right\rceil - 1.$$

Remember that by Theorem 2, Equation 5, if G is regular then $\mathcal{E}(G) \geq n+1$.

To prove this theorem, we first examine the case of a odd number of vertices. The underlying technique in the odd case is to define a good Eulerian circuit in the complete digraph from a Hamiltonian decomposition of K^+_{n-1}. This is done by using the B_n matrix. To obtain graph with degree less than $n-1$, we remove Hamiltonian circuits from K^+_{n-1}, two by two in order to preserve the symmetry property of the digraph and we define an Eulerian circuit from the remaining Hamiltonian circuits. The main problem is to evenly remove the Hamiltonian circuits.

When the size of the digraph is even, the main problem is to define a good Eulerian circuit in the complete digraph. Then, similar techniques can be used for smaller degree as in the odd case.

4.2.1 Proof of Theorem 3: digraphs with odd number of vertices

Let us consider $n = 2p+1$. Before proving Theorem 3 for n odd, we give a technical result involving K^+_n.

Lemma 7 *Let $n > 5$ be an odd integer. The Eulerian diameter $\mathcal{E}$ of K^+_n is $n+1$. The Eulerian diameter of K^+_3 is 3.*

Proof: First of all, the only Eulerian circuit in K^+_3 is

$$\mathbf{0\ \ 1\ \ 2\ \ 0\ \ 2\ \ 1}$$

whose Eulerian diameter is clearly 3.

For the general case, let consider B_{2p}, and the associated permutations $\Pi_0, \ldots \Pi_{2p-1}$. Let define the Hamiltonian circuits C_i, $0 \leq i < 2p$, as $(2p, \Pi_i, 2p)$. Let consider $\mathcal{C}_n$ the circuit as the succession of the C_i's. An example is given in Figure 9. As shown in [20], the Hamiltonian cycles, $C_0, \ldots, C_{2p-1}$, is a Hamiltonian decomposition of K^*_n. Thus, $\mathcal{C}_n$ is an Eulerian circuit.

In order to compute the Eulerian diameter of $\mathcal{C}_n$, we consider the stretch of $\mathcal{C}_n$.

Since $\mathcal{C}_n$ is composed by the succession of Hamiltonian cycles, the stretch of symbol $2p$ is exactly $2p+1 = n$. From Lemma 6, given an occurrence of symbol i, $0 \leq i < 2p$, in row j, this symbol appears at most two columns after in the next row ($(j+1) \bmod 2p$) in B_{2p}. Thus the maximal distance between two occurrences of symbol i in $\mathcal{C}_n$ is exactly $n+2$. Lemma 1 gives the value $n+1$ for the Eulerian diameter of $\mathcal{C}_n$.

Theorem 2 shows that $n+1$ is the best value that can be obtained in that case. □

Note that the construction of the Eulerian circuit can be seen as inserting a column into B_{2p}, in the last position with the element $2p$ as shown in Figure 9.

Proof of Theorem 3 : case where n is odd.

Let us start with two simple remarks.

If $d = p$, then the theorem follows from Lemma 7. Thus, let consider $\mathcal{C}_n$ as the Eulerian circuit in K^+_n given in the previous proof, composed by Hamiltonian cycles called C_i, $0 \leq i < 2p$.

As a second remark, let consider $\mathcal{C}_n$ in which we have removed two circuits, namely C_0 and C_p. By Lemma 5, these two Hamiltonian circuits are opposite, i.e., the second one uses the reverse arcs of the first one. Thus, their removal does not affect the symmetry property of the resulting digraph. In terms of Eulerian diameter, we need to apply once more Lemma 6, the stretch of this Eulerian circuit is increased by 2 (compared with the Eulerian circuit of the complete graph), so

(C_0)	**0**	9	1	8	2	7	3	6	4	5	10
(C_1)	1	**0**	2	9	3	8	4	7	5	6	10
(C_2)	2	1	3	**0**	4	9	5	8	6	7	10
(C_3)	3	2	4	1	5	**0**	6	9	7	8	10
(C_4)	4	3	5	2	6	1	7	**0**	8	9	10
(C_5)	5	4	6	3	7	2	8	1	9	**0**	10
(C_6)	6	5	7	4	8	3	9	2	**0**	1	10
(C_7)	7	6	8	5	9	4	**0**	3	1	2	10
(C_8)	8	7	9	6	**0**	5	1	4	2	3	10
(C_9)	9	8	**0**	7	1	6	2	5	3	4	10

