Gossiping in Chordal Rings under the Line Model

Lali Barrière
Dept. Mat. Aplicada i Telemàtica
UPC Barcelona **and** Johanne Cohen*
LRI, Univ. Paris-Sud
91405 Orsay cedex
France
and Margarida Mitjana
Dept. Matemàtica Aplicada I
UPC Barcelona

Abstract

This paper is devoted to the gossip (or all-to-all) problem in the chordal ring under the one-port *line model*. The line model assumes long distance calls between non neighboring processors. In this sense, the line model is strongly related to circuit-switched networks, wormhole routing, optical networks supporting wavelength division multiplexing, ATM switching, and networks supporting connected mode routing protocols.

Since the chordal rings are competitors of networks as meshes or the tori because of theirs short diameter and bounded degree, it is of interest to ask whether they can support intensive communications (typically a all-to-all) as efficiently as these networks. We propose polynomial algorithms to derive optimal or near optimal gossip protocols in the chordal ring.

1 Introduction

In the study of the properties of interconnection networks, the problem of dissemination of information is an important and a very active research area [12, 27]. Indeed, the ability of an interconnection network to effectively disseminate the information among its processors (e.q., accumulation,

*Additional support by the DRET of the DGA.

broadcast or gossip) is a “pertinent” measure to determine the best communication structures for parallel and/or distributed computers. Assume that every node of a network holds a piece of information. Broadcast is the information dissemination problem that consists, for one node of a network, to send its piece of information to all the other nodes. The accumulation problem can be considered as the reverse of broadcast problem. In the accumulation problem, every vertex has to send its piece of information to one specific vertex of the network. Finally, gossiping is a simultaneous broadcast from every node of the network. Due to their complexity, these three communication primitives are often provided at the software level. Most of the communication libraries available on parallel systems (as MPI [25]) provide access to such communication procedures. More generally, these three communication patterns are fundamental primitives used in many algorithms for the programming, and for the control of parallel and distributed systems. For example, they are used for barrier synchronization or cache coherence [29], for parallel search algorithm [7], and for linear algebra algorithms [8].

In [10], Farley introduced the model called *Line Model* which satisfies the following: **(i)** a *call* involves exactly two nodes (these two nodes can be at distance more than 1), **(ii)** any two paths corresponding to simultaneous calls must be edge-disjoint. Furthermore, Farley assumed that nodes satisfy the $1-$*port* hypothesis, that is: **(iii)** a node can take part in one call at a time. The *vertex-disjoint paths mode* can be defined analogously to the line model by replacing hypothesis (ii) by the following **(iv)** any two paths corresponding to simultaneous calls must be vertex-disjoint. The calls are subject to different possible constraints: when two nodes are involved in the same call, they can either exchange all the informations they are aware of (*full-duplex* mode) or alternatively, the information can only flow in one direction (*half-duplex* mode).

A *round* is the set of all calls carried out simultaneously. The complexity of our communication protocols will be measured by the number of communication rounds required to complete these protocols. For a given graph $G = (V, E)$, and for any arbitrary node u in G, we denote by $b(G, u)$ (resp. $a(G, u)$) the minimum number of rounds for broadcasting from the source node u (resp. for accumulation) in the graph G. Similarly, the gossip time of G, denoted by $g(G)$, is the minimum number of rounds necessary to perform a gossip in G.

In [10], Farley proved that, in the $1-$port model, broadcast from any node in any $n-$node network can be performed in $\lceil \log_2 n \rceil$ rounds. His proof makes use of routing along the edge of a spanning tree of the network.

However, the gossip problem is still open for arbitrary networks, that is, the complexity of gossiping in the $1-$port line model in arbitrary networks is not known. Hromkovic et al. [18] gave a lower bound for the gossip problem, and some results have been derived for tree-networks [9] and for planar graphs [17].

Chordal ring networks were introduced in [6]. They form a family of generalized loop networks [3]. The *chordal ring* of N vertices and chord c, denoted by $\mathbf{C}(\mathbf{N},\mathbf{c})$, is the graph with vertices labeled in $\mathbb{Z}_N$, and adjacencies given by $i \sim i \pm 1$, $i \sim i + c$ for every even vertex i. The structure of these graph has been extensively studied. For example, Arden and Lee [1] studied the problem of the maximization of the number of nodes for a given diameter, and Yebra et al. [30] found a relationship between a certain type of plane tessellation and the chordal ring. Moreover, due to their simple structure, and their short diameter, chordal ring graphs are attractive topologies for interconnection networks. Chordal ring can support compact [26] and fault-tolerance [2] routing functions. Finally, Comellas and Hell [5] presented an optimal solution for the broadcast problem in chordal ring under the telephone model.

