Complexity of Determining the b-continuity Property of Graphs

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Abstract. In all this paper we use the graph theory definitions and notations form [1]. This paper deals with b-colorings of a graph G, i.e., proper colorings in which for each color, there exists at least one vertex it is assigned to such that each other color is assigned to at least one of its neighboor. The maximal cardinality of such a b-coloring is denoted by $b(G)$, and each proper coloring with cardinal $\chi(G)$ is a b-coloring.
We say that G is b-continuous iff for each k, $\chi(G) \leq k \leq b(G)$, there exists a b-coloring with cardinal k. It is well known that no all graphs are b-continuous. Calling b-spectrum of G the set of cardinals of all the b-colorings of G, we first show that for any integer set I, there exists a graph which b-spectrum is I. Then, we show that, even if b-colorings of cardinal $\chi(G)$ and $b(G)$ are given, the problem of knowing if G is b-continuous is NP-complete. At end, we show that interval graphs are b-continuous.

1 Introduction

A b-coloring of a graph G is a proper coloring π of the vertices of G [1] such that for each color c, there exists a vertex v with $\pi(v) = c$ such that for any color $c' \neq c$, there exists $v' \in \Gamma_G(v)$ with $\pi(v') = c'$ (where $\Gamma_G(v)$is the neighborhood of v). Such a vertex v is called a *b-chromatic vertex for c*. We define $|\pi| = |\{c : \exists v \in V(G),\ \pi(v) = c\}|$ the *cardinality* of the coloring. If $|\pi| = k$, then π is called a (k)*b-coloring*.

The b-coloring of graphs has been defined in [2] from the a-coloring of graphs [5] (i.e., a proper coloring π in G in which for each pair of colors c and c', there exists an edge $[u, v]$ in G such that $\pi(u) = c$ and $\pi(v) = c'$). Irving and Manlove also give an algebraic definition of a b-coloring [2].

Given a graph G, the b-chromatic number $b(G)$ is the greater integer k such that there exists a (k)b-coloring. Given an integer k, knowing if $b(G) \geq k$ is a NP-complete problem [2], even if G is bipartite [3]. This problem has been shown to be in P for trees, and some lower bounds of the b-chromatic number has been given for the cartesian product of two graphs [2, 4].

It is easy to see that any proper coloring with $\chi(G)$ colors (the chromatic number) is a b-coloring. One peculiar carateristic of b-colorings is that for some graphs G, there exists some integers k, $\chi(G) < k < b(G)$, for which there is no (k)b-coloring in G (see for example the hypercube $H(3)$ with $k = 3$ [2]). As far as we know, this is the first coloring definition with such a characteristic. Thus, we say that a graph G is *b-continuous* iff for any k, $\chi(G) \leq k \leq b(G)$, there exists a (k)b-coloring in G. This carateristic has been first studied in the paper of [3] in which they give an infinite class of graphs being not b-continuous.

We focus here on the b-continuity of graphs. We thus define the *b-spectrum* $S_b(G)$ of a graph G as the set of integers k such that there exists a (k)b-coloring of G. In this paper we mainly answer two questions :

1. For any subset of integers I, does there exist a graph G with $S_b(G) = I$?

2. Is the problem of knowing if a given graph G is b-chromatic be NP-complete?

The first question is asked in [3]. We define a composition of graphs with which, from elementary bipartite and complete graphs, for any integer set I we give a graph G with $S_b(G) = I$.

About the second question, given a graph G and an integer k, knowing if $k \in S_b(G)$ is NP-complete [3].Note that this does not answer question 2. We show that the problem of knowing if a given graph G is b-continuous is NP-complete. This problem also remains NP-complete if a $(\chi(G))$b-coloring and a $(b(G))$b-coloring are given. We then study some classes of graphs in which determining the chromatic number and/or the b-chromatic number is easy. In this sense using a similar idea in [8], we show that interval graphs are b-continuous. Note that we know the the a-chromatic number problem is NP-complete for interval graphs [6] but that the question is still open for the b-chromatic number problem.

In Section 3 we deal with the Question 1, by showing that for any set of integers $I \subset \mathbb{N} \setminus \{0, 1\}$ there exist a graph with b-spectrum I, In Section 4 we deal with the Question 2 to show that the problem is NP-complete even if b-colorings of cardinal $\chi(G)$ and $b(G)$ are given. We conclude by showing in Section 5 that interval graphs and a special family of graphs are b-continuous.

