About the b-continuity property of graphs

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Abstract

This paper deals with *b-colorings* of a graph G, that is, proper colorings in which for each color c, there exists at least one vertex colored by c such that its neighbors are colored by each other color. The *b-chromatic number* $b(G)$ of a graph G is the maximum of colors for which G has a b-coloring. It is easy to see that every G has a b-coloring using $\chi(G)$ colors.

We say that G is *b-continuous* iff for each k, $\chi(G) \leq k \leq b(G)$, there exists a b-coloring with k colors. It is well known that not all graphs are b-continuous. We call *b-spectrum* $S_b(G)$ of G the set of integers k for which there is a b-coloring of G by k colors. We show that for any finite integer set I, there exists a graph whose b-spectrum is I and we investigate the complexity of the problem to decide whether a graph G is b-continuous, even if b-colorings using $\chi(G)$ and $b(G)$ colors are given.

Key words: Complexity, Graph, Coloring, b-chromatic

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1 Introduction

Throughout this paper, we follow the notation and terminology in [2]. A *b-coloring* of a graph G is a proper coloring π of the vertices of G such that for each color c, there exists a vertex v with $\pi(v) = c$ such that for any color $c' \neq c$, there exists $v' \in \Gamma_G(v)$ such that $\pi(v') = c'$ (where $\Gamma_G(v)$ is the neighborhood of v). Such a vertex v is called a *b-chromatic vertex for color c*. We denote by $|\pi| = |\pi(V(G))|$ the number of colors used in the coloring π. If $|\pi| = k$, then π is called a *(k)b-coloring*. The *b-chromatic number* $b(G)$ of a graph G is the maximum k for which G has a (k)b-coloring. Obviously, a coloring of G with $\chi(G)$ colors where $\chi(G)$ is the chromatic number of G is a b-coloring.

The b-chromatic number was introduced in [7]. As said in [12], the motivation, similarly as for the previously studied achromatic number (cf. e.g., [3,6,8,9]), comes from algorithmic graph theory. Suppose one colors a given graph properly, but in an arbitrary way. After all vertices are colored, one would wish to reduce the number of colors. The simplest operation at hand is recoloring all vertices of the same color with another color. In an achromatic coloring there is an edge between any two color classes, and hence such recoloring is impossible. The achromatic number of a graph is thus the worst case number of colors that may be needed to color the graph under the above described heuristics.

A slightly more involved operation would take one color class and recolor its vertices, but not necessarily each with the same color. Obviously, such recoloring is impossible if each color class contains a b-chromatic vertex. Hence the b-chromatic number of the graph serves as the tight upper bound for the number of colors used by this more sophisticated coloring heuristics.

Given a graph G and an integer k, the problem to decide whether $b(G) \geq k$ is NP-complete [7], even if G is bipartite [14]. In [12] they strengthened these results by proving that this problem is NP-complete even if G is bipartite and $k = \Delta(G) + 1$.

Considering b-colorings we observe further interesting properties. There are some graphs for which there exist b-colorings by k_1 colors and k_2 color but no b-colorings by k colors where $k_1 < k < k_2$. A similar behavior has been noted for the strict colorings of mixed hypergraphs [10]. Thus, we define the *b-spectrum* $S_b(G)$ of a graph G as the set of integers k for which that there exists a (k)b-coloring of G. We call a graph G *b-continuous* if $S_b(G)$ is an interval that is there are (k)b-colorings of G for all k, with $\chi(G) \leq k \leq b(G)$. This characteristic has been first studied in [12] where the authors give an infinite class of graphs that are not b-continuous. In [1] it is proved that interval graphs are b-continuous, this result was generalized in [4,11] by showing that

Graphs G and H Graph $\zeta(G,H)$

Fig. 1. Example of the join $\zeta(G,H)$.

even all chordal graphs are b-continuous.

We focus here on the b-continuity of graphs. In this paper we mainly answer two questions:

Question 1. For any finite set of integers I, does there exist a graph G with $S_b(G) = I$?

Question 2. Is the problem of deciding if a given graph G is b-continuous NP-complete?