As [5], this paper is devoted to the study of communication problems in chordal rings. In particular, our aim is to find an algorithm for the gossip problem in the full-duplex edge-disjoint paths mode since the model is appropriate to networks supporting long distance calls such as wormhole or circuit-switched routing. The next section describes the method to find the gossip time in the chordal ring. Section 3 deals with some properties of the chordal ring and, finally, in Section 4 gossiping algorithms are described in order to give the upper bound of the gossip time in chordal ring.

2 Basic concepts

An interconnection network is modeled by a connected undirected graph $G = (V, E)$, where the vertices in V correspond to the processors, and the edges in E represent the communication links of the network. Our gossip algorithms are based on the so-called 3-phase gossip method [15]. For that purpose, we conclude the section with a decomposition of the chordal ring into disjoint cycles. This decomposition is the base of all our gossip protocols of Section 4. In the full-duplex line model, Farley has shown the following:

Lemma 1 (Farley [10]) *Let G be a graph of n nodes. In the 2-way mode* *line model,*

$$b(G) = a(G) = \lceil \log_2 n \rceil.$$

Moreover, Hromkovic et al. [18] gave a lower bound for the gossip problem:

Lemma 2 (Hromkovic et al. [18]) *Let G be a graph of N nodes and of edge-bisection B. In the 2-way mode line model*

$$g(G) \geq 2\lceil \log_2 N \rceil - \log_2 B - \log_2 \log_2 N - 2.$$

In order to get upper bounds, we use the "three–phase algorithm" method as in [15]. The three–phase algorithm is composed of an accumulation phase, a gossiping phase, and a broadcasting phase.

Algorithm 1 Three-phase gossip algorithm

Divide G into r connected components containing exactly one accumulation node. These components are called *accumulation components.* $A(G)$ is the set of accumulation node.
/* *Accumulation phase* */
Each vertex $u \in A(G)$ accumulates the information from the nodes of its component.
/* *Gossip phase* */
Perform a gossip among the set $A(G)$ of accumulation nodes.
/* *Broadcast phase* */
Every node in $A(G)$ broadcasts information in its components.

To obtain an effective algorithm, we will look for a set of accumulation node such that the gossip phase can be performed as quickly as gossiping in a complete graph, and such that the size of the accumulation components is sufficiently small in order to minimize the time for the first and third phases. Moreover, these accumulation components should be connected so that the optimal $\lceil \log_2 N \rceil$-round accumulation and broadcast algorithms described in [10] (see Lemma 1) can be independently applied in all the components. For the gossip phase, our algorithms will be based on the two following algorithms 2 and 3. K_N stands for the complete graph of N nodes.

Algorithm 2 Gossiping in a K_N, N even

For j:=1 **to** $\lceil \log_2 N \rceil$ **do**
    For each vertex i, i even, **do** in parallel

exchange information between node i and node $i + 2^j - 1 \pmod N$;

Algorithm 3 Gossiping in a K_N, N odd

```
m:= ⌊N/2⌋;
For each node i, 0 < i < N/2, do in Parallel
    exchange information between node i and node i + m;
if m is odd then n' = m + 1 else n' = m + 2;
Gossip in the complete graph of vertices {0, ..., n'};
For each node i, i < N/2, do in Parallel
    exchange information between node i and node i + m;
```

In our algorithms for the chordal rings, a call between vertices i and j are replaced by a call between the accumulation node of the ith component and the accumulation node of the jth component. For a given call between the accumulation node x_i of the ith component and the accumulation node x_j of the jth component, that is for a given path $\mathcal{P}$ between x_i and x_j, the *length* of the call is defined as the number of components traversed by $\mathcal{P}$ plus one.

Now, we will present some properties of the chordal rings.

3 Definition of the chordal rings

Definition 1 *Let N be an even integer and c an odd integer between 1 and $N/2$. The* chordal ring *graph of order N and chord c, $\mathbf{C(N,c)}$, is the graph of order N, with vertices labeled in $\mathbb{Z}_N$, and adjacencies given by $i \sim i \pm 1$, $i \sim i + c$ for all even vertex i.*

Chordal ring graphs are connected and 3-regular. They are bipartite, with partition sets $V_0 = \{0, 2, 4, \ldots N-2\}$ and $V_1 = \{1, 3, 5, \ldots, N-1\}$.