2 Preliminary Result

Definition 1. *Let G and H be two graphs. We note by $\zeta(G, H)$ the graph defined by :*

- $V(\zeta(G, H)) = V(G) \cup V(H)$
- $E(\zeta(G, H)) = E(G) \cup E(H) \cup \{[g, h], \forall g \in V(G), \forall h \in V(H)\}$

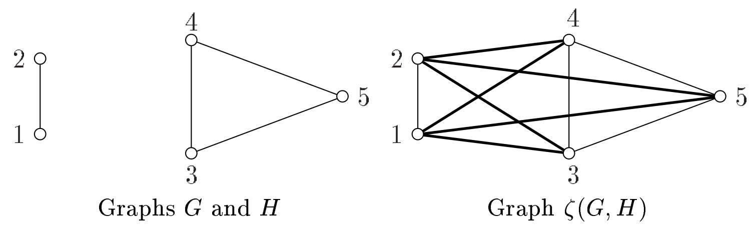

Fig. 1. Example of function ζ.

Proposition 1. *Let G and H be two graphs. The b-spectrum of $\zeta(G, H)$ is the set $\{k + k' : k \in S_b(G), k' \in S_b(H)\}$*

Proof. Let π be a (k)b-coloring of the graph G. Let π' be a (k')b-coloring of the graph H.

We construct a coloring π'' of a the graph $\zeta(G, H)$, defined by
$\forall x \in V(G),\ \pi''(x) = \pi(x)$ and $\forall x \in V(H),\ \pi''(x) = \pi'(x) + k$.

It is clear to see that π'' is a proper coloring and also a $(k + k')$b-coloring.

Let π'' be a (k'')b-coloring of the graph $\zeta(G, H)$. It is clear that the sets of colors $C_1 = \pi''(V(G))$ and $C_2 = \pi''(V(H))$ are disjoint. Thus any b-chromatic vertex for a color $c \in C_1$ (resp. $c \in C_2$) and all its neighbors of color belonging to C_1 (resp. C_2) are in $V(G)$ (resp $V(H)$). Consequently the restriction $\pi'' : V(G) \to C_1$ (resp. $\pi'' : V(H) \to C_2$ is a b-coloring of G (resp. of H), and we have $|C_1| \in S_b(G)$, $|C_2| \in S_b(H)$ with $k'' = |C_1| + |C_2|$. $\square$

We denote by $K'_{n,n}$ the graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.

Proposition 2. *[3] The b-spectrum of $K'_{n,n}$ is the set $\{2, n\}$.*

3 Graphs with a given b-spectrum

Theorem 1. *For any finite nonempty set $I \subset \mathbb{N} \setminus \{0, 1\}$ there exists a graph G such that $S_b(G) = I$*

Proof. To prove the theorem it suffices to consider sets I with $min(I) = 2$. Indeed, for $min(I) = \alpha > 2$, if there exists a graph G such that $S_b(G) = \{2, n_1-(\alpha-2), \ldots, n_p-(\alpha-2)\}$ then by Proposition 1 we have $S_b(\zeta(G, K_{\alpha-2})) = \{\alpha, n_1, \ldots, n_p\}$.

Thus we only consider three cases function of the cardinality of I.

Case 1: For $I = \{2\}$, it suffices to consider $G = K_2$.

Case 2: For $I = \{2, n\}$, by Proposition 2 the b-spectrum of $K'_{n,n}$ is $\{2, n\}$.

Case 3: $I = \{2, n_1, \ldots, n_p\}$, with $2 < n_1 < \ldots < n_p$ and $p \geq 2$. We Consider a graph G being a bipartite graph with n_p independant vertex set. this graph $G = (\cup_{i=0}^{p} V_i, \cup_{i=1}^{p} E_i)$ is defined as follows (see Fig 2) :

1. $V_0 = \{v_0^1, \ldots, v_0^{n_p}\}$, $V_p = \{v_p^1, \ldots, v_p^{n_p}\}$,
2. $\forall i \in \{1, \ldots, p-1\}$, $V_i = \{v_i^1, \ldots, v_i^{n_i-1}\}$,
3. $\forall \ell, j$, with $(1 \le j \le n_p)$ and $(1 \le \ell \le n_p)$, $v_0^\ell v_p^j \in E_p \Leftrightarrow (\ell \ne j)$
4. $\forall i \in \{1, \ldots, p-1\}$, $\forall \ell, j$, with $(2 \le j \le n_i - 1)$ and $(1 \le \ell \le n_p)$, $v_0^\ell v_i^j \in E_i \Leftrightarrow (\ell \ne j)$
5. $\forall i \in \{1, \ldots, p-1\}$, $\forall \ell,$, $v_0^\ell v_i^1 \in E_i \Leftrightarrow (2 \le \ell \le n_i - 1)$