In Section 3, we we deal with Question 1, this question was posed in [12]. Given any finite set of integers $I \subset \mathbb{N} \setminus \{0,1\}$, we use a graph composition to define a graph G such that $I = S_b(G)$. In Section 4, we deal with Question 2. Given a graph G and an integer k, it is known that the problem of deciding whether $k \in S_b(G)$ is NP-complete [7,12]; however this result alone does not answer Question 2. We show that the problem of deciding if a given graph G is b-continuous is NP-complete. One could ask if the difficulty of this problem is inherent in the fact that it is hard to determine $\chi(G)$ and $b(G)$. However, this problem also remains NP-complete even if a $(\chi(G))$b-coloring and a $(b(G))$b-coloring are given.

2 Preliminary Results

We define the join of two graphs as follows:

Definition 1 *Let G and H be two graphs. The join $\zeta(G,H)$ of G and H is the graph defined by :*

- $V(\zeta(G,H)) = V(G) \cup V(H)$
- $E(\zeta(G,H)) = E(G) \cup E(H) \cup \{[g,h], g \in V(G) \wedge h \in V(H)\}$.

The following Proposition gives the b-spectrum of the join of two graphs.

Proposition 1 *Let G and H be two graphs. The b-spectrum of $\zeta(G,H)$ is the set*
$\{k + k' : k \in S_b(G), k' \in S_b(H)\}$.

Proof. Let π be a (k)b-coloring of the graph G. Let π' be a (k')b-coloring of the graph H.

We construct a coloring π'' of the graph $\zeta(G,H)$, defined by $\forall x \in V(G),\ \pi''(x) = \pi(x)$ and $\forall x \in V(H),\ \pi''(x) = \pi'(x) + k$. It is clear to see that π'' is a proper coloring and also a $(k+k')$b-coloring.

Conversely, let π'' be a (k'')b-coloring of the graph $\zeta(G,H)$. It is clear that the sets of colors $C_1 = \pi''(V(G))$ and $C_2 = \pi''(V(H))$ are disjoint. Thus given any b-chromatic vertex for a color $c \in C_1$ (resp. $c \in C_2$), all of its neighbors whose color belongs C_1 (resp. C_2) are in $V(G)$ (resp $V(H)$). Consequently the restriction $\pi'' : V(G) \to C_1$ (resp. $\pi'' : V(H) \to C_2$) is a b-coloring of G (resp. of H), and we have $|C_1| \in S_b(G)$, $|C_2| \in S_b(H)$ with $k'' = |C_1| + |C_2|$. $\square$

In particular, taking the join of a graph G with the complete graph K_n has the effect of "shifting" the b-spectrum of G to the right by n units, adding n to each element of $S_b(G)$.

We denote by $K'_{n,n}$ the graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.

Proposition 2 *[12] The b-spectrum of $K'_{n,n}$ is the set $\{2, n\}$.*

3 Graphs with a given b-spectrum

In this section we prove that for any set of integers I, there exists a graph G such that $S_b(G) = I$.

Theorem 1 *For any finite nonempty set $I \subset (\mathbb{N} \setminus \{0,1\})$ there exists a graph G such that $S_b(G) = I$.*

Proof. It suffices to consider sets I with $min(I) = 2$. Indeed, for $min(I) = \alpha > 2$, if there exists a graph G with b-spectrum $S_b(G) = \{2, n_1 - (\alpha - 2), \ldots, n_p - (\alpha - 2)\}$ then by Proposition 1 we have $S_b(\zeta(G, K_{\alpha-2})) = \{\alpha, n_1, \ldots, n_p\}$.

Thus we only consider three cases with respect to the cardinality of I.

Case 1: For $I = \{2\}$, it suffices to consider $G = K_2$.

Case 2: For $I = \{2, n_1\}$, by Proposition 2, I is the b-spectrum of K'_{n_1,n_1}.