Figure 9: Eulerian circuit for K_{11} composed by the succession of the Hamiltonian circuits C_0, $\ldots C_9$.

the Eulerian diameter. Thus, we have built a digraph of degree $2d = n - 3 = 2(p-1)$, having Eulerian diameter less than $n+3$. And the theorem follows in this case also.

In order to obtain the general case, we perform the same way. However, if we remove simply from the previous digraph (and also from the Eulerian circuit) the two following Hamiltonian circuits: C_1 and C_{p+1}, then the Eulerian diameter would be increased again by 2, and lead to an Eulerian diameter too important. The idea is to balance the removal of the Hamiltonian circuits and choose to remove C_0 and C_p, $C_{p/2}$ and $C_{3p/2}$. Considering a ring of size p, we need to solve the following subproblem. Mark k elements in the ring such that the distance between two unmarked elements is minimum.

Claim 3 *We can mark k vertices of the ring of size p such that the maximal distance between two consecutive unmarked elements is* $\left\lceil \frac{p}{p-k} \right\rceil$.

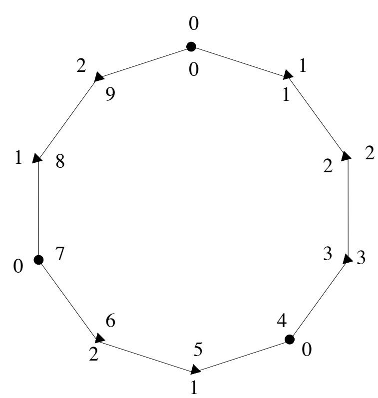


Figure 10: Two marking for $p = 10$, first for $k = 3$ marked with $\bullet$, and the second one for $k = 7$, marked with triangles.

Assume first that $k \leq p/2$. Let $L_1 = \left\lfloor \frac{p}{k} \right\rfloor$ and $L_2 = \left\lceil \frac{p}{k} \right\rceil$. Then, we use the following process to mark the elements in the ring. First, mark one element every L_1, $p \bmod k$ times and afterwards one element every L_2 $(k - p \bmod k)$ times.

Using this process, we have the following simple property. Two marked elements are separated by at most β elements, and two unmarked elements are separated by at most 2 elements. This is simply due to:

$$p = L_1 \cdot (k - p \bmod k) + L_2(p \bmod k).$$

Then, if $k \leq p/2$, then the maximal distance between two consecutive unmarked elements is 2, i.e., $\left\lceil \frac{p}{p-k} \right\rceil$.

When, $k > p/2$, we simply reverse the previous marking obtained for $k' = p - k$. Then the maximal distance between two unmarked elements in the final marking is exactly $\lceil \frac{p}{k'} \rceil$, i.e., $\left\lceil \frac{p}{p-k} \right\rceil$.

An example of this marking process is given in Figure 10. This ends the proof of this claim. •

In order to built a digraph of degree $2d$, we remove from the Eulerian circuit $2(n - d) = 2k$ Hamiltonian cycles. In order to preserve the symmetry property of the resulting digraph, we need to remove associated pairs of cycles, i.e., C_i and $C_{(p+i) \bmod (2p)}$. Let consider the marked elements by the previous claim. We remove the cycles C_i and C_{p+i}, if i is a marked element in the ring.