We can check that the graph G satisfying the following properties :

a. Any edge has one and only one endpoint in V_0 (see 3,4,5).
b. The graph induced by $V_0 \cup V_{n_p}$ is a K'_{n_p,n_p} (see 3).
c. $\forall i \in \{1, \ldots, p-1\}$, the graph induced by $V_i \cup \{v_0^1, \ldots, v_0^{n_i-1}\}$ is a K'_{n_i-1,n_i-1} (see 4).
d. $\forall i \in \{1, \ldots, p-1\}$, the graph induced by $\{v_0^{n_i}, \ldots, v_0^{n_p}\} \cup (V_i \setminus \{v_i^1\})$ is the complete bipartite graph $K_{n_p-(n_i-1),n_i-2}$ (see 4 and 5).
e. $\forall i \in \{1, \ldots, p-1\}$, there is no edge lacking a vertex in the set $\{v_0^1, v_0^{n_i}, \ldots, v_0^{n_p}\}$ with v_i^1 (see 4).

In the remaining of this section we show that the b-spectrum of G is I.

By (a.), G is bipartite. Hence $\chi(G) = 2$ and thus any 2-coloring of G is a b-coloring. Since $\Delta(G) + 1 = n_p$ there is no b-coloring of G with more than n_p colors.

First, $\forall i \in \{1, \ldots, p-1\}$, we define a b-coloring π_i of G using n_i colors.

Let $i_0 \in \{1, \ldots, p-1\}$. Consider π_{i_0} the following coloring :

- $\forall j \in \{1, \ldots, n_{i_0} - 1\}$, $\pi_{i_0}(v_0^j) = \pi_{i_0}(v_{i_0}^j) = j$
- $\forall j \in \{n_{i_0}, \ldots, n_p\}$, $\pi_{i_0}(v_0^j) = 1$
- $\forall i \in \{1, \ldots, p\}$, with $i \ne i_0$, $\forall v \in V_i, \pi_{i_0}(v) = n_{i_0}$

Obviously $\forall i \in \{1, \ldots, p-1\}, \pi_i$ is a b-coloring of G. The b-chromatic vertices are v_0^j, $\forall j \in \{1, \ldots, n_{i_0} - 1\}$ and $v \in V_i$ with $i > i_0$.
Let π_p be the following coloring of G.

- $\forall j \in \{1 \ldots n_p\}$, $\pi_p(v_0^j) = \pi_p(v_p^j) = j$
- $\forall i, 1 \le i < p$, $\forall j \in \{1, \ldots, n_i - 1\}$, $\pi_p(v_i^j) = j$

Obviously π_p is a (n_p)b-coloring of G.

So, we have shown that $I \subseteq S_b(G)$. It remains to prove the equality between these sets. It suffices to show that for any (k)b-coloring π of G, $k < n_1$ implies that $k = 2$ and $n_r \le k < n_{r+1}$, with $r \ge 1$ implies $k = n_r$.

Claim. If there exists a (k)b-coloring of G such that $k < n_p$, then exactly one color is not in $\pi(V_0)$ and all the other colors have all their b-chromatic vertices in V_0.

Indeed, since $k < n_p$ and the graph induced by $V_0 \cup V_p$ is K'_{n_p,n_p}, thus at least one color is not in $\pi(V_0)$. Otherwise there exists at least one vertex in V_p with all the k colors in its neighbohood. Hence there is no available color for this vertex.

Since every edge has one and only one endpoint in V_0, then if at least two colors have b-chromatic vertices in $\cup_{i=1}^{p} V_i$ then all the k colors are in $\pi(V_0)$ which is not possible as described above. So the b-chromatic vertices of exactly $k-1$ colors are in V_0.

Suppose first that $k < n_1$. Since the graph induced by $V_0 \cup V_p$ is K'_{n_p,n_p}, and $k < n_1 < n_p$, then at least one color is not in $\pi(V_0)$. Without loss of generality we may suppose $k \notin \pi(V_0)$. Therefore all the b-chromatic vertices of colors 1 to $k-1$ are in V_0. As $k-1 < n_1 - 1$, at least two vertices of the set $\{v_0^1, \ldots, v_0^{n_1-1}\}$ have the same color, say for instance color $k-1$. By construction, no vertex of $V(G) \setminus V_0$ can be of color $k-1$, therefore there is no b-chromatic vertex of color $k-2$ at all, and k must be equal to 2 in order that π be a b-coloring.

Suppose now that $n_r \leq k < n_{r+1}$, with $r \geq 1$. By the claim given above, at least one color lacks in V_0, say $k \notin \pi(V_0)$. The b-chromatic vertices of colors 1 to $k-1$ are all in V_0. As above, at least two vertices of $\{v_0^1, \ldots, v_0^{n_{r+1}-1}\}$ have the same color, say color 1. By construction, no vertex of V_j with $j \geq r+1$, can have color 1, and the only vertex of V_j with $j \leq r$ which may accept this color is the vertex v_j^1. Therefore the b-chromatic vertices of colors 2 to $k-1$ are in the set $\{v_0^2, \ldots, v_0^{n_r-1}\}$, hence $k-2 \leq n_r - 2$, so $k = n_r$.