Case 3: $I = \{2, n_1, \ldots, n_p\}$, with $2 < n_1 < \ldots < n_p$ and $p \geq 2$. We construct a graph with $p+1$ independent vertex sets. This graph $G = (\cup_{i=0}^{p} V_i, \cup_{i=1}^{p} E_i)$ is defined as follows (see Fig 2) :

(1) $V_0 = \{v_0^1, \ldots, v_0^{n_p}\}$, $V_p = \{v_p^1, \ldots, v_p^{n_p}\}$,
(2) $\forall i \in \{1, \ldots, p-1\}$, $V_i = \{v_i^1, \ldots, v_i^{n_i - 1}\}$,

(3) $\forall \ell, j$, with $(1 \leq j \leq n_p)$ and $(1 \leq \ell \leq n_p)$, $[v_0^\ell, v_p^j] \in E_p \Leftrightarrow (\ell \neq j)$
(4) $\forall i \in \{1, \ldots, p-1\}$, $\forall \ell, j$, with $(2 \leq j \leq n_i - 1)$ and $(1 \leq \ell \leq n_p)$, $[v_0^\ell, v_i^j] \in E_i \Leftrightarrow (\ell \neq j)$
(5) $\forall i \in \{1, \ldots, p-1\}$, $\forall \ell \in \{2, \ldots, n_i - 1\}$, $[v_0^\ell, v_i^1] \in E_i$.

We can check that the graph G satisfies the following properties:

a. Any edge has exactly one endpoint in V_0 (see 3,4 and 5).
b. The graph induced by $V_0 \cup V_p$ is a K'_{n_p,n_p} (see 3).
c. $\forall i \in \{1, \ldots, p-1\}$, the graph induced by $V_i \cup \{v_0^1, \ldots, v_0^{n_i-1}\}$ is a K'_{n_i-1,n_i-1} (see 4 and 5).
d. $\forall i \in \{1, \ldots, p-1\}$, the graph induced by $\{v_0^{n_i}, \ldots, v_0^{n_p}\} \cup (V_i \setminus \{v_i^1\})$ is the complete bipartite graph $K_{n_p-(n_i-1),n_i-2}$ (see 4).
e. $\forall i \in \{1, \ldots, p-1\}$, there is no edge connecting a vertex in the set $\{v_0^1, v_0^{n_i}, \ldots, v_0^{n_p}\}$ with v_i^1 (see 4 and 5).

In the remaining of this section we show that the b-spectrum of G is I.

By Property (a.), G is bipartite. Hence $\chi(G) = 2$ and thus any 2-coloring of G is a (2)b-coloring. Since G contains only n_p vertices of degree $\geq n_p$, there is no b-coloring of G with more than n_p colors. First, $\forall i_0 \in \{1, \ldots, p-1\}$, we define a b-coloring π_{i_0} of G using n_{i_0} colors.

- $\forall j \in \{1, \ldots, n_{i_0} - 1\}$, $\pi_{i_0}(v_0^j) = \pi_{i_0}(v_{i_0}^j) = j$
- $\forall j \in \{n_{i_0}, \ldots, n_p\}$, $\pi_{i_0}(v_0^j) = 1$
- $\forall i \in \{1, \ldots, p\}$, with $i \neq i_0$, $\forall v \in V_i, \pi_{i_0}(v) = n_{i_0}$

Obviously $\forall i \in \{1, \ldots, p-1\}$, π_i is a b-coloring of G. The b-chromatic vertices can be taken as $v_0^1, \ldots, v_0^{n_{i_0}-1}, v_p^1$.

Finally, let π_p be the following coloring of G.

- $\forall j \in \{1, \ldots, n_p\}$, $\pi_p(v_0^j) = \pi_p(v_p^j) = j$
- $\forall i, 1 \leq i < p$, $\forall j \in \{1, \ldots, n_i - 1\}$, $\pi_p(v_i^j) = j$

Obviously π_p is a (n_p)b-coloring of G.

So, we have shown that $I \subseteq S_b(G)$. It remains to prove the equality between these sets. It suffices to show that for any (k)b-coloring π of G, $k < n_1$ implies that $k = 2$ and $n_r \leq k < n_{r+1}$, with $1 \leq r \leq p-1$, implies $k = n_r$.