For any two vertices x, y, we define $\alpha_{x,y} : \mathbb{Z}_N \to \mathbb{Z}_N$ as follows: if $x - y \equiv 0 \pmod 2$ then $\alpha_{x,y}(i) = y - x + i$, otherwise $\alpha_{x,y}(i) = y + x - i$. In both cases, $\alpha_{x,y}$ is an automorphism and it verifies $\alpha_{x,y}(x) = y$. So, $\mathbf{C(N,c)}$ is vertex-transitive.

For more details on these graphs, see [2].

In this section we present upper bounds for the edge bisection width of a chordal ring $C(N,c)$. The *edge bisection width*, B, is the minimum number of edges which separate the graph into two sets of vertices of the same cardinality.

Lemma 3 *Let* $\mathbf{C(N,c)}$ *be the chordal ring of order* N *and chord* c*. Let* B *be the edge bisection width of* $\mathbf{C(N,c)}$*, then* $B \le c+2$*.*

Proof.

First, we observe that the natural order for integer numbers gives a natural partition of the vertices, so we have an immediate bound for k :

$$\mathbb{Z}_N = [0, N/2-1] \cup [N/2, N-1] \text{ where } [a,b] = \{a, a+1, \ldots, b\}.$$

Let $A = \{(i,j) \in E | \, i \in [0, N/2-1], \, j \in [N/2, N-1]\}$ be the set of edges between $[0, N/2-1]$ and $[N/2, N-1]$. Clearly, from the definition of the edge bisection width, we have that the number of edges of A is an upper bound for B.

For $N/2$ even we have
$A = \{(N-c+1, 1), (N-c+3, 3), \ldots, (N-2, c-2)\} \cup$
$\cup \{(N/2-c+1, N/2+1), (N/2-c+3, N/2+3), \ldots, (N/2-2, N/2+c-2)\} \cup$
$\cup \{(N-1, 0), (N/2-1, N/2)\}$.
Thus $|A| = (c-1)/2 + (c-1)/2 + 2 = c+1 \ge B$.

For $N/2$ odd, we apply the same argument as previously and we get $|A| = c+2 \ge B$.

$\square$

¿From Lemmas 3 and 2, we can deduce a lower bound of the gossip time in chordal rings $\mathbf{C(N,c)}$.

Lemma 4 *Let* $\mathbf{C(N,c)}$ *be a chordal ring of* N*. In the 2-way mode line model*

$$g(G) \ge 2\lceil \log_2 N \rceil - \log_2(c+2) - \log_2 \log_2 N - 2.$$

3.1 Decomposition of the chordal rings into cycles

Let us introduce a decomposition of the chordal ring $\mathbf{C(N,c)}$ of order N and chord c that will be particularly helpful for the design of our gossip algorithm. $\mathbf{C(N,c)}$ can be decomposed into a union of cycles $C_0, \ldots, C_{\lfloor \frac{N}{c+1} \rfloor - 1}$:

$$C_i = \{i(c+1), i(c+1)+1, \ldots, i(c+1)+c\}.$$

If $N = a(c+1)+b$, then the chordal ring $\mathbf{C(N,c)}$ consists of a cycles of length $c+1$, labeled from 0 to $a-1$, and a path of b vertices, b even.

Given a cycle C_i, a vertex is said to be the jth vertex of cycle C_i, $j = 0, \ldots, c$, if its label is equal to $i(c+1)+j$. Note that the vertex 0 of C_i is connected to vertex 1 and c of C_i. Moreover, for each i, $0 \leq i < \lfloor \frac{N}{c+1} \rfloor - 1$, cycles C_i and C_{i+1} share $(c+1)/2$ edges. Among these edges, $(c-1)/2$ edges are of type $[x, x+c]$ and one edge is of type $[x, x+1]$. In other words, for any even value j, the jth vertex in C_i is adjacent to the $j-1$st vertex in C_{i+1} by a chordal edge. See the decomposition of $\mathbf{C(64, 7)}$ in Fig. 1.

4 Gossip Algorithm for the chordal ring $\mathbf{C(N, c)}$

In this section, we describe a polynomial time algorithm to compute an optimal communication scheme for gossiping in any chordal ring $\mathbf{C(N, c)}$. We consider two cases: $N = a(c+1)$ and $N = a(c+1)+b$ where $2 \leq b \leq c-1$. For these two cases, we split the graph into a certain number of components, say δ components, such that the number of vertices of each component is almost equal to $\lceil N/\delta \rceil$. Then, we explain how to choose the accumulation node for each component. Finally, we set the paths corresponding to calls such that the gossip phase between the accumulation nodes perform almost as quickly as gossiping in the complete graph.

By convention, the integers r and s represent respectively the number of components, and their maximum size.