Hence the equality $S_b(G) = I$ is established. □

Figure 2 gives a graph with b-spectrum $\{2, 4, 6\}$.

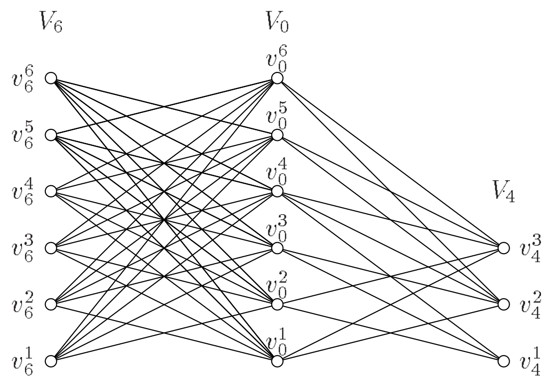

Fig. 2. A graph with b-spectrum $\{2, 4, 6\}$.

4 NP-completeness results

In [2] the authors show that the following problem called B-CHROMATIC NUMBER Problem is NP-complete.

B-CHROMATIC NUMBER **Problem**
Instance: a graph G and an integer k

Question: Does there exist a b-coloring of size c such that $k \leq c$?

Problem X3C may be defined as follows:

Exact Cover by 3-Sets (X3C)
Instance: A set $S = \{s_1, s_2, \cdots, s_n\}$ and a collection $T = \{T_1, T_2, \cdots, T_m\}$ where $\forall i,\ |T_i| = 3$
Question: Does T contain an exact cover for S, i.e, is there a set T' ($T' \subset T$) of pairwise disjoint sets whose union is S?

A polynomial transformation $\mathcal{R}$ is provided. $\mathcal{R}$ transforms any instance of X3C defined by (S,T) into an instance of B-CHROMATIC NUMBER Problem defined by (G,k).

Let $I = (S,T)$ be an arbitrary instance of the X3C problem with $S = \{s_1, s_2, \cdots, s_n\}$, and $T = \{T_1, T_2, \cdots, T_m\}$. We describe the polynomial transformation $\mathcal{R}$ see [2], such that $\mathcal{R}(S,T) = (G,k)$.

Let $G = (V,E)$. Let $V = \{u_1, \ldots, u_n, v, w_1, \ldots, w_m, x_1, \ldots, x_n, y_1, \ldots, y_m\}$ and consider the set E containing the elements

$[u_i, v]$ for $1 \leq i \leq n$,
$[v, w_i]$ for $1 \leq i \leq m$,
$[w_i, w_j]$ for $1 \leq i < j \leq m$,
$[w_i, x_j]$ for $1 \leq i \leq m$, $1 \leq j \leq n$,
$[x_i, x_j]$ for $1 \leq i < j \leq n$,
$[x_i, y_j]$ for $1 \leq i \leq n$, $1 \leq j \leq m \Leftrightarrow s_i \in T_j$,
$[y_i, y_j]$ for $1 \leq i < j \leq m \Leftrightarrow T_j \cap T_i \neq \emptyset$.

More precisely, in [2], the authors prove

- **Fact 0:** $m + n + 1$ vertices of the graph obtained by $\mathcal{R}$ have degree at least $m + n$ and all the other vertices have degree less than $m + n$.
- **Fact 1:** T contains an exact cover for S if and only if there exists two b-colorings of graph G of cardinal $m + n + 1$ and $m + n$.
- **Fact 2:** T does not contain an exact cover for S if and only if there exists one and only b-coloring of G of cardinal $m + n$.
- **Fact 3:** The minimum number of colors having no b-chromatic vertices for any coloring of graph G of size $m + n + 1$ is equal to the minimum number of elements of S not covered.

The purpose of this section is to prove that the Problem B-CERNABLE is NP-complete.

B-CERNABLE **Problem**
Instance: a graph G, $\chi(G)$, and $b(G)$.
Question: Is G b-continuous?

Before, giving the main result of this section, we show the following technical lemma.