Claim 1 *If there exists a (k)b-coloring of G such that $k < n_p$, then exactly one color is not in $\pi(V_0)$ and all the other colors have all their b-chromatic vertices in V_0.*

Proof. Since $k < n_p$ and since the graph induced by $V_0 \cup V_p$ is K'_{n_p,n_p}, at

Fig. 2. A graph with b-spectrum $\{2, 4, 6\}$.

least one color is not in $\pi(V_0)$. Otherwise there exists at least one vertex in V_p with all the k colors in its neighborhood. Hence there is no available color for this vertex.

Since every edge has exactly one endpoint in V_0, if two or more colors have b-chromatic vertices in $\cup_{i=1}^{p} V_i$ then all the k colors are in $\pi(V_0)$ which is not possible as described above. So the b-chromatic vertices of exactly $k-1$ colors are in V_0. This concludes the proof of the Claim. $\square$

Suppose firstly that $k < n_1$. By Claim 1, $|\pi(V_0)| = k - 1$. Without loss of generality we may suppose $k \notin \pi(V_0)$. As $k - 1 < n_1 - 1$, at least two vertices of the set $\{v_0^1, \dots, v_0^{n_1-1}\}$ have the same color, say for instance color $k-1$. By construction, no vertex of $V(G) \setminus V_0$ can be of color $k - 1$, and by Claim 1 all the b-chromatic vertices of colors 1 to $k-1$ are in V_0. Therefore, if $k > 2$, there is no b-chromatic vertex of color $k - 2$ at all. Thus if π is a b-coloring, then k must be equal to 2.

Suppose now that $n_r \leq k < n_{r+1}$, with $1 \leq r \leq p - 1$. By Claim 1, $|\pi(V_0)| = k-1$. Without loss of generality we may suppose $k \notin \pi(V_0)$. As $k-1 < n_{r+1}-1$, at least two vertices of $\{v_0^1, \dots, v_0^{n_{r+1}-1}\}$ have the same color, assume that this color is 1. By construction, no vertex of V_j with $j \geq r + 1$, can have color 1, and the only vertex of V_j with $j \leq r$ which may accept this color is the vertex v_j^1. Therefore the b-chromatic vertices of colors 2 to $k-1$ are in the set $\{v_0^2, \dots, v_0^{n_r-1}\}$, hence $k - 2 \leq n_r - 2$, so $k = n_r$.

Hence $S_b(G) = I$ is established. $\square$

4 NP-completeness results

In this section we deal with the complexity of deciding whether a given graph is b-continuous. Given a graph G, One may ask how knowing b-colorings using $\chi(G)$ and $b(G)$ would help to decide the b-continuity of G. The main result of this section is the NP-completeness of the problem to decide whether a given graph G is b-continuous even if b-colorings using $\chi(G)$ and $b(G)$ colors are given.

We consider the following decision problems:

B-CHROMATIC NUMBER
Instance: a graph G and an integer k

Question: Does there exist a b-coloring using at least k colors?

Exact Cover by 3-Sets (X3C)
Instance: A set $S = \{s_1, s_2, \cdots, s_n\}$ and a collection $T = \{T_1, T_2, \cdots, T_m\}$ where $\forall i,\ |T_i| = 3$
Question: Does T contain an exact cover for S, i.e., is there a set T' ($T' \subset T$) of pairwise disjoint sets whose union is S?

B-CERNABLE
Instance: a graph G, $(\chi(G))$b-coloring, and $(b(G))$b-coloring.
Question: Is G b-continuous?

Irving and Manlove [7] proved that B-CHROMATIC NUMBER is NP-complete. To prove the NP-completeness of B-CHROMATIC NUMBER , the authors provided a polynomial transformation $\mathcal{R}$. $\mathcal{R}$ transforms any instance of X3C defined by (S, T) into an instance of B-CHROMATIC NUMBER defined by (G, k). the proof of our result makes appeal to this transformation $\mathcal{R}$.