4.1 A simple case: $N = a(c+1)$

Since the graph is composed of a cycles of $c+1$ vertices (see Section 3.1), the components are naturally defined by groups of these cycles. Actually, there are two types of decompositions according to the parameter a:

- If $a \leq c+1$, then there are a components ($r = a$): each component is a cycle of $c+1$ vertices ($s = c+1$).
- If $a > c+1$, then there are $c-1$ components ($r = c-1$). More precisely, assuming that $a = \alpha(c-1)+\beta$, β of these components are an union of $\alpha+1$ cycles of $c+1$ vertices, and $c-1-\beta$ of these components are an union of α cycles of $c+1$ vertices ($s = (c+1)(\alpha+1)$).

We label the components between 0 and $r-1$. In the kth component, the cycles are labeled between 1 and Γ, where Γ is the number of cycles in the component k. Thus, the vertex j of the ith cycle in the kth component is denoted by (k, i, j). The kth component has $(k, 1, R = c-1)$ as accumulation node.

Since each component is connected, the accumulation phase and the broadcast phase can be performed using the algorithm in [10]. Now, we focus on the gossip phase, and we are interested in a path $\mathcal{P}_{i \to i+\ell \pmod r}$ corresponding to a call between the two accumulation nodes of component i and $i+\ell \pmod r$, $i = 0, \ldots, r-1$ and $\ell = 1, \ldots, R/2$. We consider two cases.

Case 1. The number of components is even ($r = 2r'$): The path $\mathcal{P}_{i \to i+\ell}$ is an union of paths denoted, $P(i,0), P(i,1), \ldots, P(i,\ell)$ such that $P(i,k)$ is as follow (in the following we set $R = c-1$):

1. if $0 \le k \le \ell - 1$, then we consider two cases:
   - if the component $i+k$ is composed of one cycle of $c+1$ vertices, then the path $P(i,k)$ is $\{(i+k, 1, R-2k), (i+k+1, 1, R-2k-1), (i+k+1, 1, R-2k-2)\}$
   - if the component $i+k$ is composed of Γ cycles of $c+1$ vertices, then the path $P(i,k)$ is $\{(i+k, s, R-2k), (i+k, s+1, R-2k-1) | s = 1, \ldots, \Gamma-1\} \cup \{(i+k, \Gamma, R-2k-1), (i+k+1, 1, R-2k-2)\}$.
2. if $k = \ell$, then the path $P(i,\ell)$ is a path of component $i+\ell$, from vertex $(i+\ell, 1, R-2\ell+1)$ to $(i+\ell, 1, R)$ passing through vertex $(i+\ell, 1, 0)$.

Note that, at any round of the algorithm 2 applied to the accumulation nodes, the calls have the same length, and there are calls between two accumulation nodes of components whose labels have a different parity. Let us prove that the calls are pairwise edge–disjoint.

Lemma 5 *Let $\epsilon = 0$ or 1. For any odd value of ℓ, $\ell < (c-1)/2$, all calls $\mathcal{P}_{i \to i+\ell}$ between accumulation nodes of the component $2i+\epsilon$ and accumulation nodes of the component $2i+\epsilon+\ell$, $i = 0, \ldots, r/2$, are edge disjoint.*

Proof. Let Γ_k be the number of cycles of $c+1$ vertices in the component k. We focus on the calls that cross the component k.

Assume w.l.g. that $\epsilon = 0$. Assume $k = 2i+\ell$, where $0 \le i \le r/2$. The call $\mathcal{P}_{k-1 \to k-1+\ell}$ crosses component k by definition (see case 1). Moreover, $\mathcal{P}_{k-1 \to k-1+\ell}$ contains vertices in $\{(k, s, R-1), (k, s, R-2) | s = 1, \ldots, \Gamma_k\}$. The call $\mathcal{P}_{k-3 \to k-3+\ell}$ also crosses component k. Moreover, $\mathcal{P}_{k-3 \to k-3+\ell}$ contains vertices in $\{(k, s, R-5), (k, s, c-6) | s = 1, \ldots, \Gamma_k\}$. It is easy to prove by induction that, for any odd integer h in $\{0, \ldots, \ell-1\}$, $\mathcal{P}_{k-h \to k-h+\ell}$ crosses the component k using the vertices in $\{(k, s, R-2h+1), (k, s, R-2h) | s = 1, \ldots, \Gamma_k\}$. Finally, the path

$P_{2i \to k}$ contains vertices of $\{(k, 1, s) | s = 0, \ldots, R-2\ell+1\}$, and vertices of $\{(k, 1, s) | s = 0, \ldots, R\}$ (see case 2). Hence, all calls passing through component k are vertex–disjoint, and thus they are edge–disjoint.

