

# Tables for the paper: A Computational Study for Bilevel Quadratic Programs using Semidefinite Relaxations

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In this file, we report all the tables we use for drawing Figures 1-4 in the paper. In particular, the data used in Figures 1-2 are shown in Table 3. In Tables 1-3 and Tables 4-6, we present numerical results for QQBP and QLBP using one sample for the input data. In Tables 1 and 4 we use a density level of 100% whereas in Tables 2 and 5 we use 50%. In Tables 3 and 6, we consider instances where  $n_1 \geq n_2$  as these instances are significantly harder to solve with CPLEX. In Tables 1-6 we show the same column information. More precisely, in column 1 we show the instance number. Columns 2-5 present the instance dimensions. Column 6 shows the optimal solution or best solution found with CPLEX in at most one hour of CPU time while column 7 shows the exact CPU time to get that solution. Column 8 reports the number of branch and bound nodes used by CPLEX. Columns 9 and 10 show the best solutions found with VNS Algorithm 5.1 and its CPU time in seconds respectively. Columns 11-13 present the optimal solution of the LP relaxation, the CPU time in seconds and number of iterations required by CPLEX to solve it respectively. Column 14 presents the nearest feasible solutions obtained using the output solutions of the LP relaxation. Similarly, columns 15-17 present the optimal solution of the SDP relaxation, the CPU time in seconds and the nearest feasible solutions obtained using the output solutions of the SDP relaxation, respectively. Finally, columns 18-19, 20-21, and 22 present the gaps we compute for the SDP and LP relaxations, for the nearest feasible solutions and for the VNS algorithm, respectively. The gaps shown in columns 18-19 are computed by  $\frac{(C15)-(C6)}{(C6)} * 100$  and  $\frac{(C11)-(C6)}{(C6)} * 100$  where (C6), (C11) and (C15) refers to columns 6, 11 and 15, respectively. The gaps shown in columns 20-21 are computed by  $\frac{(C6)-(C17)}{(C6)} * 100$  and  $\frac{(C6)-(C14)}{(C6)} * 100$  where (C6), (C14)

and (C17) refers to columns 6, 14 and 17, respectively. Similarly, the gaps in column 22 are calculated by  $\frac{(C6)-(C9)}{(C6)} * 100$  where (C6) and (C9) refers to columns 6 and 9, respectively. In Tables 1-6, we boldface the minimum gaps obtained with SDP and LP relaxations in columns 18-19. Similarly, we boldface the minimum gaps obtained with the nearest feasible or VNS procedures in columns 20-22.

The data in Tables 7-8 is used to draw Figures 3-4 in the paper, respectively. In Tables 7-10 we present average numerical results using 10 samples of the input data for large scale instances when  $n_1 \geq n_2$ . In particular, Tables 7-8 present numerical results for QQBP using density levels of 100% and 50%, respectively. Tables 9-10 present numerical results for QLBP using density levels of 100% and 50%, respectively. All these tables show the same column information. More precisely, in column 1 we show the instance number. Columns 2-5 show the instance dimensions. Columns 6-9 present the optimal solutions, the CPU time in seconds, the number of iterations required by CPLEX and the feasible solutions obtained with the nearest feasible procedure for the LP relaxations, respectively. Similarly, columns 10-12 present the optimal solutions, the CPU time in seconds, and the feasible solutions obtained with the nearest feasible procedure for the SDP relaxations, respectively. Finally, in columns 13-14, we show the gaps for the LP and SDP relaxations, respectively. These gaps are computed by  $\frac{(C6)-(C9)}{(C6)} * 100$  and  $\frac{(C10)-(C12)}{(C10)} * 100$  where (C6), (C9), (C10) and (C12) refers to columns 6, 9, 10, and 12, respectively. In Tables 7-10, we boldface the minimum gaps obtained with the LP and SDP relaxations in columns 13-14, respectively.