Lemma 1. *Let $K'_{n,n} = (U \cup V, E)$ be the complete bipartite graph minus a perfect matching. In every coloring π of $K'_{n,n}$ by k colors, with $2 < k < n$,each bipartition classes contains a b-chromatic vertices for at most one color.*

Proof. Suppose the contrary. By symmetry, say U contains b-chromatic vertices for at least two colors. This implies that $|\pi(V)| = k$. Since $k < n$, at least one vertex of U would have all the k colors in its neighborhood, a contradiction. □

Theorem 1. *The problem* B-CERNABLE *is NP-complete.*

Before proving Theorem 1, we give the upper bound of $b(G)$ presented in [2]. Assume that the vertices $v_1, \ldots, v_n$ of G are ordered such that $d(v_1), \ldots, d(v_n)$, where $d(x)$ denotes the degree of x. Let $t(G) := max\{i : d(v_i) \geq i - 1\}$. Then $t(G) \geq b(G)$.

Proof. Problem B-CERNABLE is in NP. Since for a graph G, for each integer k between $\chi(G)$ and $b(G)$, a non-deterministic polynomial time algorithm can determine if there exists a (k)b-coloring of G.

The proof involves a transformation $\mathcal{W}$ from the NP-complete problem X3C. We provide the transformation $\mathcal{W}$ from the X3C problem using the transformation $\mathcal{R}$ described in [2] given above. We describe the polynomial transformation $\mathcal{W}$, $\mathcal{W}(S,T) = (G, \chi(G), b(G))$.

- **Input: An instance of X3C, i.e.,** $S = \{s_1, s_2, \cdots, s_n\}$**, and a collection** $T = \{T_1, T_2, \cdots, T_m\}$ **where** $\forall i,\ |T_i| = 3, T_i \subseteq S$**.**
- **Output: A graph** G**,** $\chi(G)$**, and** $b(G)$**.**

1. $S' = \{s'_1, s'_2, \cdots, s'_{3n}\}$ and $T' = \{T'_1, T'_2, \cdots, T'_{3m}\}$
   for $1 \leq i \leq m$, if $T_i = \{s_j, s_k, s_\ell\}$ then $T'_i = \{s'_j, s'_k, s'_\ell\}$, $T'_{i+m} = \{s'_{j+n}, s'_{k+n}, s'_\ell + n\}$, and $T'_{i+2m} = \{s'_{j+2n}, s'_{k+2n}, s'_\ell + 2n\}$
2. $\mathcal{R}(S', T') = (G', k' = 3n + 3m + 1)$.
3. $F = \zeta(G', (\{u\}, \emptyset))$
4. $B = \zeta((\{u\}, \emptyset), K'_{3(m+n)+2, 3(m+n)+2})$.
5. $G = F \cup B$ (with a common vertex u)
6. return $(G, 3m + 3n + 1, 3m + 3n + 3)$

(The obtained graph G is shown in Fig. 3. In Fig. 3 the graph G' is the resulting graph from the transformation $\mathcal{R}$ and $a = 3m + 3n + 2$.)

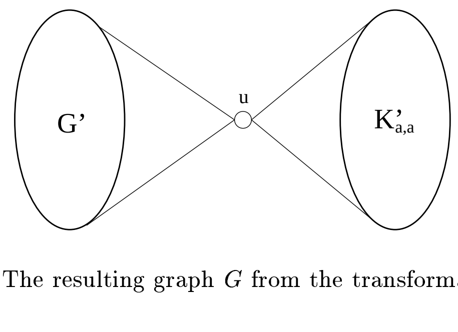

Fig. 3. The resulting graph G from the transformation W

Since the transformation $\mathcal{R}$ is polynomial, the transformation $\mathcal{W}$ is polynomial too. Now, we show that T contains an exact cover for S if and only if the graph G is b-continuous .

First we prove that $\chi(G) = 3m + 3n + 1$ and that $b(G) = 3m + 3n + 3$.

Obviously G' obtained by $\mathcal{R}$ contains a clique of size $3m+3n$. Thus $\zeta(G', (\{u\}, \emptyset))$ contains a clique of size $3m + 3n + 1$, and so $\chi(G) \geq 3m + 3n + 1$. From Facts 1 and 2, the graph G' obtained by $\mathcal{R}$ has a $(3m+3n)$b-coloring and from Proposition 1, the graph $\zeta(G', (\{u\}, \emptyset))$ has a $(3m+3n+1)$b-coloring r. Moreover, from Proposition 1 and Proposition 2 the graph $\zeta((\{u\}, \emptyset), K'_{3(m+n)+2,3(m+n)+2})$ has only two b-colorings c' and c'' of size resptively 3, $3m+3n+3$. W.l.o.g.. Assume that $c'(u) = c''(u) = r(u)$. The graph G has $(3m + 3n + 1)$b-coloring d, defined as follows:

- $d(v) = c'(v)$ if $v \in V(\zeta((\{u\}, \emptyset), K'_{3(m+n)+2,3(m+n)+2}))$.
- $d(v) = r(v)$ otherwise.