For this digraph of degree $2d$, the distance between two consecutive settings of the same symbol is at most n plus twice the number of consecutive Hamiltonian cycles that have been deleted from $\mathcal{C}_n$, using Lemma 6 as shown in Figure 11. Thus, we directly obtain an Eulerian circuit for which the Eulerian diameter is $n + 2\left\lceil \frac{p}{p-k} \right\rceil - 1$, i.e., $n + 2\lceil \frac{p}{d} \rceil - 1$. □

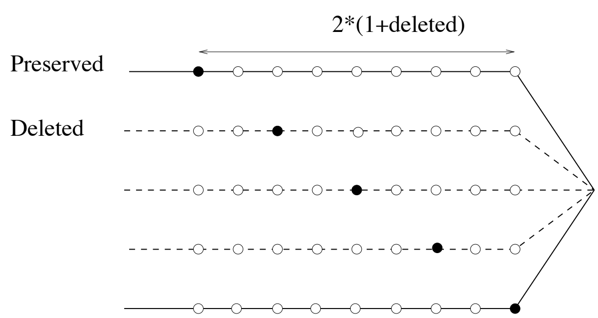


Figure 11: Maximal distance between two consecutive settings of the same symbol in the Eulerian circuit.

Figure 12 shows a digraph obtained by the process described in the previous proof.

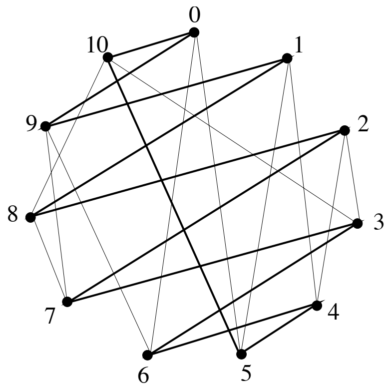
4.2.2 Proof of Theorem 3: digraphs with even number of vertices

When the number of vertices is even, the previous strategy cannot be applied as is. The idea developed afterwards is to take an Eulerian circuit for the odd case and add correctly the missing edges in it.

As in the previous section, before proving Theorem 3 for n even, we give a technical result involving K_n^+.

Lemma 8 *If n is even, the Eulerian diameter of K_n^+ is less than $n + 4$.*

Proof: Let n be $2p + 2$. Consider the B_{2p} matrix construction as before. We build an Eulerian circuit $\mathcal{C}_n$ as the concatenation of $2p$ arc-disjoint circuits in K_n^+, in two phases. First we insert two



(C_0)	0	9	1	8	2	7	3	6	4	5	10
(C_1)					*deleted*						
(C_2)					*deleted*						
(C_3)	3	2	4	1	5	0	6	9	7	8	10
(C_4)					*deleted*						
(C_5)	5	4	6	3	7	2	8	1	9	0	10
(C_6)					*deleted*						
(C_7)					*deleted*						
(C_8)	8	7	9	6	0	5	1	4	2	3	10
(C_9)					*deleted*						

Figure 12: Graph with 11 vertices and degree 4, and an Eulerian circuit resulting from the decomposition of K_{11} given in Figure 9. In bold, C_0 (and C_5) is shown.

columns into B_{2p} between the two first columns: one is a complete column of symbol $2p+1$, and the second one is the duplication of Column 1. The second step consists in inserting at the end of the first row the pair $2p+1, 2p$. This process is shown in Figure 13.

(C'_0)	0	**7**	**0**	5	1	4	2	3	6	**7**	**6**
(C'_1)	1	**7**	**1**	0	2	5	3	4	6		
(C'_2)	2	**7**	**2**	1	3	0	4	5	6		
(C'_3)	3	**7**	**3**	2	4	1	5	0	6		
(C'_4)	4	**7**	**4**	3	5	2	0	1	6		
(C'_5)	5	**7**	**5**	4	0	3	1	2	6		

Figure 13: Extension of the B_6 matrix leading to an Eulerian circuit of K_8.

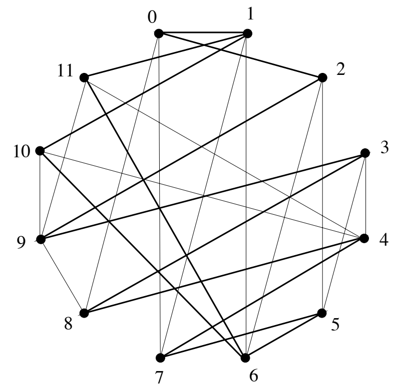
The first point is that $\mathcal{C}_n$ is an Eulerian circuit. This is due to the fact that it is built from an Eulerian circuit in K^+_{n-1} in which we have added circuits of length 2. Thus, the primary circuit in K^+_{n-1} is also a circuit in K^+_n and uses all the arcs of the form (i,j), $0 \le i,j \le 2p$, $i \ne j$. Since any column of B_{2p} is a permutation (see Lemma 3), the addition of the first columns covers all the arcs of the form $(i, 2p+1)$ or $(2p+1, i)$, $0 \le i < 2p$. The extension of the first circuit using vertices $2p$ and $2p+1$ completes the description of the Eulerian circuit.