| #              | Instance Size  |                |                |                | MLP <sub>1</sub> |         | VNS    |           | LP <sub>1</sub> |           | F(SDP) |         | Gaps Ubs % |                  | Gaps NFS %       |                 | Gap <sub>NFS</sub> % |        |       |
|----------------|----------------|----------------|----------------|----------------|------------------|---------|--------|-----------|-----------------|-----------|--------|---------|------------|------------------|------------------|-----------------|----------------------|--------|-------|
|                | m <sub>1</sub> | m <sub>2</sub> | n <sub>1</sub> | n <sub>2</sub> | Bf               | Time(s) | B.Sol. | Time(s)   | Ub              | Time(s)   | F(LP)  | Iter.   | F(LP)      | SDP <sub>1</sub> | SDP <sub>1</sub> | LP <sub>1</sub> |                      |        |       |
| 1              | 5              | 5              | 10             | 10             | 28.2098          | 0.22    | 0      | 28.2098   | 3.23            | 34.2055   | 92     | 0.13    | -3.8947    | 0.38             | 13.14            | 13.81           | 0                    |        |       |
| 2              | 5              | 5              | 10             | 15             | 40.1997          | 0.25    | 0      | 40.1997   | 56.60           | 49.1313   | 155    | 0.16    | -14.0800   | 0.38             | 13.14            | 13.81           | 0                    |        |       |
| 3              | 5              | 5              | 10             | 25             | 64.1304          | 0.19    | 14     | 55.3596   | 23.01           | 72.8743   | 156    | 0.14    | -1.7274    | 1.14             | 32.3424          | 35.03           | 13.68                |        |       |
| 4              | 5              | 5              | 10             | 50             | 101.3429         | 0.92    | 0      | 101.3429  | 5.07            | 101.9174  | 148    | 0.44    | 50.7842    | 10.98            | 44.09            | 102.69          | 0                    |        |       |
| 5              | 5              | 5              | 10             | 100            | 183.4443         | 0.25    | 0      | 182.9454  | 49.25           | 184.6001  | 239    | 0.17    | 61.5130    | 140.89           | 47.45            | 49.89           | 0                    |        |       |
| 6              | 5              | 5              | 15             | 10             | 113.9089         | 0.22    | 0      | 113.9089  | 25.26           | 131.3711  | 311    | 0.13    | 25.5706    | 0.39             | 69.25            | 66.47           | 0.27                 |        |       |
| 7              | 5              | 5              | 25             | 10             | 275.9347         | 0.75    | 40     | 275.9347  | 173.04          | 362.6656  | 771    | 0.17    | 21.4942    | 0.58             | 7.97             | 78.98           | 6.35                 |        |       |
| 8              | 5              | 5              | 50             | 10             | 460.1259         | 87.91   | 2051   | 310.0704  | 68.79           | 1078.6839 | 2869   | 0.70    | 5.2826     | 1.12             | 3.80             | 92.21           | 19.57                |        |       |
| 9              | 5              | 5              | 100            | 10             | 1325.6715        | 1415.03 | 499    | 345.0305  | 135.20          | 4223.5855 | 11054  | 5.32    | 21.1801    | 2.34             | 3.59             | 98.85           | 32.61                |        |       |
| 10             | 5              | 5              | 40             | 40             | 333.5363         | 8.62    | 101    | 297.3339  | 94.38           | 719.6996  | 1065   | 0.36    | -3.6904    | 8.10             | 1.28             | 98.40           | 73.97                |        |       |
| 11             | 5              | 5              | 60             | 60             | 687.0716         | 195.61  | 1622   | 527.0621  | 68.28           | 1602.9359 | 4125   | 0.98    | -14.1588   | 36.97            | 13.82            | 101.11          | 10.85                |        |       |
| 12             | 5              | 5              | 80             | 80             | 1141.7225        | 3600    | 5086   | 828.2675  | 21.30           | 2973.3891 | 2.77   | 6634    | 36.0520    | 137.30           | 10.05            | 102.06          | 23.29                |        |       |
| 13             | 5              | 5              | 100            | 100            | 1364.0053        | 2284.39 | 928    | 1077.3925 | 48.96           | 4329.7125 | 4.65   | 11159   | 43.7991    | 341.72           | 8.42             | 96.79           | 27.45                |        |       |
| 14             | 10             | 10             | 10             | 10             | 34.9150          | 0.19    | 11     | 26.5371   | 16.72           | 46.6162   | 139    | 0.14    | 30.4852    | 1.12             | 16.56            | 12.69           | 24.00                |        |       |
| 15             | 10             | 10             | 15             | 10             | 98.6332          | 0.22    | 10     | 90.3113   | 71.64           | 113.0857  | 0.16   | 339     | 95.1339    | 105.9197         | 3.50             | 3.55            | 8.44                 |        |       |
| 16             | 10             | 10             | 25             | 10             | 182.2096         | 1.17    | 49     | 76.8984   | 82.23           | 281.6830  | 819    | 0.19    | -11.7955   | 1.14             | 0                | 106.47          | 57.80                |        |       |
| 17             | 10             | 10             | 50             | 10             | 442.6112         | 79.00   | 1427   | 100.68    | 1075.9257       | 0.83      | 3273   | -0.3254 | 488.2302   | 1.12             | 1.69             | 100.07          | -                    |        |       |
| 18             | 10             | 10             | 100            | 10             | 1131.9131        | 3600    | 1759   | 100.45    | 4163.3441       | 6.01      | 10476  | 0       | 1308.6213  | 3.74             | -2.62            | 100.00          | -                    |        |       |
| 19             | 10             | 10             | 15             | 15             | 35.1297          | 0.19    | 9      | 34.9503   | 42.88           | 47.6009   | 0.14   | 138     | 2.2972     | 38.4259          | 9.38             | 35.50           | 0.51                 |        |       |
| 20             | 10             | 10             | 25             | 10             | 46.8608          | 0.17    | 0      | 46.8608   | 2.04            | 53.0831   | 0.14   | 197     | -0.5495    | 49.4002          | 5.42             | 13.28           | 0                    |        |       |
| 21             | 10             | 10             | 50             | 10             | 123.3956         | 0.22    | 0      | 123.3956  | 24.62           | 128.7934  | 0.16   | 260     | 4.4996     | 123.6014         | 0.17             | 42.48           | 0.08                 |        |       |
| 22             | 10             | 10             | 100            | 10             | 156.9117         | 0.31    | 0      | 154.9044  | 17.18           | 160.4050  | 0.19   | 317     | -8.9150    | 157.2814         | 0.24             | 2.23            | 1.28                 |        |       |
| 23             | 10             | 10             | 20             | 20             | 107.9319         | 0.72    | 5      | 101.6910  | 39.22           | 179.4016  | 0.19   | 499     | 3.4758     | 115.0900         | 1.45             | 66.22           | 5.78                 |        |       |
| 24             | 10             | 10             | 30             | 30             | 185.5452         | 5.24    | 106    | 157.7529  | 6.16            | 410.3252  | 0.23   | 1017    | 0.9671     | 202.0968         | 6.15             | 158.5912        | 99.48                |        |       |
| 25             | 10             | 10             | 40             | 40             | 387.2704         | 10.31   | 105    | 216.1889  | 19.28           | 731.8874  | 0.37   | 1909    | 8.8019     | 407.9579         | 11.50            | 365.4651        | 44.18                |        |       |
| 26             | 10             | 10             | 50             | 50             | 460.5015         | 67.59   | 1107   | 418.3966  | 34.10           | 1112.1761 | 0.62   | 2635    | -11.2983   | 486.1204         | 32.35            | 406.7601        | 9.14                 |        |       |
| 27             | 10             | 10             | 60             | 60             | 608.6118         | 245.93  | 1580   | 533.2806  | 202.23          | 1600.1876 | 1.08   | 3682    | 15.3063    | 652.5314         | 48.41            | 544.2684        | 12.38                |        |       |
| 28             | 10             | 10             | 80             | 80             | 876.5662         | 447.85  | 479    | 842.1377  | 47.69           | 2758.2804 | 2.99   | 6883    | 36.9908    | 1019.5418        | 149.42           | 827.1666        | 3.93                 |        |       |
| 29             | 10             | 10             | 100            | 100            | 1256.5405        | 1678.40 | 494    | 757.6477  | 36.13           | 4334.7503 | 6.72   | 10604   | -10.0297   | 1408.2074        | 446.13           | 1223.8057       | 39.70                |        |       |
| 30             | 5              | 10             | 20             | 20             | 116.0406         | 1.03    | 8      | 102.6424  | 6.83            | 194.7713  | 0.16   | 452     | 1.5731     | 125.4226         | 1.34             | 101.8979        | 11.55                |        |       |
| 31             | 5              | 10             | 30             | 30             | 212.8583         | 3.79    | 116    | 205.7644  | 19.78           | 418.0583  | 0.22   | 1010    | -6.0037    | 229.0393         | 5.73             | 189.8694        | 3.33                 |        |       |
| 32             | 5              | 10             | 50             | 50             | 532.2305         | 34.87   | 546    | 493.1035  | 177.83          | 1113.0958 | 0.58   | 2645    | 9.3256     | 557.6540         | 33.66            | 476.1099        | 7.35                 |        |       |
| 33             | 5              | 10             | 100            | 100            | 1466.2272        | 1413.83 | 636    | 1240.9057 | 87.19           | 4307.9043 | 5.96   | 10725   | 1.4738     | 1591.0761        | 390.22           | 1409.9650       | 15.37                |        |       |
| 34             | 10             | 5              | 20             | 20             | 139.5095         | 0.58    | 7      | 130.3833  | 73.64           | 200.2820  | 0.16   | 488     | 8.2256     | 145.8375         | 1.12             | 133.8776        | 6.54                 |        |       |
| 35             | 10             | 5              | 30             | 30             | 205.8333         | 3.37    | 27     | 197.1814  | 89.26           | 406.3610  | 0.22   | 976     | -6.6960    | 220.1545         | 4.95             | 172.2758        | 4.20                 |        |       |
| 36             | 10             | 5              | 50             | 50             | 527.3863         | 116.03  | 2443   | 505.8825  | 106.95          | 1162.0994 | 0.61   | 2624    | 18.7733    | 574.2924         | 29.30            | 489.6611        | 4.08                 |        |       |
| 37             | 10             | 5              | 100            | 100            | 1615.7601        | 1456.46 | 510    | 1257.1087 | 86.49           | 4512.9628 | 5.88   | 10877   | 6.3604     | 1776.1785        | 383.65           | 1575.9052       | 22.20                |        |       |
| 38             | 20             | 20             | 50             | 50             | 414.4010         | 282.41  | 6094   | 349.0619  | 69.62           | 1084.5514 | 0.78   | 2737    | -2.7384    | 455.7355         | 31.29            | 380.5037        | 15.77                |        |       |
| 39             | 20             | 20             | 80             | 80             | 909.6791         | 3600    | 4543   | 629.5854  | 66.38           | 2747.2482 | 3.52   | 6744    | 49.1000    | 999.8464         | 204.89           | 898.0002        | 30.79                |        |       |
| 40             | 20             | 20             | 100            | 100            | 1147.2265        | 1428.46 | 514    | 827.6981  | 148.86          | 4373.3311 | 8.38   | 10416   | 39.2014    | 1380.5541        | 578.47           | 998.9173        | 27.85                |        |       |
| Minimum values |                |                |                |                | 28.2098          | 0.17    | 0      | 26.5371   | 2.04            | 34.2055   | 0.13   | 92      | -14.1588   | 29.7773          | 0.38             | 24.5030         | 0.01                 | 0.57   | -     |
| Maximum values |                |                |                |                | 1615.7601        | 3600    | 6094   | 1257.1087 | 2022.23         | 4512.9628 | 8.38   | 11159   | 95.1339    | 1776.1785        | 578.47           | 1575.9052       | 20.34                | 281.21 | 73.97 |
| Average values |                |                |                |                | 488.74           | -       | -      | -         | -               | 1339.57   | 1.56   | 3306.47 | 12.54      | 535.17           | 80.44            | 452.58          | 7.15                 | 105.99 | -     |

“-”: No solution found.