We can apply the same arguments for the case $k = 2i + \ell + 1$. And the lemma holds. $\square$

At each round $t = 1, \ldots \lceil \log_2 r \rceil - 1$ of the gossip phase (instruction 1), the call $\mathcal{P}_{k \to k+\ell}$ has length $\ell = 2^t - 1$. Thus ℓ is at most $2^{\lceil \log_2 r \rceil - 1} - 1 < r/2 < R/2$. The paths corresponding to the last round, $t = \lceil \log_2 r \rceil$, have length $\ell = 2^{\lceil \log_2 r \rceil} - 1 \pmod r = 2^{\lceil \log_2 r \rceil} - 1 - r < R/2$. In all cases Lemma 5 holds and at each round the calls are pairwise edge–disjoint.

In Fig. 1 there is an exemple of the paths $\mathcal{P}_{i \to i+3}$ corresponding to the round $t = 2$, in the chordal ring $\mathbf{C(64, 7)}$. Since in the decomposition of this graph all the groups have only one cycle, the vertex j in cycle i is labeled by $(i, 1, j)$.

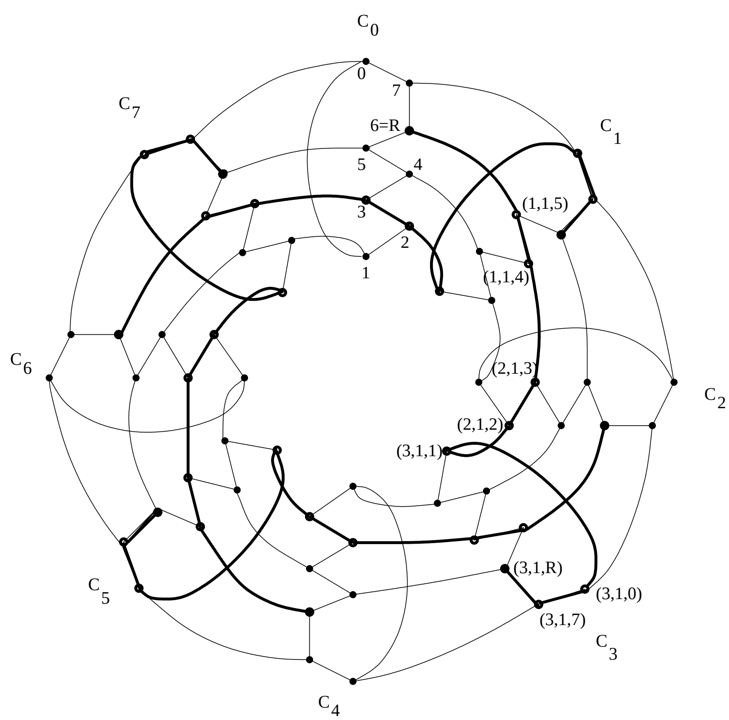

Figure 1: The chordal ring $\mathbf{C(64, 7)}$.

Therefore, we conclude that the decomposition in cycles as defined in Section 3.1 allows a gossip among accumulation nodes that performs as quickly as the gossip in the complete graph.

Case 2. The number of components is odd ($r = 2r' + 1$). The first round and the last round in algorithm 3, are identical (see instructions 2 and 6). During these rounds, all calls have the same length $r' < r/2$. After round 1, only the accumulation nodes of the components 0 to n' take part in a call where $n' = r'$ if r' is odd and $n' = r' + 1$ otherwise. (see the instruction 5). We merge the components $n', \ldots, r-1$ into a single n'–component. After this merging, the number of components is even and we can apply the same algorithm as in Case 1. Hence Lemma 5 holds and at each round the calls are pairwise edge–disjoint.

4.2 The upper bound in case $N = a(c+1)$

Since the gossip algorithm performs in $2\lceil \log_2 s \rceil + \lceil \log_2 r \rceil$, the calculation of the number of rounds gives:

- $a > c+1$, then $r = c-1$ and $s = (\alpha+1)(c+1)$.

$$g(\mathbf{C}(\mathbf{N}, \mathbf{c})) \leq 2\lceil \log_2 N \rceil - \lceil \log_2 c \rceil + O(1)$$

- $a \leq (c+1)$, then $r = a$ and $s = c+1$.

$$g(\mathbf{C}(\mathbf{N}, \mathbf{c})) \leq 2\lceil \log_2 N \rceil - \lceil \log_2 a \rceil + O(1)$$

4.3 The general case: $N = a(c+1) + b$, $0 < b < c$.

Note that, since N and $(c+1)$ are even, b is necessarily even. We consider two cases: $a \geq b/2$ and $a < b/2$.