The stretch of any symbol can be computed as before. However, we have to take into account that the length of C'_0 is $n+3$ (instead of n in the odd case). Lemma 6 still applies, leading to a further 2 additive constant to the stretch of any element. Thus, the maximal stretch of an element in this Eulerian circuit is $n+5$. Using Lemma 1 leads to the desired result. □

Even if this result is near from the optimal, it is still open to know the exact value of the Eulerian diameter of K^+_n.

Proof of Theorem 3 for n even.

The same process as shown for the proof of Theorem 3 for n odd can be used here. The starting point is not the Eulerian circuit $\mathcal{C}_n$ used for the completed digraph, but a circuit $\mathcal{C}'_n$ obtained from $\mathcal{C}_n$ by removing the small cycle of length 2: $(2p, 2p+1, 2p)$ (i.e., we use the circuit obtained after



(C'_0)							*deleted*						
(C'_1)	1	11	1	0	2	9	3	8	4	7	5	6	10
(C'_2)							*deleted*						
(C'_3)							*deleted*						
(C'_4)	4	11	4	3	5	2	6	1	7	0	8	9	10
(C'_5)							*deleted*						
(C'_6)	6	11	6	5	7	4	8	3	9	2	0	1	10
(C'_7)							*deleted*						
(C'_8)							*deleted*						
(C'_9)	9	11	9	8	0	7	1	6	2	5	3	4	10

Figure 14: Graph with 12 vertices and minimum degree 4, and an Eulerian circuit resulting from the decomposition of K_{12} given in Figure 9. In bold, C_1 (and C_6) is shown.

the first step of the construction of $\mathcal{C}_n$ in the previous proof).

However, we have to notice that the obtained digraph is not regular. Then, the stretch of any element is exactly equal to the length of one circuit (i.e., $n+1$) plus $2(1+k)$, where $2k$ is the number of deleted circuits. The other points of the proof have been given previously. We don't precise them here. □

4.2.3 A conjecture for K_n^+

In Lemma 7 and Lemma 8, we have shown that $\mathcal{E}(K_n^+) = n+1$ when n is odd and $n+1 \leq \mathcal{E}(K_n^+) \leq n+4$ when n is even. In fact, we conjecture that for any $n \geq 4$, $\mathcal{E}(K_n^+) = n+1$. With the following computational experiments, we have shown that this conjecture is true for $n = 4, 6, 8$:

- $\mathcal{E}(K_4^+) = 5$: this is obtained for the following Eulerian circuit: 0 1 2 3 0 2 1 0 3 1 3 2.
- $\mathcal{E}(K_6^+) = 7$: this is obtained for: 0 1 2 3 4 5 0 3 1 4 2 5 1 0 4 3 5 2 1 3 0 2 4 1. 5 3 2 0 5 4
- $\mathcal{E}(K_8^+) = 9$: obtained for:

0	1	2	3	4	5	6	7		
0	2	1	4	6	3				
0	5	7	2	6	4	1	3	2	5
0	4	7	3	6	1	5	4	2	
0	3	7	1	6	5	2	4		
0	7	5	3	1					
0	6	2	7	4	3	5	1	7	6.