Table 1: Upper and lower bounds for QQP using Dense Data

| #              | Instance Size  |                |                |                | MLP <sub>1</sub> |         | B&Bn  |          | VNS      |           | LP <sub>1</sub> |          | F(LP)    |           | SDP <sub>1</sub> |                  | F(SDP) |                  | Gaps Ubs %      |                  | Gaps NFS %      |                  | GapvNs % |                 |  |        |  |       |  |        |  |       |  |
|----------------|----------------|----------------|----------------|----------------|------------------|---------|-------|----------|----------|-----------|-----------------|----------|----------|-----------|------------------|------------------|--------|------------------|-----------------|------------------|-----------------|------------------|----------|-----------------|--|--------|--|-------|--|--------|--|-------|--|
|                | m <sub>1</sub> | m <sub>2</sub> | n <sub>1</sub> | n <sub>2</sub> | BF               | Time(s) | B&Bn  | B.Sol.   | Time(s)  | Ub        | Time(s)         | Iter.    | F(LP)    | Ub        | Time(s)          | SDP <sub>1</sub> | F(SDP) | SDP <sub>1</sub> | LP <sub>1</sub> | SDP <sub>1</sub> | LP <sub>1</sub> | SDP <sub>1</sub> |          | LP <sub>1</sub> |  |        |  |       |  |        |  |       |  |
| 1              | 5              | 5              | 10             | 10             | 25.6984          | 0.20    | 0     | -        | 100.09   | 29.0036   | 0.14            | 145      | 0        | 27.4786   | 0.33             | 22.5319          | 6.93   | 15.20            | 12.32           | 100.00           | -               | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 2              | 5              | 5              | 10             | 15             | 39.9225          | 0.22    | 0     | 36.5176  | 134.27   | 61.6934   | 0.14            | 101      | 29.3465  | 41.8472   | 1.12             | 22.9500          | 4.82   | 4.44             | 42.51           | 26.49            | 8.53            | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 3              | 5              | 5              | 10             | 25             | 59.6592          | 0.16    | 0     | 59.6592  | 105.48   | 63.4062   | 0.14            | 157      | 32.8005  | 63.4062   | 1.12             | 28.3607          | 6.28   | 6.28             | 52.46           | 45.02            | 0               | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 4              | 5              | 5              | 10             | 50             | 57.2499          | 0.22    | 0     | 57.2499  | 132.12   | 61.2665   | 0.16            | 151      | 8.8750   | 61.2216   | 8.13             | 4.0706           | 6.94   | 7.02             | 92.89           | 84.50            | 0               | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 5              | 5              | 5              | 10             | 100            | 163.1826         | 0.41    | 15    | 161.5557 | 216.94   | 163.2250  | 0.17            | 252      | 14.8222  | 163.6387  | 113.60           | 19.7340          | 0.28   | 0.03             | 87.91           | 90.92            | 1.00            | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 6              | 5              | 5              | 15             | 10             | 52.4541          | 0.20    | 0     | 29.8914  | 11.06    | 56.0275   | 0.16            | 190      | 41.0956  | 55.5463   | 0.42             | 37.8837          | 5.89   | 6.87             | 27.78           | 21.65            | 43.01           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 7              | 5              | 5              | 25             | 10             | 83.1546          | 0.36    | 26    | 48.9262  | 115.58   | 89.9181   | 0.21            | 1254     | 69.1164  | 89.6475   | 1.12             | 73.1103          | 8.13   | 12.08            | 16.88           | 88.75            | 41.16           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 8              | 5              | 5              | 50             | 10             | 208.7631         | 3.92    | 259   | 100.68   | 293.9575 | 1.22      | 7430            | 23.4930  | 233.2371 | 1.12      | 191.7430         | 11.72            | 40.81  | 8.15             | 88.75           | -                | -               |                  |          |                 |  |        |  |       |  |        |  |       |  |
| 9              | 5              | 5              | 100            | 10             | 700.3613         | 3600    | 33214 | -        | 102.35   | 1154.5643 | 14.15           | 29348    | 15.3568  | 747.5751  | 2.33             | 691.3155         | 6.74   | 64.85            | 1.29            | 97.81            | -               | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 10             | 5              | 5              | 40             | 40             | 226.8426         | 0.61    | 0     | 200.7566 | 92.40    | 247.8089  | 0.67            | 5155     | 65.1646  | 240.6330  | 6.04             | 188.1444         | 6.08   | 9.24             | 17.06           | 71.27            | 11.50           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 11             | 5              | 5              | 60             | 60             | 407.1249         | 10.52   | 964   | 329.6141 | 25.68    | 485.4833  | 2.59            | 13244    | 7.0844   | 444.8179  | 29.84            | 360.0939         | 9.26   | 19.25            | 11.55           | 98.26            | 19.04           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 12             | 5              | 5              | 80             | 80             | 658.6102         | 64.49   | 968   | 548.2150 | 116.45   | 893.5369  | 8.72            | 19514    | 4.4254   | 708.3672  | 108.04           | 581.0746         | 7.55   | 35.67            | 11.77           | 99.94            | 16.76           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 13             | 5              | 5              | 100            | 100            | 745.4745         | 3600    | 45478 | 556.9815 | 171.12   | 1200.5491 | 15.35           | 25664    | 8.4929   | 838.9855  | 295.32           | 685.7276         | 12.54  | 61.09            | 8.01            | 98.86            | 25.28           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 14             | 10             | 10             | 10             | 10             | 12.2987          | 0.14    | 0     | 7.7339   | 62.81    | 14.8039   | 0.14            | 73       | 3.3384   | 13.4872   | 1.12             | 6.7699           | 9.66   | 20.37            | 44.95           | 72.86            | 37.12           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 15             | 10             | 10             | 15             | 10             | 62.4674          | 0.19    | 0     | 101.24   | 67.5291  | 101.24    | 0.14            | 439      | 61.5228  | 67.7726   | 1.12             | 61.1828          | 8.49   | 8.10             | 2.06            | 1.51             | -               | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 16             | 10             | 10             | 25             | 10             | 114.4292         | 0.23    | 0     | 101.68   | 127.8627 | 127.8627  | 0.19            | 1037     | 112.5160 | 128.7759  | 1.12             | 108.1757         | 12.54  | 11.74            | 5.46            | 1.67             | -               | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 17             | 10             | 10             | 50             | 10             | 193.0252         | 4.68    | 320   | 100.58   | 275.9778 | 1.13      | 5995            | 2.1084   | 216.6136 | 1.10      | 174.7907         | 12.22            | 42.98  | 9.45             | 98.91           | -                | -               |                  |          |                 |  |        |  |       |  |        |  |       |  |
| 18             | 10             | 10             | 100            | 10             | 384.0296         | 3600    | 24916 | 100.33   | 947.7654 | 18.20     | 24913           | -36.7491 | 517.5670 | 3.96      | 433.7443         | 34.77            | 146.79 | -12.95           | 109.57          | -                | -               |                  |          |                 |  |        |  |       |  |        |  |       |  |