It is clear that d is a $(3m + 3n + 1)$-coloring of G. So $\chi(G) = 3m + 3n + 1$ and d is a $(3m + 3n + 1)$b-coloring of G. The graph G has $(3m + 3n + 3)$b-coloring d' defined by :

- $d'(v) = c''(v)$ if $v \in V(\zeta((\{u\}, \emptyset), K'_{3(m+n)+2,3(m+n)+2}))$.
- $d'(v) = r(v)$ otherwise.

It is easy to see that coloring d' is a $(3m + 3n + 3)$b-coloring. From Fact 0, G' contains $3m+3n+1$ vertices of degree $3m+3n$ and all other vertices have degree less than $3m + 3n$, and it is obvious that $V(\zeta((\{u\}, \emptyset), K'_{3(m+n)+2,3(m+n)+2}))$ contains $2(3m+3n+2)+1$ vertices of degree $3m+3n+2$. Hence $t(G) = 3m+3n+3$, so $|d'| = b(G)$.

- Assume that T contains an exact cover for S. So, from Fact 1, there exists a $(3m + 3n + 1)$b-coloring ℓ of G'. Then $\zeta(G', (\{u\}, \emptyset))$ has a b-coloring ℓ' of size $3m + 3n + 2$, and $(3m + 3n + 2)$b-coloring c of G can be defined by:
  - $c(v) = c'(v)$ if $v \in V(\zeta((\{u\}, \emptyset), K'_{3(m+n)+2,3(m+n)+2}))$.
  - $c(v) = \ell'(v)$ otherwise.

  Thus G is b-continuous .
- Assume that graph G is b-continuous .
  We prove by contradiction that T contains an exact cover for S. Suppose that T does not contain an exact cover for S. So, by definition, T' does not contain an exact cover for S'. Let c be a $(3m + 3n + 2)$b-coloring of G.
  The vertex u is adjacent to all the other vertices in G. So no vertex can be of the same color as u. Assume that $c(u) = 0$. So, u is a b-chromatic vertex for the color 0 and there are $3m + 3n + 1$ colors for all other vertices in G. Since T does not contain an exact cover for S, T' does not contain an exact for S' and by construction, any cover of S' has at least 3 elements of S' not covered. From Fact 3, any coloring of graph G' of size $3m + 3n + 1$ has at least 3 colors having no b-chromatic vertex in G'. So, these 3 colors must have their b-chromatic vertices in $K'_{3(m+n)+2,3(m+n)+2}$, but by Lemma 1 in

every $(3m+3n+1)$-coloring of $K'_{3(m+n)+2,3(m+n)+2}$ at most two colors may have b-chromatic vertices , a contradiction So, T contains an exact cover for S. $\square$

A similar proof can be used to show that Given only a graph G the problem of knowing if G is b-continuous is NP-complete (see The Apendix).

5 b-continuity of some sets of graphs

5.1 Interval graphs

In this section we prove that some classes of graphs are b-continuous. Let G be a graph. If one can constuct an algorithm which reduce any (k)b-coloring, for any $k > \chi(G)$, to a $(k-1)$b-coloring, than G is b-continuous.

Definition 1. *A graph $G = (V, E)$ is an interval graph, if one can associate to each vertex $v \in V$ an interval $[i_x, s_x] \subseteq \mathbb{R}$, such that $(x, y) \in E \Leftrightarrow [i_x, s_x] \cap [i_y, s_y] \neq \emptyset$.*

Definition 2. *Let $G = (V, E)$ be an interval graph and $\pi : V(G) \to \{1, \ldots, p\}$ a coloring of G. Let $S \subset V(G)$ a set of vertices. We call a vertex $x \in S$ left-maximum in S, if for any $y \in S$, we have $i_y \leq i_x$. Moreover, a color $k \in \pi(S)$ will be said left-maximum in S, if for any $x \in S$ with $\pi(x) = k$ and any $j \in \pi(S)$ there exists a vertex $y \in S$ with $\pi(y) = j$ and $i_y \leq i_x$.*

Theorem 1. *Every interval graph G is b-continuous*

Proof. Let π be a b-coloring of G with p colors, $p > \chi(G)$. We give an algorithm which reduces π to a b-coloring π' of G using only $p-1$ colors. Recall that the extended neighborhood $\Gamma_G[x]$ of a vertex x results from the adjonction of x itself to its ordinary neighborhood $\Gamma_G(x) = \{y \in V \ : \ (x, y) \in E\}$.