Let $I = (S, T)$ be an arbitrary instance of the X3C problem with $S = \{s_1, s_2, \cdots, s_n\}$, and $T = \{T_1, T_2, \cdots, T_m\}$. We describe the polynomial transformation $\mathcal{R}$ presented and proved in [7] such that $\mathcal{R}(S, T) = (G, k)$.

Let $G = (V, E)$. Let $V = \{u_1, \ldots, u_n, v, w_1, \ldots, w_m, x_1, \ldots, x_n, y_1, \ldots, y_m\}$ and consider the set E containing the elements

$[u_i, v]$ for $1 \leq i \leq n$,

$[v, w_i]$ for $1 \leq i \leq m$,

$[w_i, w_j]$ for $1 \leq i < j \leq m$,

$[w_i, x_j]$ for $1 \leq i \leq m$, $1 \leq j \leq n$,

$[x_i, x_j]$ for $1 \leq i < j \leq n$,

$[x_i, y_j]$ for $1 \leq i \leq n$, $1 \leq j \leq m \Leftrightarrow s_i \in T_j$,

$[y_i, y_j]$ for $1 \leq i < j \leq m \Leftrightarrow T_j \cap T_i \neq \emptyset$.

More precisely, in [7], the authors prove

- **Fact 0:** $m + n + 1$ vertices of the graph obtained by $\mathcal{R}$ have degree at least $m + n$ and all the other vertices have degree less than $m + n$.
- **Fact 1:** T contains an exact cover for S if and only if there exist two b-colorings of graph G of cardinality $m + n + 1$ and $m + n$.
- **Fact 2:** T does not contain an exact cover for S if and only if there exists one b-coloring of G of cardinality $m + n$, but no b-coloring of cardinality $m + n + 1$.
- **Fact 3:** The minimum number of colors having no b-chromatic vertices for any coloring of graph G of size $m + n + 1$ is equal to the minimum number

of elements of S not covered.

Before giving the main result of this section, we prove the following technical lemma.

Lemma 1 *Let $K'_{n,n} = (U \cup V, E)$ be the complete bipartite graph minus a perfect matching. In every proper coloring π of $K'_{n,n}$ by k colors, with $2 < k < n$, each bipartition class contains b-chromatic vertices for at most one color.*

Proof. We prove this lemma by contradiction. Assume without loss of generality that U contains b-chromatic vertices for at least two colors. This implies that $|\pi(V)| = k$. Since $k < n$, at least one vertex of U would have all the k colors in its neighborhood. This contradicts the fact that π is a proper coloring.

Theorem 2 *The problem* B-CERNABLE *is NP-complete.*

Proof. Problem B-CERNABLE is in NP: given a graph G, for each integer k between $\chi(G)$ and $b(G)$, a non-deterministic polynomial time algorithm can determine if there exists a (k)b-coloring of G.

The proof involves a transformation $\mathcal{W}$ from the NP-complete problem X3C. We provide the transformation $\mathcal{W}$ from the X3C problem using the transformation $\mathcal{R}$ presented in [7] given above. We firstly describe the polynomial transformation $\mathcal{W}$, where $\mathcal{W}(S,T) = (G, \chi(G), b(G))$. We describe later a $(\chi(G))$b-coloring and $(b(G))$b-coloring of G.

Transformation $\mathcal{W}$:

- **Input:** An instance of X3C, i.e., $S = \{s_1, s_2, \cdots, s_n\}$, and a collection $T = \{T_1, T_2, \cdots, T_m\}$ where $\forall i,\ |T_i| = 3, T_i \subseteq S$.
- **Output:** A graph G, $\chi(G)$, and $b(G)$.