| 19             | 10             | 10             | 15             | 15             | 17.9739          | 0.19    | 0     | 11.7920  | 33.93    | 19.7191   | 0.14            | 139      | 6.3750   | 18.0994   | 1.12             | 11.9758          | 0.70   | 9.71             | 33.37           | 64.53            | 34.39           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 20             | 10             | 10             | 25             | 30             | 30.6740          | 0.19    | 32    | 30.6740  | 123.45   | 34.1866   | 0.13            | 155      | 17.1446  | 33.0513   | 1.35             | 18.6754          | 7.75   | 11.45            | 39.12           | 44.11            | 0               | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 21             | 10             | 10             | 50             | 50             | 90.8327          | 0.25    | 0     | 90.8327  | 117.08   | 92.8102   | 0.16            | 210      | 30.4346  | 93.3325   | 9.86             | 29.5896          | 2.75   | 2.18             | 67.42           | 66.49            | 0               | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 22             | 10             | 10             | 100            | 100            | 122.0795         | 0.33    | 0     | 116.3552 | 114.79   | 123.0423  | 0.19            | 263      | 25.3325  | 122.5938  | 119.93           | 21.1671          | 0.42   | 0.79             | 82.66           | 79.25            | 4.69            | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 23             | 10             | 10             | 20             | 20             | 53.2054          | 0.30    | 21    | 43.5767  | 109.36   | 60.1533   | 0.16            | 660      | 29.5450  | 58.1978   | 1.14             | 36.4500          | 9.38   | 13.06            | 31.49           | 44.47            | 18.10           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 24             | 10             | 10             | 30             | 30             | 133.4849         | 0.45    | 26    | 121.9393 | 213.52   | 141.6776  | 0.34            | 3129     | 118.8467 | 140.7586  | 3.87             | 118.3075         | 5.45   | 6.14             | 11.37           | 10.97            | 8.65            | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 25             | 10             | 10             | 40             | 40             | 217.1710         | 0.75    | 15    | 205.5410 | 293.24   | 227.2342  | 0.62            | 4577     | 173.4153 | 226.0745  | 7.46             | 181.6836         | 4.10   | 4.63             | 16.34           | 20.15            | 5.36            | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 26             | 10             | 10             | 50             | 50             | 280.2296         | 2.62    | 79    | 181.6985 | 122.03   | 314.0183  | 1.48            | 8004     | 24.6109  | 293.3038  | 22.31            | 224.3744         | 4.67   | 12.06            | 19.93           | 91.22            | 35.16           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 27             | 10             | 10             | 60             | 60             | 333.6322         | 6.91    | 165   | 219.9933 | 130.90   | 425.7203  | 1.92            | 7513     | 1.7608   | 349.9772  | 36.29            | 249.1516         | 4.90   | 27.60            | 25.32           | 99.47            | 34.06           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 28             | 10             | 10             | 80             | 80             | 531.7820         | 855.43  | 16537 | 455.5880 | 197.69   | 785.9048  | 7.55            | 16616    | -31.3033 | 574.6099  | 120.81           | 440.0249         | 8.05   | 47.79            | 17.25           | 105.89           | 14.33           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 29             | 10             | 10             | 100            | 100            | 657.8562         | 99.89   | 988   | 513.7103 | 174.28   | 1115.2906 | 11.82           | 21190    | -5.3558  | 741.1384  | 300.77           | 569.7095         | 12.66  | 69.53            | 13.40           | 100.81           | 21.91           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 30             | 5              | 10             | 20             | 20             | 64.6834          | 0.26    | 17    | 62.5286  | 270.99   | 70.4404   | 0.19            | 829      | 55.1946  | 69.1102   | 1.13             | 55.3193          | 6.84   | 8.90             | 14.48           | 14.67            | 3.33            | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 31             | 5              | 10             | 30             | 30             | 156.5221         | 0.41    | 20    | 145.2900 | 254.22   | 164.7974  | 0.34            | 3213     | 126.7790 | 170.0961  | 4.29             | 137.5036         | 8.67   | 5.29             | 12.15           | 19.00            | 7.18            | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 32             | 5              | 10             | 50             | 50             | 322.0708         | 2.51    | 65    | 259.8424 | 128.45   | 372.7927  | 1.56            | 8700     | 14.1608  | 336.1307  | 21.76            | 267.0491         | 4.37   | 15.75            | 17.08           | 95.60            | 19.32           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 33             | 5              | 10             | 100            | 100            | 817.6352         | 3600    | 43582 | 673.8946 | 45.23    | 1303.8215 | 18.20           | 24480    | 34.5911  | 898.2545  | 298.81           | 723.0033         | 9.86   | 59.46            | 11.57           | 95.77            | 17.58           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 34             | 10             | 5              | 20             | 20             | 89.6737          | 0.22    | 0     | 79.2270  | 177.37   | 91.6137   | 0.16            | 668      | 87.0306  | 92.1642   | 1.12             | 77.5780          | 2.78   | 2.16             | 13.49           | 2.95             | 11.65           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 35             | 10             | 5              | 30             | 30             | 123.0971         | 0.30    | 0     | 104.4967 | 110.32   | 129.4647  | 0.30            | 2067     | 41.1508  | 129.4659  | 3.76             | 92.3908          | 5.17   | 5.17             | 24.94           | 64.13            | 15.11           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 36             | 10             | 5              | 50             | 50             | 226.7862         | 8.10    | 1822  | 215.8611 | 388.83   | 280.9375  | 1.12            | 5876     | -14.9060 | 256.6907  | 13.79            | 200.6434         | 13.19  | 23.88            | 11.53           | 106.57           | 4.82            | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 37             | 10             | 5              | 100            | 100            | 633.6206         | 3600    | 48040 | 553.7922 | 236.91   | 1190.8229 | 13.06           | 21819    | -12.3781 | 726.8855  | 320.88           | 601.2978         | 14.72  | 87.94            | 5.10            | 101.95           | 12.60           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 38             | 20             | 20             | 50             | 50             | 236.1134         | 6.05    | 406   | 163.9032 | 288.87   | 310.4640  | 1.08            | 5380     | 0        | 258.3828  | 19.69            | 217.8083         | 9.29   | 31.32            | 7.87            | 100.00           | 30.67           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 39             | 20             | 20             | 80             | 80             | 576.4881         | 87.46   | 1070  | 395.8838 | 159.33   | 848.0195  | 7.81            | 15089    | 27.2001  | 634.6253  | 139.26           | 508.2188         | 10.08  | 47.10            | 11.84           | 95.28            | 31.33           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| 40             | 20             | 20             | 100            | 100            | 814.5009         | 140.88  | 1028  | 568.6750 | 250.65   | 1278.0772 | 16.50           | 24117    | 86.2144  | 883.6439  | 375.69           | 742.4055         | 8.49   | 56.92            | 8.85            | 89.42            | 30.18           | -                |          |                 |  |        |  |       |  |        |  |       |  |
| Minimum values |                |                |                | 12.2987        |                  | 0.14    |       | 7.7339   |          | 14.8039   |                 | 0.13     |          | 73        |                  | -36.7490         |        | 13.4872          |                 | 0.28             |                 | -12.95           |          | 1.51            |  |        |  |       |  |        |  |       |  |
| Maximum values |                |                |                | 817.6352       |                  | 3600    |       | 48040    |          | 673.8946  |                 | 388.83   |          | 1303.8215 |                  | 18.20            |        | 29348            |                 | 173.4153         |                 | 742.4095         |          | 34.77           |  | 146.79 |  | 92.89 |  | 109.57 |  | 43.01 |  |
| Average values |                |                |                | 268.12         |                  | -       |       | -        |          | -         |                 | -        |          | -         |                  | -                |        | -                |                 | -                |                 | -                |          | -               |  | -      |  | -     |  | -      |  | -     |  |
|                |                |                |                | 5501.82        |                  | -       |       | -        |          | -         |                 | -        |          | -         |                  | -                |        | -                |                 | -                |                 | -                |          | -               |  | -      |  | -     |  | -      |  | -     |  |
|                |                |                |                | 7743.90        |                  | 3.71    |       | 7743.90  |          | 32.44     |                 | 294.18   |          | 60.05     |                  | 230.39           |        | 8.12             |                 | 26.44            |                 | 23.28            |          | 68.43           |  | -      |  | -     |  | -      |  | -     |  |