Let X be the set of b-chromatic vertices for the coloring π. First we shall apply the following subroutine, in order to eliminate the b-chromatic vertices for at least one color :

Pruning X

While $|\pi(X)| = p$ **do**

- (a) Choose a left-maximum vertex $x \in X$. Call $S(x) = \{y \in V(G) \ : \ i_y > i_x\}$, $\Gamma'_G[x] = \Gamma_G[x] \setminus S(x)$ and $S'(x) = \{y \in S(x) \ : \ \pi(y) \notin \pi(\Gamma'_G[x])\}$.
- (b) Select a color k which is left-maximum in $S'(x)$, and color each vertex having this color in $S'(x)$ with another color, in such a way that the coloring π remains a good one.
- (c) Delete from X the vertices which are no more b-chromatic for the (modified) coloring π.

od.

Before going further we must show that this subroutine is consistent, and that it gives as output a good coloring c for which at least one color has lost all its b-chromatic vertices, *i.e.* for which $|\pi(X)| < p$.

As regards part (a), note that by definition of a left-maximum vertex, no vertex in $S(x)$ is b-chromatic. Also, the set $\Gamma'_G[x]$ is a clique of G, since every interval I_y with $y \in \Gamma_G[x]$ contains i_x. Therefore we have $|\Gamma'_G[x]| \leq \chi(G) < p$, and the set $\{1,\ldots,p\} \setminus \pi(\Gamma'_G[x])$ is nonempty. But, since vertex x is b-chromatic, the colors of the last set must be found in $S(x)$ and the set $S'(x)$ also is nonempty.

As regards part (b), since $y \in S'(x)$ is not b-chromatic, we have $|\pi(\Gamma_G[y])| < p$ and the recoloring of y is possible. Note also that the set of vertices in $S'(x)$ having color k is a stable set of G, so we may recolor these vertices independently.

As regards part (c), color k is no longer present in $\Gamma_G[x] \subset \Gamma'_G[x] \cup S(x)$, so at least x may be deleted from X, but we must prove that no new b-chromatic vertex can arise from the recoloring of part (b). This is clear for any vertex $y \in S(x)$ since $\Gamma_G[y] \subset \Gamma'_G[x] \cap S(x)$ in which color k no longer appears. But it could happen that some vertex $y \in \Gamma'_G[x]$ keeps a neighbor of color k outside of the previous set, while its neighbors of the same color in $S'(x)$ were recolored in part (b). Let therefore $y \in \Gamma'_G[x]$ and $z \in \Gamma_G[y] \cap S'(x)$ with $\pi(z) = k$ before recoloring, and $\pi(z) = j \neq k$ afterwards. It suffices to prove that j is not a new color in the neighborhood of y, which is easy if this color already was in $\pi(\Gamma'_G[x])$ since $\Gamma'_G[x] \subset \Gamma_G[y]$. Otherwise, $\{j,k\}$ is a subset of $\pi(S'(x))$. By the left-maximality of k there is a vertex $z' \in S'(x)$, already of color j before step (b), with $i_{z'} \leq i_z$, implying that z' is in $\Gamma_G[y]$, and we are done.

After the last application of the previous subroutine, we get a new set X of b-chromatic vertices such that at least one color lacks in $\pi(X)$. We have to consider two cases.

Case 1. We obtain $|\pi(X)| = p - 1$. So only one color, say k, lost all its b-chromatic vertices at the end of the previous subroutine. It is now easy, for the same reason as given above for part (b), to recolor all the vertices remaining in V with color k, by another color, and obtain by this way a b-coloring π' of G using only $p-1$ colors, namely the set $\{1,\ldots,p\} \setminus \{k\}$.

Case 2. $|\pi(X)| \leq p - 2$. Let x be the b-chromatic vertex selected in the last application of part (a), with $\pi(x) = j$. After application of (b), together with x, other b- chromatic vertices, necessarily in $\Gamma'_G[x]$, lost all their neighbors of color k. If we now give to x this color k, these vertices may reintegrate X for a coloring without j. In fact, since x was the last b-chromatic vertex of color j we may recolor all the vertices having this color, by another color, and the result is a b-coloring π' of G with set of colors $\{1,\ldots,p\} \setminus \{j\}$. □

5.2 A special family of graphs

Let G be a connected graph, $(T_v)_{v\in V(G)}$ a family of nonempty trees. For each T_v, select a vertex w_v, and consider the graph H obtained by identification of each $v \in V(G)$ with v.

Definition 3. *Let π be a b-coloring of H, we say that $v \in V(H)$ is an extreme vertex of π if it is a b-chromatic vertex such that the graph induced by $V(H) \setminus \{v\}$ has at least two connected component, and only one connected component called C_v contains all the b-chromatic vertices of π.*

Proposition 1. *If G is b-continuous, so is H*

Proof. G is b-continuous, so for any k, $\chi(G) \leq k \leq b(G)$, G has a (k)b-coloring. As trees are bipartite we may extend any (k)b-coloring of G to a (k)b-coloring of H. It is clair that $b(G) \leq b(H)$.