4.3.1 Case 3. $a \geq b/2$

A graph of type $\mathcal{A}$ is a subgraph of the chordal ring that contains $c+3$ consecutive vertices. A subgraph of type $\mathcal{A}$ contains two chordal edges of type $[x, x+c]$, and $c+2$ edges of type $[x, x+1]$. If i is the smallest label of nodes of a subgraph H of type $\mathcal{A}$, then the vertex of label $i+j$ is called the jth vertex of H (the vertex 0 of H is incident to vertices 1 and c in H).

We split the graph in $a - b/2$ cycles of $c+1$ vertices and in $b/2$ graphs of type $\mathcal{A}$ of $c+3$ vertices. We note $\mathcal{S} = \{\mathcal{S}_i | i = 0, \dots, a-1\}$ the set of subgraphs which constitute this decomposition. There are two cases according to the parameter a:

- If $a \leq c+1$, then there are a components: each component is one element of $\mathcal{S}$.
- If $a > c+1$, then there are $c-1$ components. Assuming that $a = \alpha(c-1) + \beta$, β of these components are an union of $\alpha+1$ elements of $\mathcal{S}$, and $c-1-\beta$ other components are an union of α elements of $\mathcal{S}$.

Each component contains at most one subgraph of type $\mathcal{A}$ and possibly many cycles as defined in Section 3.1. The subgraph of type $\mathcal{A}$ is labeled 1, and the remaining cycles are labeled $2, 3, \dots$

The component k has $(k, 1, c-1)$ as its accumulation node if component k does not contain a subgraph of type $\mathcal{A}$, and $(k, 1, c+1)$ otherwise. We set $R = c-1$.

The accumulation phase and the broadcast phase can be performed using the algorithm in [10] since each component is connected. Now, we focus on the gossip phase, and we are interested in the call $\mathcal{P}'_{i \to i+\ell}$ between the two accumulation nodes of the component i and the component $i+\ell$. Again, we consider two cases according to the parity of the number of components.

Case 3.a. the number of components is even ($r = 2r'$): This path $\mathcal{P}'_{i \to i+\ell}$ is an union of paths denoted by $P'(i,0), P'(i,2), \dots, P'(i,\ell)$ such that $P'(i,k)$, $k = 0 \leq k \leq \ell$, is as follows:

1. if the component $i+k$ is composed of cycles of $c+1$ vertices only, then $P'(i,k) = P(i,k)$ [1];
2. if the component $i+k$ contains a graph of type $\mathcal{A}$, then we consider three sub-cases:
   - if $k \neq 0$ and $k \neq \ell$, then $P'(i,k) = P(i,k)$
   - if $k = 0$, then the path $P'(i,1)$ is $\{(i,1,c+1), (i+1,1,c-2), (i+1,1,c-3)\}$ if the component $i+k$ is composed of a subgraph of type $\mathcal{A}$ only. If the component $i+k$ is contains also some cycles, say Γ cycles, then the path $P'(i,0)$ is set to $\{(i,1,c+1), (i,s,c-2), (i,s,c-1), (i+1,1,c-3) | s = 2, \dots \Gamma\}$.

[1] defined in Case 1

- if $k = \ell$, then the path $P'(i,\ell)$ is set to $\{[(i+\ell,1,s)|s=0,\ldots,c-2\ell\} \cup \{(i+\ell,1,0),(i+\ell,1,c),(i+\ell,1,c+1)]\}$.

Let us prove that the calls performed at the same round are pairwise edge-disjoint.

Lemma 6 *Let $\epsilon = 0$ or 1. For any odd value of ℓ, $\ell < (c-1)/2$, all calls between the accumulation node of the component $2i+\epsilon$ and the accumulation node of the component $2i+\epsilon+\ell$, $i = 0,\ldots,r/2$, are edge-disjoint.*

Proof. To prove this lemma, let us consider the kth component. Let Γ_{i+k} be the number of cycles of $c+1$ vertices in the component k. If the component k does not contain a graph of type $\mathcal{A}$, then by similar arguments as in the proof of Lemma 5, it is easy to prove that the calls passing through component k are edge disjoint. So, we mainly focus on the case where the component k contains a graph of type $\mathcal{A}$.