“-”: No solution found.

Table 2: Upper and lower bounds for QQP using Sparse Data

| #                 | Instance Size  |                |                | MILP <sub>1</sub> |           | VNS      |           | LP <sub>1</sub> |           |           | SDP <sub>1</sub> |          | Gaps Ubs % |           | Gaps NFS %       |                 | GapVNS % |                  |                 |        |       |
|-------------------|----------------|----------------|----------------|-------------------|-----------|----------|-----------|-----------------|-----------|-----------|------------------|----------|------------|-----------|------------------|-----------------|----------|------------------|-----------------|--------|-------|
|                   | m <sub>1</sub> | m <sub>2</sub> | n <sub>1</sub> | n <sub>2</sub>    | Bf        | Time(s)  | B&Bn      | B.Sol.          | Time(s)   | Iter.     | F(LP)            | Ub       | Time(s)    | F(SDP)    | SDP <sub>1</sub> | LP <sub>1</sub> |          | SDP <sub>1</sub> | LP <sub>1</sub> |        |       |
| 1                 | 5              | 5              | 100            | 10                | 1266.5244 | 2358.43  | 1009      | 291.4347        | 233.29    | 4279.5137 | 5.18             | 11187    | 8.4062     | 1370.9014 | 2.33             | 1255.4226       | 8.24     | 237.89           | 0.88            | 99.34  | 76.99 |
| 2                 | 5              | 5              | 100            | 10                | 1232.2103 | 1179.26  | 510       | 100.74          | 4295.5404 | 4.06      | 11185            | 43.7133  | 1369.2901  | 2.22      | 1228.6578        | 11.12           | 248.60   | 0.28             | 96.45           | 96.45  | -     |
| 3                 | 5              | 5              | 100            | 10                | 1174.6083 | 2140.53  | 948       | -               | 102.88    | 4122.9933 | 6.41             | 10296    | -83.3734   | 1327.2296 | 3.76             | 1055.9746       | 12.99    | 251.01           | 10.09           | 102.84 | -     |
| 4                 | 5              | 5              | 100            | 10                | 1260.3391 | 33987.77 | 2119      | -               | 103.22    | 4325.8268 | 7.00             | 11793    | 226.5020   | 1405.5449 | 3.85             | 1232.9018       | 11.52    | 243.23           | 2.18            | 82.03  | -     |
| 5                 | 10             | 10             | 100            | 10                | 1386.7380 | 43200    | 21241     | -               | 100.98    | 4309.7498 | 7.47             | 11004    | -2.1628    | 1511.6937 | 3.96             | 1370.4843       | 9.01     | 210.78           | 1.17            | 100.16 | -     |
| 6                 | 10             | 10             | 100            | 10                | 1195.9622 | 1722.62  | 680       | -               | 100.37    | 4163.1434 | 6.32             | 10431    | 107.4105   | 1357.7587 | 3.87             | 1195.9622       | 13.53    | 248.10           | 0               | 91.02  | -     |
| 7                 | 10             | 10             | 100            | 10                | 1117.9014 | 1537.71  | 474       | -               | 102.05    | 4130.8219 | 6.60             | 10270    | 73.1967    | 1262.9853 | 3.85             | 1115.2909       | 12.98    | 269.52           | 0.23            | 93.45  | -     |
| 8                 | 10             | 10             | 100            | 10                | 1428.4078 | 1036.92  | 474       | -               | 105.33    | 4341.5791 | 4.88             | 12079    | 71.3038    | 1635.4258 | 3.87             | 1384.9754       | 14.49    | 203.95           | 3.04            | 95.01  | -     |
| 9                 | 10             | 10             | 60             | 60                | 709.3626  | 350.83   | 2516      | 542.2799        | 122.83    | 1634.2326 | 1.05             | 3759     | 1.8717     | 738.9968  | 60.20            | 652.1688        | 4.18     | 130.38           | 8.06            | 99.74  | 23.55 |
| 10                | 10             | 10             | 60             | 60                | 671.8542  | 223.35   | 1575      | 455.9048        | 58.91     | 1573.4793 | 1.11             | 3740     | 15.7372    | 699.2405  | 51.26            | 586.9313        | 4.08     | 134.20           | 12.64           | 97.66  | 32.14 |
| 11                | 10             | 10             | 60             | 60                | 555.5562  | 179.73   | 1090      | 338.5449        | 37.22     | 1580.3135 | 1.06             | 3563     | -18.8303   | 595.9495  | 52.45            | 494.3418        | 7.27     | 184.46           | 11.02           | 103.39 | 39.06 |
| 12                | 10             | 10             | 60             | 60                | 643.0517  | 163.58   | 1351      | 528.3561        | 158.79    | 1567.6491 | 1.09             | 3940     | -28.6218   | 693.1134  | 46.55            | 579.8128        | 7.79     | 143.78           | 9.83            | 104.45 | 17.84 |
| 13                | 10             | 10             | 100            | 100               | 1123.4381 | 1481.29  | 494       | 789.7448        | 86.75     | 4212.1288 | 4.32             | 11036    | -2.2420    | 1285.4952 | 386.49           | 1002.1876       | 14.43    | 274.93           | 10.79           | 100.20 | 29.70 |
| 14                | 10             | 10             | 100            | 100               | 1198.6579 | 1581.97  | 540       | 748.5892        | 63.73     | 4195.9334 | 6.07             | 10269    | 34.7366    | 1359.5241 | 396.76           | 1125.3464       | 13.42    | 250.05           | 6.12            | 97.10  | 37.55 |
| 15                | 10             | 10             | 100            | 100               | 1046.5891 | 1441.49  | 570       | 882.3456        | 52.43     | 4258.3849 | 5.51             | 10184    | -77.5087   | 1274.2041 | 375.13           | 1008.5157       | 21.75    | 306.88           | 3.64            | 107.41 | 15.69 |
| 16                | 10             | 10             | 100            | 100               | 1142.7455 | 1291.83  | 494       | 1040.6666       | 76.27     | 4318.2086 | 4.46             | 11368    | -54.1194   | 1377.1646 | 369.13           | 1079.3694       | 20.51    | 277.88           | 5.55            | 104.74 | 8.93  |
| Average values    |                |                |                |                   |           |          |           |                 |           |           |                  |          |            |           |                  |                 |          |                  |                 |        |       |
|                   |                |                |                |                   | 1072.12   | -        | 2255.31   | -               | -         | 3581.84   | 4.53             | 9135.87  | 22.87      | 1204.03   | 110.35           | 1023.02         | 11.70    | 225.97           | 5.34            | 98.43  | -     |
| Using sparse data |                |                |                |                   |           |          |           |                 |           |           |                  |          |            |           |                  |                 |          |                  |                 |        |       |
| 1                 | 5              | 5              | 100            | 10                | 572.3962  | 43200    | 439634    | 96.8028         | 173.33    | 1059.0180 | 13.49            | 21787    | -15.7253   | 637.3006  | 2.32             | 560.6819        | 11.34    | 85.01            | 2.05            | 102.75 | 83.09 |
| 2                 | 5              | 5              | 100            | 10                | 776.4855  | 150.32   | 1021      | -               | 104.35    | 1138.5350 | 11.26            | 20706    | 40.3139    | 837.3014  | 2.32             | 726.9487        | 7.83     | 46.63            | 6.37            | 94.81  | -     |
| 3                 | 5              | 5              | 100            | 10                | 590.6323  | 43200    | 426849    | -               | 100.06    | 1136.0205 | 19.53            | 25938    | 0          | 667.4799  | 2.54             | 579.1917        | 13.01    | 92.34            | 1.94            | 100.00 | -     |
| 4                 | 5              | 5              | 100            | 10                | 584.6340  | 43200    | 288817    | -               | 100.09    | 1052.9340 | 10.86            | 20632    | 15.5370    | 666.5156  | 2.32             | 618.2633        | 14.01    | 80.10            | -5.75           | 97.34  | -     |
| 5                 | 10             | 10             | 100            | 10                | 582.4194  | 43200    | 335730    | -               | 102.35    | 1057.0174 | 21.57            | 28378    | 29.6308    | 688.7765  | 3.76             | 577.1285        | 18.26    | 81.49            | 0.90            | 94.91  | -     |
| 6                 | 10             | 10             | 100            | 10                | 491.5244  | 43200    | 318005    | -               | 100.12    | 1068.3949 | 15.51            | 25389    | 7.0169     | 600.1164  | 3.98             | 477.3256        | 22.09    | 117.36           | 2.88            | 98.57  | -     |
| 7                 | 10             | 10             | 100            | 10                | 569.9921  | 43200    | 340571    | -               | 100.06    | 1105.5194 | 13.62            | 24017    | 38.4334    | 665.9559  | 4.09             | 567.4914        | 16.84    | 93.95            | 0.44            | 93.26  | -     |
| 8                 | 10             | 10             | 100            | 10                | 586.9255  | 43200    | 378286    | -               | 100.07    | 1098.0634 | 14.51            | 24309    | -50.0125   | 678.0269  | 3.90             | 559.1764        | 15.52    | 87.09            | 4.73            | 108.52 | -     |
| 9                 | 10             | 10             | 60             | 60                | 420.3089  | 9.92     | 214       | 359.3281        | 234.56    | 492.5427  | 3.14             | 12982    | -13.6192   | 438.3975  | 37.60            | 350.3112        | 4.30     | 17.19            | 16.65           | 103.24 | 14.51 |
| 10                | 10             | 10             | 60             | 60                | 384.8141  | 8.00     | 115       | 355.8930        | 146.95    | 466.9076  | 2.39             | 10055    | -7.1144    | 407.5359  | 36.61            | 329.4940        | 5.90     | 21.33            | 14.38           | 101.85 | 7.52  |
| 11                | 10             | 10             | 60             | 60                | 337.6191  | 11.14    | 685       | 258.8003        | 144.95    | 470.8486  | 2.31             | 8779     | -12.4923   | 363.0245  | 38.14            | 302.6208        | 7.52     | 39.46            | 10.37           | 103.70 | 23.35 |
| 12                | 10             | 10             | 60             | 60                | 390.9005  | 4.90     | 67        | 331.9232        | 147.90    | 458.0281  | 3.14             | 14367    | 33.9822    | 408.9025  | 34.32            | 331.9387        | 4.61     | 17.17            | 15.08           | 91.31  | 15.09 |
| 13                | 10             | 10             | 100            | 100               | 781.8445  | 73.16    | 557       | 571.4267        | 218.34    | 1266.7561 | 15.75            | 26355    | 12.5453    | 847.9040  | 317.48           | 663.7781        | 8.45     | 62.02            | 15.10           | 98.40  | 26.91 |
| 14                | 10             | 10             | 100            | 100               | 809.6117  | 43200    | 541620    | 590.9015        | 206.84    | 1267.5068 | 19.55            | 25924    | -8.3993    | 857.1348  | 293.90           | 676.8418        | 5.87     | 56.56            | 16.40           | 101.04 | 27.01 |
| 15                | 10             | 10             | 100            | 100               | 790.9876  | 163.63   | 988       | 565.6298        | 197.43    | 1185.4138 | 17.75            | 24788    | -12.3230   | 854.1306  | 307.46           | 723.1531        | 7.98     | 49.87            | 8.58            | 101.56 | 28.49 |
| 16                | 10             | 10             | 100            | 100               | 664.3987  | 114.55   | 988       | 431.3438        | 158.89    | 1170.4901 | 17.32            | 22708    | 2.2462     | 729.6685  | 310.07           | 577.0692        | 9.82     | 76.17            | 13.15           | 99.66  | 35.08 |
| Average values    |                |                |                |                   |           |          |           |                 |           |           |                  |          |            |           |                  |                 |          |                  |                 |        |       |
|                   |                |                |                |                   | 583.46    | -        | 192134.18 | -               | -         | 968.37    | 12.60            | 21069.62 | 3.75       | 646.76    | 87.55            | 538.83          | 10.83    | 63.98            | 7.70            | 99.43  | -     |