(1) $S' = \{s'_1, s'_2, \cdots, s'_{3n}\}$, $T' = \{T'_1, T'_2, \cdots, T'_{3m}\}$ where
for $1 \leq i \leq m$, $T'_i = \{s'_j, s'_k, s'_\ell\}$, $T'_{i+m} = \{s'_{j+n}, s'_{k+n}, s'_{\ell+n}\}$, and $T'_{i+2m} = \{s'_{j+2n}, s'_{k+2n}, s'_{\ell+2n}\}$ where $T_i = \{s_j, s_k, s_\ell\}$
(2) $\mathcal{R}(S', T')(G', k')$, where $k' = 3n + 3m + 1$.
(3) $F = \zeta(G', (\{u\}, \emptyset))$, where $(u, \emptyset)$ is the graph with a single vertex u.
(4) $B = \zeta((\{u\}, \emptyset), K'_{3(m+n)+2, 3(m+n)+2})$.
(5) $G = F \cup B$ (with a common vertex u).
(6) return $(G, 3m + 3n + 1, 3m + 3n + 3)$.

(The obtained graph G is shown in Fig. 3. In Fig. 3 graph G' is the resulting graph from transformation $\mathcal{R}$ and $a = 3m + 3n + 2$.)

Fig. 3. The resulting graph G from the transformation $\mathcal{W}$

First we prove that $\chi(G) = 3m + 3n + 1$ and that $b(G) = 3m + 3n + 3$; we also describe a $(\chi(G))$b-coloring and a $(b(G))$b-coloring of G.

By the definition of $\mathcal{W}$, the graph G' obtained via $\mathcal{R}$ contains a clique of size $3m + 3n$. Thus $\zeta(G', (\{u\}, \emptyset))$ contains a clique of size $3m + 3n + 1$, and $\chi(G) \geq 3m + 3n + 1$.

From Facts 1 and 2, G' has a $(3m + 3n)$b-coloring and from Proposition 1, $\zeta(G', (\{u\}, \emptyset))$ has a $(3m+3n+1)$b-coloring r. Moreover, from Proposition 1 and Proposition 2, $\zeta((\{u\}, \emptyset), K'_{3(m+n)+2,3(m+n)+2})$ has only two b-colorings c' and c'' using respectively 3 and $3m + 3n + 3$ colors. Assume without loss of generality that $c'(u) = c''(u) = r(u)$. Graph G has $(3m+3n+1)$b-coloring h, defined as follows:

- $h(v) = c'(v)$ if $v \in V(\zeta((\{u\}, \emptyset), K'_{3(m+n)+2,3(m+n)+2}))$.
- $h(v) = r(v)$ otherwise.

It is clear that h is a $(3m+3n+1)$-coloring of G. So $\chi(G) = 3m+3n+1$ and h is a $(3m + 3n + 1)$b-coloring of G. Moreover G has $(3m + 3n + 3)$b-coloring h' defined by :

- $h'(v) = c''(v)$ if $v \in V(\zeta((\{u\}, \emptyset), K'_{3(m+n)+2,3(m+n)+2}))$.
- $h'(v) = r(v)$ otherwise.

It is easy to see that coloring h' is a $(3m + 3n + 3)$b-coloring. From Fact 0, G' contains $3m + 3n + 1$ vertices of degree at least $3m + 3n$ and the degree of all other vertices is less than $3m+3n$. Also $V(\zeta((\{u\}, \emptyset), K'_{3(m+n)+2,3(m+n)+2}))$ contains $2(3m + 3n + 2) + 1$ vertices; $2(3m + 3n + 2)$ vertices have degree $3m+3n+2$, whilst u is of degree $2(3m+3n+2)$. There are at most $3m+3n+2$ vertices of degree at least $3m + 3n + 3$, hence $b(G)$ cannot be lager than $3m + 3n + 3$ (otherwise there would be at least $3m + 3n + 4$ vertices of degree at least $3m + 3n + 3$). Recall that the instance of our problem is graph G, a $\chi(G)$b-coloring and $b(G)$b-coloring of G.

Since transformation $\mathcal{R}$ is polynomial(see [7]), transformation $\mathcal{W}$ is polynomial too. Now, we show that T contains an exact cover for S if and only if the graph G is b-continuous.