Assume w.l.g. that $\epsilon = 0$. Let $k = 2i+\ell$ where $0 \le i \le r/2$. Applying the same argument as in lemma 5, we get that, for any odd integer h in $\{1,\ldots,\ell-1\}$, $\mathcal{P}_{k-h\to k-h+\ell}$ crosses the kth component using vertices $\{(k,s,c-2h),(k,s,c-2h-1)|s=1,\ldots,\Gamma_k\}$. For $h=\ell$, the call $\mathcal{P}'_{2i\to k}$ contains vertices $\{[(i+\ell,1,s)|s=0,\ldots,c-2\ell\}\cup\{(i+\ell,1,0),(i+\ell,1,c),(i+\ell,1,c+1)]\}$, by definition. Thus all calls crossing component k are vertex-disjoint and, hence, edge-disjoint. $\square$

Case 3.b. The number of components is odd ($r = 2r'+1$). This case can be treated in a similar way as Case 2 of Section 4.1.

4.3.2 Case 4. $a < b/2$

Since $b < c+1$, it implies that $a < (c+1)/2$. By using the cycle decomposition, we split the graph into $r = \lfloor a/2 \rfloor + 1$ components 1. The components $1 \ldots \lfloor a/2 \rfloor$ contain two $c+1$–cycles and the component 0 is a path of b vertices if a is even, or a union of a path of b vertices and a cycle of $c+1$ vertices if a is odd. The vertex j of the b–path in component 0 has label $(0,1,j)$ and the vertices in the other components are labeled as in Section 4.1. We take $(k,1,R=b-2)$ as accumulation vertices (in Section 4.1, $R=c-1$). Let us notice that there are $b/2+1$ edges connecting component $r-1$ with component 0.

This decomposition allows us to define an algorithm as in Section 4.1.

4.4 The upper bound in case $N = a(c+1) + b$

Since the gossip algorithm performs in $2\lceil \log_2 s \rceil + \lceil \log_2 r \rceil$, the calculation of the number of rounds gives:

- $a \geq b/2$:

  $a > c + 1$, then $r = c - 1$ and $s = (\alpha + 1)(c + 3)$.

  $$g(\mathbf{C}(\mathbf{N}, \mathbf{c})) \leq 2\lceil \log_2 N \rceil - \lceil \log_2 c \rceil + O(1)$$

  $a \leq (c+1)$, then $r = a$ and $s = c + 3$.

  $$g(\mathbf{C}(\mathbf{N}, \mathbf{c})) \leq 2\lceil \log_2 N \rceil - \lceil \log_2 a \rceil + O(1)$$

- $a < b/2 < (c+1)/2$, then $r = \lfloor a/2 \rfloor + 1$ and $s = 2(c+1)$:

  $$g(\mathbf{C}(\mathbf{N}, \mathbf{c})) \leq 2\lceil \log_2 N \rceil - \lceil \log_2 a \rceil + O(1)$$

5 Conclusion

We have decomposed the chordal ring $\mathbf{C}(\mathbf{N}, \mathbf{c})$ into r components of size s in ordre to apply the three–phase algorithm. This enables us to give an upper bound for the gossiping time under the half-duplex line model:

$$g(\mathbf{C}(\mathbf{N}, \mathbf{c})) \leq 2\lceil \log_2 s \rceil + \lceil \log_2 r \rceil$$

According to the different values of c and $N = a(c+1) + b$ we can reduce the results into the following two cases:

- $a \geq (c+1)$

  $$g(\mathbf{C}(\mathbf{N}, \mathbf{c})) \leq 2\lceil \log_2 N \rceil - \lceil \log_2 c \rceil + O(1)$$

- $a < (c+1)$

  $$g(\mathbf{C}(\mathbf{N}, \mathbf{c})) \leq 2\lceil \log_2 N \rceil - \lceil \log_2 a \rceil + O(1)$$

¿From Lemma 2 and Lemma 3 we can conclude that in the first of the above cases our bound is optimal. In the second one, we expect that a better approximation of the edge–bisection width could prove the optimality of this algorithm.

References

[1] B.W. Arden and H. Lee. Analysis of chordal ring network. *IEEE Trans. Comput.*, C-30(4):291–295, April 1981.

[2] L. Barrière, J. Fàbrega, E. Simó, and Zaragozá. Fault-tolerant routings in chordal ring networks. Submitted to *Networks*, 1997.

[3] J.-C. Bermond, F. Comellas, and D. F. Hsu. Distributed loop computer networks: A survey. *J. Parallel Distributed Comput.*, pages 2–10, 1995.

[4] N.K. Cheung and al. Special issue on dense WDM networks. *IEEE Journal Selected Areas in Communications*, 8, 1990.

[5] F. Comellas and P. Hell. Broadcasting in chordal rings. IWIN'97, Praga, 1997.