"-": No solution found.

Table 3: Upper and lower bounds for QQP instances

| #              | Instance Size  |                |                |                | MLP <sub>2</sub> |         | B&Bn    |           | VNS       |           | LP <sub>2</sub> |         |          | SDP <sub>2</sub> |         | F(SDP)    |         | Gaps Ubs %       |                  | Gaps NFS %      |                 | GapVNS % |       |   |
|----------------|----------------|----------------|----------------|----------------|------------------|---------|---------|-----------|-----------|-----------|-----------------|---------|----------|------------------|---------|-----------|---------|------------------|------------------|-----------------|-----------------|----------|-------|---|
|                | m <sub>1</sub> | m <sub>2</sub> | n <sub>1</sub> | n <sub>2</sub> | Time(s)          | Bf      | Time(s) | B&Bn      | B.Sol.    | Time(s)   | Ub              | Time(s) | F(LP)    | Iter.            | Time(s) | Ub        | F(SDP)  | SDP <sub>2</sub> | SDP <sub>2</sub> | LP <sub>2</sub> | LP <sub>2</sub> |          |       |   |
| 1              | 5              | 5              | 10             | 10             | 37.1142          | 0.23    | 12      | 37.1142   | 101.99    | 39.1896   | 0.16            | 143     | 34.3990  | 39.9934          | 0.36    | 27.8880   | 7.76    | 5.59             | 24.86            | 7.32            | 0               |          |       |   |
| 2              | 5              | 5              | 10             | 15             | 52.4691          | 0.22    | 77      | 52.4691   | 130.07    | 61.3968   | 0.16            | 161     | 24.3513  | 56.5294          | 1.12    | 33.6355   | 7.74    | 17.02            | 35.89            | 7.32            | 0               |          |       |   |
| 3              | 5              | 5              | 10             | 25             | 114.1747         | 0.19    | 13      | 114.1747  | 120.59    | 120.59    | 0.16            | 126     | 125.0765 | 125.0765         | 1.47    | 92.0954   | 9.55    | 9.55             | 19.34            | 100.00          | 0               |          |       |   |
| 4              | 5              | 5              | 10             | 50             | 80.9424          | 0.30    | 6       | 80.9424   | 134.02    | 91.7183   | 0.16            | 171     | 1.6703   | 84.0199          | 2.23    | 38.0187   | 3.80    | 13.31            | 53.03            | 97.94           | 0               |          |       |   |
| 5              | 5              | 5              | 10             | 100            | 130.4261         | 0.23    | 0       | 130.4261  | 113.68    | 151.3364  | 0.27            | 201     | 7.6329   | 141.071          | 7.25    | 30.5193   | 8.16    | 16.03            | 76.60            | 94.15           | 0               |          |       |   |
| 6              | 5              | 5              | 15             | 10             | 21.8967          | 0.31    | 0       | 21.8967   | 130.34    | 96.9536   | 0.17            | 239     | 55.9583  | 55.9583          | 1.14    | 49.9263   | 9.10    | 89.03            | 2.66             | 100.00          | 57.31           |          |       |   |
| 7              | 5              | 5              | 25             | 10             | 143.7133         | 1.42    | 39      | 143.7133  | 177.13    | 288.6498  | 0.17            | 694     | -0.9190  | 178.3933         | 0.47    | 149.2575  | 8.15    | 74.98            | 9.52             | 100.56          | 12.88           |          |       |   |
| 8              | 5              | 5              | 50             | 10             | 393.5570         | 59.86   | 1192    | 360.7495  | 229.15    | 1033.8927 | 0.59            | 2661    | -14.6865 | 441.0759         | 1.13    | 385.7142  | 12.07   | 162.70           | 1.99             | 103.73          | 8.34            |          |       |   |
| 9              | 5              | 5              | 100            | 10             | 1291.1254        | 3600    | 2030    | 356.5529  | 216.64    | 4249.7004 | 4.90            | 11113   | -31.1405 | 1486.1738        | 2.00    | 1327.3464 | 15.11   | 229.15           | -2.81            | 102.41          | 72.38           |          |       |   |
| 10             | 5              | 5              | 40             | 40             | 362.8493         | 5.85    | 17      | 362.8493  | 127.93    | 715.0145  | 0.31            | 1629    | 1.8553   | 381.0729         | 2.68    | 336.7773  | 5.02    | 97.06            | 7.19             | 99.49           | 12.18           |          |       |   |
| 11             | 5              | 5              | 60             | 60             | 542.8041         | 155.97  | 1180    | 389.6081  | 122.31    | 1560.2296 | 0.70            | 3887    | -28.0842 | 608.0549         | 5.41    | 478.3737  | 12.02   | 187.44           | 11.87            | 105.17          | 28.22           |          |       |   |
| 12             | 5              | 5              | 80             | 80             | 854.6622         | 464.04  | 570     | 779.7105  | 139.71    | 2761.0313 | 2.29            | 6308    | 37.1153  | 968.2970         | 12.17   | 757.4016  | 13.30   | 223.06           | 11.38            | 95.66           | 8.77            |          |       |   |
| 13             | 5              | 5              | 100            | 100            | 1145.8789        | 1478.31 | 625     | 816.3867  | 143.10    | 4212.0413 | 2.93            | 11114   | -20.2726 | 1327.5434        | 21.44   | 1090.0570 | 15.85   | 267.58           | 4.87             | 101.77          | 28.75           |          |       |   |
| 14             | 10             | 10             | 10             | 10             | 36.6114          | 0.17    | 11      | 26.9253   | 148.51    | 39.9469   | 0.14            | 141     | 36.4918  | 38.9251          | 1.14    | 36.0711   | 6.32    | 6.38             | 1.48             | 0.33            | 26.46           |          |       |   |
| 15             | 10             | 10             | 15             | 10             | 86.1085          | 0.27    | 0       | 54.6473   | 167.20    | 223.0802  | 0.14            | 271     | 11.2864  | 89.9561          | 0.41    | 84.9948   | 4.47    | 42.94            | 1.29             | 86.89           | 36.54           |          |       |   |
| 16             | 10             | 10             | 25             | 10             | 114.8968         | 2.59    | 112     | 5.9747    | 105.38    | 243.2758  | 0.19            | 605     | 21.7692  | 125.2858         | 1.12    | 75.8305   | 9.04    | 111.73           | 34.00            | 81.05           | 94.80           |          |       |   |
| 17             | 10             | 10             | 50             | 10             | 391.9525         | 147.55  | 3577    | -         | 100.08    | 1058.4495 | 0.83            | 2705    | -23.9010 | 454.8014         | 1.13    | 318.2968  | 16.03   | 170.05           | 18.79            | 106.10          | -               |          |       |   |
| 18             | 10             | 10             | 100            | 10             | 810.9480         | 3600    | 2648    | -         | 100.11    | 4258.7215 | 5.71            | 11103   | -9.6191  | 1336.3581        | 3.45    | 1100.1791 | 64.79   | 425.15           | -35.67           | 101.19          | -               |          |       |   |
| 19             | 10             | 10             | 15             | 15             | 37.5618          | 0.17    | 6       | 23.1040   | 113.13    | 46.6960   | 0.14            | 153     | -15.3526 | 38.9369          | 1.14    | 23.0914   | 3.66    | 24.32            | 38.52            | 140.87          | 38.49           |          |       |   |
| 20             | 10             | 10             | 25             | 10             | 95.2042          | 0.41    | 0       | 95.2042   | 171.37    | 108.7412  | 0.16            | 178     | 93.7791  | 103.2236         | 1.34    | 92.0262   | 8.42    | 14.22            | 3.34             | 1.50            | 0               |          |       |   |