If $b(G) < b(H)$, it is suffices to construct an algorithm which reduce, for any k, $b(G) < k < b(H)$, a (k)b-coloring π of H to to a b-coloring π' using only $k-1$ colors. For this, if there are b-chromatic vertices outside of $V(G)$, choose an extreme vertex v of π in this set. One can easily recolor all connected components $C \neq C_v$ in such away that all the neighbors of v take the color of its neighbor in C_v. We iterate this recoloring process until one color has lost all its b-chromatic vertices or that all the b-chromatic vertices are in $V(G)$. We must now consider two cases.

Case 1. If one color c lost all its b-chromatic vertices then it is easy to recolor any vertex of this color by a missing color in its neighborhood.

Case 2. $V(G)$ contains at least one b-chromatic vertex of each color. As $b(G) < k$ there exists at least one b-chromatic vertex v such that not all the k colors appear in $\Gamma_G[v]$ and by recoloring T_v with the set $\pi(\Gamma_G[v])$, v becomes a non b-chromatic vertex. We iterate this until one color has lost all its b-chromatic vertices, so this case reduces to the previous one. $\square$

Corollary 1. *The graphs containing only one cycle are b-continuous.*

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Appendix

B-CONTINUITY **Problem**
Instance: a graph G
Question: Does G be b-continuous?

Theorem 2. *The problem* B-CONTINUITY *is NP-complete*

Proof.
First, we prove that the problem B-CONTINUITY is in NP. For a graph G, $\chi(G)$ and $b(G)$ can be computed using non-deterministic polynomial time algorithms. For each integer i between $\chi(G)$ and $b(G)$, a non-deterministic polynomial time algorithm can determine if there exists a b-coloring of G by i colors.

The proof involves a transformation $\mathcal{T}$ from the NP-complete problem X3C. We provide the transformation $\mathcal{T}$ from the X3C problem using the transformation $\mathcal{R}$ described in [2].

We suppose that $S = \{s_1, s_2, \cdots, s_n\}$, and a collection $T = \{T_1, T_2, \cdots, T_m\}$ (where $\forall i,\ |T_i| = 3, T_i \subseteq S$) is some arbitrary instance I of the X3C problem. From this instance, we obtain an instance I'' of B-CHROMATIC NUMBER problem of by using transformation $\mathcal{R}$. The instance I'' is composed of graph G and an integer k. After that, the graph $\zeta(G, H_3)$ is build where H_3 is a hypercube of 8 vertices. So, the instance of the B-CONTINUITY problem obtained by the transformation $\mathcal{T}$ is the graph $\zeta(G, H_3)$.

$$\mathcal{T}(I) = I' \text{ is if and only if}$$
$$\begin{cases} \mathcal{R}(I) = I'' & \text{where } I'' = (G, k) \text{ and} \\ I' = (\zeta(G, H_3)) & \text{where } H_3 \text{ is a hypercube of 8 vertices.} \end{cases}$$

where I is an arbitrary instance of the X3C problem such that $I = (S, T)$ with $|S| = n$ and $|T| = m$. Since the transformation $\mathcal{R}$ is polynomial, the transformation $\mathcal{T}$ is too polynomial. Now, we show that T contains an exact cover for S if and only if graph $\zeta(G, H_3)$ is b-continuous.

- Let assume that T contains an exact cover for S. So, by Fact 1, there exist two b-colorings of graph G obtained by $\mathcal{R}$ of size $m + n + 1$ and $m + n$. From Proposition 1, graph $\zeta(G, H_3)$ has four b-colorings of size $m + n + 2$, $m + n + 1 + 2$, $m + n + 4$, $m + n + 1 + 4$. So graph $\zeta(G, H_3)$ is b-continuous.
- Let assume that graph $\zeta(G, H_3)$ is b-continuous. We prove by contradiction that T contains an exact cover for S. Assume that T does not contain an exact cover for S. So, by Fact 2, there exists one b-coloring of graph G of size $m+n$. Thus, from Proposition 1, graph $\zeta(G, H_3)$ has some b-colorings of size $m + n + 2$, $m + n + 4$. As graph $\zeta(G, H_3)$ is b-continuous, it implies that graph $\zeta(G, H_3)$ has too a b-coloring of size $m + n + 3$. From Proposition 1, it implies that graph G has a b-coloring of size $m + n + 3 - 2$, $m + n + 3 - 4$ and it is a contradiction with Fact 1.so, T contains an exact cover for S. □