- $\mathcal{E}(K_{10}^+) \leq 12$: several Eulerian circuits have been found with Eulerian stretch of 12, i.e.:

0	1	2	3	4	5	6	7	8	9
0	2	1	3	5	4	6	8	7	9
1	0	3	2	4	7	5	8	6	9
2	0	4	1	4	3	6	5	7	0
8	2	9	5	1	6	4	8	3	0
7	2	5	9	6	3	1	8	4	9
7	6	0	5	2	8	1	9	3	7
4	0	6	2	7	1	5	3	8	0
9	4	2	6	1	7	3	9	8	5

4.3 Case of general regular digraphs

Theorem 4 *For any integers n and δ, with $n \geq 1$ and $2 \leq \delta \leq n-1$, there exists a δ-regular digraph G with n vertices, with Eulerian diameter verifying*

$$\begin{cases} \mathcal{E}(G) = n+1 & \text{if } r \leq q \\ n+1 \leq \mathcal{E}(G) \leq n+3 & \text{else,} \end{cases}$$

where $q = n \texttt{ div } \delta$ and $r = n \texttt{ mod } \delta$.

Proof: By Theorem 2, the Eulerian diameter of any regular digraph is greater or equal to $n+1$. In the following, for any n and δ, we construct a δ-regular digraph G with n vertices containing an Eulerian circuit $\mathcal{C}$ with stretch $S_{\mathcal{C}} = n+2$ or $S_{\mathcal{C}} = n+4$ (in fact, G is deduced from $\mathcal{C}$ as in the previous proofs). Then, Lemma 1 concludes.

Let $V = \{0, \ldots, n-1\}$ be the vertex set of G. We decompose V into $k = \lceil \frac{n}{\delta} \rceil$ ordered sequences V_i defined by

$$V_i = \delta \cdot i, \delta \cdot i\ + 1, \ldots, \delta \cdot i\ + (\delta - 1) \ \text{ with } \ 0 \leq i \leq k-1.$$

In other words, $V_i[j] = \delta \cdot i + j$, where $0 \leq j < \delta - 1$ and V_{k-1} is made of at most δ vertices. Let π be a permutation of S_δ. For any V_i, we denote by $\pi(V_i)$ the sequence of vertices $V_i[\pi(0)], V_i[\pi(1)], \ldots, V_i[\pi(\delta-1)]$.

Let us define $q = n \texttt{ div } \delta$ and $r = n (\text{mod } \delta)$. We consider three cases.

Case 1 : $r = 0$.
Consider the matrix B_δ given in Definition 2. From Lemma 3, each line of $B_\delta = (\Pi_0, \ldots, \Pi_{\delta-1})$ is a permutation of S_δ. We construct a cyclic sequence of vertices $\mathcal{C}$ of V as follows (see Figure 15 (a)). This sequence is made of consecutive subsequences

$$\mathcal{C} = C_{0,0}, C_{0,1}, \ldots, C_{0,q-1}, C_{1,0}, \ldots, C_{1,q-1}, C_{2,0}, \ldots, C_{\delta-1,q-1},$$

such that for any i, j, where $0 \leq i \leq \delta - 1$ and $0 \leq j \leq q-1$, $C_{i,j} = \Pi_i(V_j)$.

This cyclic sequence $\mathcal{C}$ defines a unique multi-digraph G from which it is a arc-covering circuit. By Lemmas 3 and 4, and by construction of $\mathcal{C}$, it is clear that no pair of vertices of V can appear more than one times as two consecutive vertices in $\mathcal{C}$. Thus, G is a simple digraph and $\mathcal{C}$ is an Eulerian circuit in G.

Consider $v \in V_j$ a vertex of V. Then in $\mathcal{C}$, there is exactly one occurrence of v in each subset $C_{i,j}$ for $0 \leq i \leq \delta - 1$, and only in them. Then, G is a δ-regular digraph, and by construction of $\mathcal{C}$, the distance in $\mathcal{C}$ between two consecutive occurrences of v is $(\delta-1)q\ + 2 = n+2$. Thus,

0	3	1	2	4	7	5	6	8	11	9	10
1	0	2	3	5	4	6	7	9	8	10	11
2	1	3	0	6	5	7	4	10	9	11	8
3	2	0	1	7	6	4	5	11	10	8	9
B_4				B_4+4				B_4+8			

(a)

0	3	1	2	12	4	7	5	6	13	8	11	9	10
1	0	2	3	12	5	4	6	7	13	9	8	10	11
2	1	3	0	12	6	5	7	4	13	10	9	11	8
3	2	0	1	12	7	6	4	5	13	11	10	8	9
B_4					B_4+4					B_4+8			

(b)

Figure 15: (a) Circuit $\mathcal{C}$ for $n = 12$ and $\delta = 4$ and (b) circuit $\mathcal{C}'$ for $n = 12$ and $\delta = 15$.