| 21             | 10             | 10             | 50             | 10             | 69.1515          | 0.42    | 0       | 69.1515   | 113.38    | 80.6851   | 0.16            | 149     | -7.6079  | 74.7879          | 3.43    | 2.4988    | 8.15    | 16.68            | 96.39            | 111.00          | 0               |          |       |   |
| 22             | 10             | 10             | 100            | 10             | 161.6892         | 0.47    | 0       | 161.6892  | 129.04    | 177.3030  | 0.17            | 255     | 3.7566   | 168.9823         | 12.50   | 51.8430   | 4.51    | 9.66             | 67.94            | 97.68           | 0               |          |       |   |
| 23             | 10             | 10             | 20             | 20             | 131.0811         | 0.81    | 10      | 129.3433  | 214.39    | 205.0330  | 0.19            | 472     | 8.3135   | 138.9944         | 1.26    | 122.6588  | 6.04    | 56.42            | 6.43             | 93.66           | 1.33            |          |       |   |
| 24             | 10             | 10             | 30             | 30             | 227.6336         | 2.06    | 3       | 227.6336  | 285.09    | 418.1175  | 0.23            | 963     | -2.3890  | 257.3342         | 1.78    | 224.4371  | 13.05   | 83.68            | 1.40             | 101.05          | 0               |          |       |   |
| 25             | 10             | 10             | 40             | 40             | 442.7060         | 7.21    | 40      | 426.9467  | 291.54    | 761.3504  | 0.36            | 1780    | -7.6797  | 470.6542         | 4.18    | 416.1496  | 6.31    | 71.98            | 6.00             | 101.73          | 3.56            |          |       |   |
| 26             | 10             | 10             | 50             | 50             | 443.2191         | 199.40  | 3018    | 416.1478  | 225.03    | 1117.7271 | 0.78            | 2495    | -7.6313  | 479.6783         | 5.52    | 386.6770  | 8.23    | 152.18           | 12.76            | 101.72          | 6.11            |          |       |   |
| 27             | 10             | 10             | 60             | 60             | 578.8155         | 157.72  | 967     | 567.0773  | 293.33    | 1592.2643 | 0.95            | 4245    | 13.9065  | 635.3771         | 9.22    | 501.3198  | 9.77    | 175.09           | 13.39            | 97.60           | 2.03            |          |       |   |
| 28             | 10             | 10             | 80             | 80             | 951.5062         | 467.97  | 502     | 793.5855  | 119.07    | 2884.8072 | 2.40            | 7263    | -15.2184 | 1061.3408        | 19.16   | 890.6139  | 11.54   | 203.18           | 6.40             | 101.60          | 16.60           |          |       |   |
| 29             | 10             | 10             | 100            | 100            | 1166.2074        | 1372.15 | 504     | 982.5003  | 138.95    | 4319.6025 | 3.68            | 11232   | 16.5189  | 1350.7111        | 35.97   | 1120.8881 | 15.82   | 270.40           | 3.89             | 98.58           | 15.75           |          |       |   |
| 30             | 5              | 10             | 20             | 20             | 108.3716         | 0.59    | 0       | 108.3716  | 199.48    | 177.1385  | 0.19            | 452     | -4.8632  | 119.1304         | 1.23    | 104.0867  | 9.93    | 63.45            | 3.95             | 104.49          | 0               |          |       |   |
| 31             | 5              | 10             | 30             | 30             | 246.5957         | 2.84    | 10      | 233.8154  | 109.46    | 451.2082  | 0.23            | 982     | 11.5263  | 274.7059         | 1.70    | 201.1049  | 11.40   | 82.97            | 18.45            | 95.33           | 5.18            |          |       |   |
| 32             | 5              | 10             | 50             | 50             | 513.0801         | 38.58   | 449     | 499.2763  | 231.32    | 1158.3140 | 0.61            | 2643    | 17.3421  | 547.1051         | 4.73    | 450.1096  | 6.63    | 125.76           | 12.27            | 96.62           | 2.69            |          |       |   |
| 33             | 5              | 10             | 100            | 100            | 1182.5778        | 1165.92 | 491     | 934.0562  | 143.18    | 4265.8242 | 5.27            | 10141   | 13.1309  | 1347.3726        | 28.02   | 1111.5808 | 13.94   | 260.72           | 6.00             | 98.89           | 21.02           |          |       |   |
| 34             | 10             | 5              | 20             | 20             | 118.7491         | 0.86    | 43      | 118.7491  | 286.91    | 183.4253  | 0.18            | 489     | -19.1790 | 132.9072         | 1.23    | 106.5871  | 11.92   | 54.46            | 10.24            | 116.15          | 0               |          |       |   |
| 35             | 10             | 5              | 30             | 30             | 239.6988         | 1.27    | 0       | 239.6988  | 234.93    | 403.1507  | 0.22            | 963     | 12.2787  | 261.7913         | 1.35    | 233.1360  | 9.22    | 68.19            | 2.74             | 94.88           | 0               |          |       |   |
| 36             | 10             | 5              | 50             | 50             | 520.8115         | 46.48   | 741     | 494.2407  | 1170.2823 | 1170.2823 | 0.57            | 2575    | 6.0896   | 562.5154         | 3.86    | 484.9361  | 8.01    | 124.70           | 6.89             | 98.83           | 5.10            |          |       |   |
| 37             | 10             | 5              | 100            | 100            | 1241.1041        | 1556.88 | 603     | 950.0321  | 178.28    | 4280.1941 | 3.03            | 11054   | 29.1615  | 1399.2550        | 25.58   | 1169.4343 | 12.74   | 244.87           | 5.77             | 97.65           | 23.45           |          |       |   |
| 38             | 20             | 20             | 50             | 50             | 542.0124         | 29.51   | 111     | 434.802   | 120.73    | 1105.9891 | 0.64            | 2793    | 0        | 578.9865         | 10.41   | 522.1135  | 6.82    | 104.05           | 3.67             | 100.00          | 19.84           |          |       |   |
| 39             | 20             | 20             | 80             | 80             | 842.8949         | 371.54  | 500     | 732.9236  | 330.29    | 2772.9357 | 2.33            | 7393    | 0        | 958.4162         | 41.30   | 790.2278  | 13.71   | 228.98           | 6.25             | 100.00          | 13.05           |          |       |   |
| 40             | 20             | 20             | 100            | 100            | 1454.1774        | 2021.32 | 1208    | 1215.6090 | 133.72    | 4361.7027 | 6.44            | 10097   | -39.7973 | 1584.2569        | 70.03   | 1394.0508 | 8.95    | 199.94           | 4.13             | 102.74          | 16.41           |          |       |   |
| Minimum values |                |                |                |                | 36.6114          | 0.17    | 0       | 5.9747    | 100.08    | 38.9469   | 0.14            | 126     | -39.7973 | 38.9251          | 0.36    | 2.49      | 3.66    | 5.59             | -35.67           | 0.33            | 0               | 0        |       |   |
| Maximum values |                |                |                |                | 1454.1774        | 3600    | 3577    | 1215.6090 | 330.29    | 4361.7027 | 6.44            | 11232   | 93.7791  | 1584.2569        | 70.03   | 1394.0508 | 64.79   | 425.15           | 96.39            | 140.87          | 15.32           | 92.24    | -     |   |
| Average values |                |                |                |                | 449.44           | -       | -       | 532.87    | -         | -         | -               | -       | -        | 1328.78          | -       | 1.22      | 3300.97 | 3.85             | 512.12           | 10.92           | 119.61          | 15.32    | 92.24 | - |