[6] H.S.M. Coxeter. Self dual configurations and regular graphs. *Bull. Amer. Math. Soc.*, pages 413–455, 1950.

[7] R.F. DeMara and D.I. Moldovan. Performance indices for parallel marker-propagation. In *Proceedings of the International Conference on Parallel Processing*, pages 658–659, 1991.

[8] J.J. Dongarra and D.W. Walker. Software librairies for linear algebra computation onhigh performances computers. *SIAM review*, 37:151–180, 1995.

[9] C. Laforest. Gossip in trees under line-communication mode. In *Euro-Par'96 Parallel Processing*, number 1123 in Lecture Notes in Computer Science, pages 371–373. Springer, August 1997.

[10] A.M. Farley. Minimum time line broadcast networks. *Networks*, 10:59–70, 1980.

[11] M. A. Fiol, J.L.A. Yebra, I. Alegre, and M. Valero. A discrete optimization problem in local networks and data alignment. *IEEE Trans. Comput.*, pages 702–713, 1987.

[12] P. Fraigniaud and E. Lazard. Methods and problems of communication in usual networks. *Discrete Appl. Math.*, 53(1):79–133, 1994.

[13] S.M. Hedetniemi, S.T. Hedetniemi, and A. Liestman. A survey of gossiping and broadcasting in communication networks. *Networks*, 18:319–349, 1986.

[14] J. Hromković, R. Klasing, B. Monien, and R. Peine. Dissemination of information in interconnection networks (broadcasting and gossiging). In D.Z. Du and F. Hsu, editors, *Combinatorial Network Theory*, pages 125–212, 1995.

[15] J. Hromković, R. Klasing, and E.A. Stöhr. Dissemination of information in vertex disjoint paths mode. part 1: General bounds and gossiping in hypercube-like network. Technical Report tr-ri-93-129, University of Paderborn. Germany., 1993.

[16] J. Hromković, R. Klasing, and E.A. Stöhr. Dissemination of information in vertex-disjoint paths mode. *Computers and Artificial Intelligence*, 15(4):295–318, 1996.

[17] J. Hromković, R. Klasing, E.A. Stöhr, and H. Wagener. Gossiping in vertex-disjoint paths mode in d-dimensional grids and planar graphs. *Information and Computation*, 123(1):17–28, 1995.

[18] J. Hromković, R. Klasing, W. Unger, and H. Wagener. Optimal algotihms for broadcast and gossip in the edge-disjoint path modes. 4th Scandinavian Workshop on Algorithm Theory (SWAT'94), *Lecture Notes in Computer Science* 824: 219–230, 1994.

[19] X. D. Hu and F. K. Hwang. Reliabilities of chordal rings. *Networks*, 22:487–501, 1992.

[20] F.K. Hwang and P.E. Wright. Survival reliability of some double-loop networks and chordal rings. *IEEE Trans. Comput.*, 44, 1995.

[21] R. Klasing. *On the complexity of broadcast and gossip in different communication modes.* PhD thesis, University of Paderborn. Germany., 1996.

[22] P. Morillo. *Grafos y digrafos asociados con teselaciones como modelos para redes de interconexión.* PhD thesis, Universitat Politècnica de Catalunya, Barcelona, Spain, 1987.

[23] P. Morillo, F. Comellas, and M. A. Fiol. Metric problems in triple loop graphs and digraphs associated to an hexagonal tessellation of the plane. Technical Report 05-0286, Departament de Matemàtica Aplicada i Telemàtica, UPC, June 1986.

[24] P. Morillo, F. Comellas, and M. A. Fiol. The optimization of chordal ring networks. *Communication Technology*, pages 295–299, 1987.

[25] P. Pacheco. *Parallel programming with MPI*. Morgan Kaufmann, 1995.

[26] D. Krizanc and F. Luccio. Boolean routing on chordal rings. In *Proceedings of the colloquium on Structural Information and Communication Complexity*, 1995.

[27] J. de Rumeur. *Communications dans les réseaux de processeurs*. Masson, 1994.

[28] C.K. Wong and D. Coppersmith. A combinatorial problem related to multimode memory organizations. *J. Assoc. Comp. Mach.*, 21:392–402, 1974.

[29] H. Xu, P. McKinley, and L. Ni. Efficient implementation of barrier synchronization in wormhole-routed hypercube multicomputers. *Journal of Parallel and Distributed Computing*, 16:172–184, 1992.

[30] J.L.A. Yebra, M.A. Fiol, P. Morillo, and I. Alegre. The diameter of undirected graphs associated to plane tessellations. *Ars Combin.*, pages 159–171, 1985.