$S_{\mathcal{C}} = n + 2$.

Case 2 : $1 \leq r \leq q$.
In this case, $k = q + 1$ and $V_{k-1} = \delta(k-1), \delta(k-1)+1, \delta(k-1)+a$, with $\delta(k-1)+a = n-1$.

Consider the circuit $\mathcal{C}$ constructed as in Case 1 from sequences $V_0, \ldots, V_{k-2}$. We construct a new circuit $\mathcal{C}'$ from $\mathcal{C}$ and vertices of V_{k-1} as follows (see Figure 15 (b)). For each t, $0 \leq j \leq a$, an occurrence of vertex $V_{k-1}[j]$ is inserted in $\mathcal{C}$ after each last vertex of the subsequence $C_{i,j}$, for any i, $0 \leq i \leq \delta - 1$.

Since by Lemma 3 each column of B_δ is a permutation, and by construction of $\mathcal{C}$ and $\mathcal{C}'$, this circuit $\mathcal{C}'$ still defines a simple digraph G in which it is an Eulerian circuit. Moreover, by the construction of $\mathcal{C}$, the maximal distance between two consecutive occurrences of a vertex in $V - V_{k-1}$ in $\mathcal{C}'$ is still less or equal to $n+2$. By construction of $\mathcal{C}'$ the maximal distance between two consecutive occurrences of a vertex of V_{k-1} in $\mathcal{C}'$ is exactly n.

Case 3 : $q + 1 \leq r \leq \delta - 1$.
In this case, $k = q+1$ and $V_{k-1} = V_a V_b$ with $V_a = \delta(k-1), \delta(k-1)+1, \ldots, \delta(k-1)+a$, $a = q-1$, $V_b = \delta(k-1)+a+1, \ldots, \delta(k-1)+b$ and $\delta(k-1)+b = n-1$. Note that $b - a \leq \delta - 1$

Consider the circuit $\mathcal{C}'$ constructed as in Case 2 from sequences $V_0, \ldots V_{k-2}, V_a$. We construct a new circuit $\mathcal{C}''$ from $\mathcal{C}'$ and vertices of V_b as follows. For each t, $0 \leq j \leq b-a-1$, an occurrence of vertex $V_b[j] = \delta(k-1)+a+1+j$ is inserted in $\mathcal{C}'$ after the $(j+1)^{th}$ vertex of each subsequence $C_{i,j}$, for any i, $0 \leq i \leq \delta - 1$. As in the previous case, $\mathcal{C}''$ is an Eulerian circuit in a δ-regular digraph G and by construction of $\mathcal{C}''$ and by Lemma 6, we show that $S_{\mathcal{C}''}$ is less or equal to $n+4$. $\Box$

Note that the δ-regular digraphs we obtain to prove Theorem 4 are almost symmetric.

5 Conclusion

To conclude, we give some open problems. We conjecture that the problem of knowing if $\mathcal{E}(G) \leq k$ is NP-complete. Another open problem is to give a tight upper bound of $\mathcal{E}(G)$, better than the one of Proposition 2. In Table 5, we summarize the results we have obtained and we give some conjectures and open questions.

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Family to which our digraphs lead to	$\mathcal{E}(G)$ for obtained digraphs	Conjectured best Eulerian diameter
K_n^+	$\begin{cases} = n+1 & \text{if } n \text{ is odd} \\ \leq n+4 & \text{else} \end{cases}$	$= n+1$
Symmetric, $\lfloor \frac{n-1}{2} \rfloor = p$ $\delta = 2d$, $p \leq d$	$n + 2\lceil \frac{p}{d} \rceil - 1$	?
Symmetric, any degree $\delta \neq 2d$	-	?
δ-regular, not symmetric, $q = n \texttt{ div } \delta$, $r = n \texttt{ mod } \delta$	$\begin{cases} = n+1 & \text{if } r \leq q, \\ \leq n+3 & \text{else} \end{cases}$	$= n+1$

Table 1: Summary of the results of Section 4

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