- Assume that T contains an exact cover for S. It is easy to see that T' contains an exact cover for S'. So, from Fact 1, there exists a $(3m+3n+1)$b-coloring ℓ of G'. Then $\zeta(G', (\{u\}, \emptyset))$ has a $(3m+3n+2)$ b-coloring ℓ'. Without loss of generality we assume that $\ell'(u) = 3m+3n+2$ and that $c'(v) \in \{1, 3\}$ for $v \in K'_{3(m+n)+2,3(m+n)+2}$. A $(3m+3n+2)$b-coloring c of G can be defined by:
  - $c(v) = c'(v)$ if $v \in V(\zeta((\{u\}, \emptyset), K'_{3(m+n)+2,3(m+n)+2}))$.
  - $c(v) = \ell'(v)$ otherwise.

  Thus G is b-continuous.

- Assume that graph G is b-continuous. We prove by contradiction that T contains an exact cover for S. Assume that T does not contain an exact cover for S. So T' does not contain an exact cover for S'. Let c be a $(3m+3n+2)$b-coloring of G. Vertex u is adjacent to all the other vertices in G. Thus no vertex can have the same color as u. Without loss of generality, assume that $c(u) = 0$. So, u is a b-chromatic vertex for the color 0 and there are $3m + 3n + 1$ colors for all other vertices in G.

  Since T does not contain an exact cover for S, T' does not contain an exact cover for S' and by construction, any cover of S' has at least 3 elements of S' not covered. From Fact 3, any coloring of graph G' of size $3m+3n+1$ has at least 3 colors having no b-chromatic vertex in G'. These 3 colors must have their b-chromatic vertices in $K'_{3(m+n)+2,3(m+n)+2}$. By Lemma 1, every $(3m + 3n + 1)$-coloring of $K'_{3(m+n)+2,3(m+n)+2}$ has at most two colors that have b-chromatic vertices. We reach a contradiction. And therefore T contains an exact cover for S.

5 Conclusion

References

[1] D. Barth, J. Cohen and T. Faik. Complexity of determining the b-continuity property of graphs, PR*i*SM Technical Report,2003.

[2] C. Berge. Graphs, North-Holland,1985.

[3] H. L. Bodlaender. Achromatic number is NP-complete for cographs and interval graphs, Infor. Process. Lett. 31 (1989) 135-138.

[4] T. Faik. About the b-continuity of graphs, Elect. Notes in Discrete Mathematics 17 (2004) 151-156

[5] F. Harary, S. Hedetniemi and G. Prins, An interpolation theorem for graphical homomorphisms, Portugal. Math 26 (1967) 453-462.

[6] F. Harary and S. Hedetniemi: The achromatic number of a graph, J. Combin. Th. 8 (1976) 154-161.

[7] R. W. Irving and D. F. Manlove, The b-chromatic number of a graph. Discrete Applied Mathematics 91 (1999) 127-141.

[8] P. Hell, D.J.Miller, Graph with given achromatic number, Discrete Mathematics, 16 (1976) 195-207.

[9] F. Hughes, G. MacGilliway, The achromatic number of graphs: A survey and some new results, Bull. Inst. Comb. Appl. 19 (1997) 27-56.

[10] T. Jiang, D. Mubayi, Zs. Tuza, V. Voloshin and D. B. West, The Chromatic Spectrum of Mixed Hypergraphs, Graphs and Combinatorics 18 (2002)309-318.

[11] J. Kára, J. Kratochvíl and M. Voigt, b-continuity, Preprint No. M14/04, Technichal University Ilmenau, Faculty for Mathematics and Natural Science, http://www.mathematik.tu-ilmenau.de/ṽoigt/chord.ps, July 2004.

[12] J. Kratochvíl, Zs. Tuza and M. Voigt, On the b-chromatic number of a graphs, WG 2002, LNCS 2573, (2002) 310-320.

[13] M. Kouider and M. Mahéo. Some bounds for the b-chromatic number of a graph. Discrete Mathematics, 256 (2002) 267-277.

[14] D. Manlove: Minimaximal and maximinimal optimization problems: a partial orderbased approch, PhD. thesis, University of Glasgow, Dept. of computing Science, June 1998.