"-": No solution found.

Table 4: Upper and lower bounds for QLBP using Dense Data

| #              | Instance Size  |                |                |                | MLP <sub>2</sub> |         | B&Bn    |          | VNS      |           | LP <sub>2</sub> |         |           | SDP <sub>2</sub> |         |          | Gaps Ubs %       |                 | Gaps NFS %       |                 | Gaps Nfs % |        |        |
|----------------|----------------|----------------|----------------|----------------|------------------|---------|---------|----------|----------|-----------|-----------------|---------|-----------|------------------|---------|----------|------------------|-----------------|------------------|-----------------|------------|--------|--------|
|                | m <sub>1</sub> | m <sub>2</sub> | n <sub>1</sub> | n <sub>2</sub> | Bf               | Time(s) | B&Bn    | B.Sol.   | Time(s)  | Ub        | Time(s)         | Iter.   | F(LP)     | Ub               | Time(s) | F(SDP)   | SDP <sub>2</sub> | LP <sub>2</sub> | SDP <sub>2</sub> | LP <sub>2</sub> |            |        |        |
| 1              | 5              | 5              | 10             | 10             | 34.1502          | 0.15    | 0       | -        | 100.13   | 37.9442   | 0.13            | 107     | 29.9477   | 37.9781          | 1.14    | 24.3563  | 11.21            | 11.11           | 28.08            | 12.31           | -          |        |        |
| 2              | 5              | 5              | 10             | 15             | 53.4395          | 0.42    | 0       | 53.3051  | 152.79   | 56.1513   | 0.17            | 117     | 40.3059   | 54.6979          | 1.26    | 38.4259  | 2.35             | 5.07            | 28.09            | 24.58           | 0.23       |        |        |
| 3              | 5              | 5              | 10             | 25             | 59.6417          | 0.14    | 0       | 50.8067  | 208.00   | 64.9719   | 0.14            | 177     | 45.9543   | 64.1638          | 1.12    | 47.7580  | 7.58             | 8.94            | 19.43            | 22.95           | 14.81      |        |        |
| 4              | 5              | 5              | 10             | 50             | 67.1810          | 0.19    | 23      | 67.1810  | 108.41   | 81.7859   | 0.16            | 208     | 5.6003    | 79.6539          | 2.01    | 25.7265  | 18.57            | 21.74           | 61.71            | 91.66           | 0          |        |        |
| 5              | 5              | 5              | 10             | 100            | 102.9261         | 0.16    | 0       | 102.6718 | 111.13   | 125.1374  | 0.16            | 169     | 15.2645   | 124.4100         | 6.15    | 25.9250  | 20.87            | 21.58           | 74.81            | 85.17           | 0.25       |        |        |
| 6              | 5              | 5              | 15             | 10             | 26.2012          | 0.17    | 7       | 100.03   | 30.4944  | 30.4944   | 0.16            | 176     | 17.0809   | 28.2108          | 1.15    | 16.1596  | 7.69             | 16.39           | 38.33            | 32.52           | -          |        |        |
| 7              | 5              | 5              | 25             | 10             | 82.4997          | 0.27    | 9       | 20.5651  | 232.39   | 85.7821   | 0.19            | 1617    | 60.6704   | 89.4066          | 1.14    | 64.9940  | 8.37             | 3.98            | 21.22            | 26.46           | 75.07      |        |        |
| 8              | 5              | 5              | 50             | 10             | 293.8953         | 2.21    | 99      | 100.09   | 362.3239 | 330.7794  | 1.30            | 7570    | 106.4043  | 330.7794         | 1.14    | 282.9472 | 12.55            | 23.28           | 3.73             | 63.80           | -          |        |        |
| 9              | 5              | 5              | 100            | 10             | 560.9800         | 3600    | 29030   | -        | 100.12   | 1076.4454 | 13.40           | 22295   | 52.2421   | 658.5683         | 2.00    | 592.6118 | 17.40            | 91.89           | -5.64            | 90.69           | -          |        |        |
| 10             | 5              | 5              | 40             | 40             | 223.3532         | 0.50    | 0       | 157.0220 | 183.83   | 243.4091  | 0.62            | 5111    | 189.3328  | 246.7761         | 2.54    | 195.2100 | 10.49            | 8.98            | 12.60            | 15.23           | 29.70      |        |        |
| 11             | 5              | 5              | 60             | 60             | 306.7181         | 4.29    | 68      | 241.1569 | 139.39   | 399.6956  | 2.12            | 10206   | 1.6713    | 335.7283         | 5.62    | 251.5710 | 9.46             | 30.31           | 17.98            | 99.46           | 21.38      |        |        |
| 12             | 5              | 5              | 80             | 80             | 502.0773         | 63.46   | 1018    | 294.0131 | 155.13   | 823.5886  | 5.74            | 14461   | -24.9294  | 583.3565         | 11.62   | 438.7834 | 16.19            | 64.04           | 12.61            | 104.97          | 41.44      |        |        |
| 13             | 5              | 5              | 100            | 100            | 777.9995         | 101.29  | 996     | 561.3215 | 147.25   | 1236.0251 | 11.83           | 23089   | -39.2158  | 841.0861         | 22.76   | 720.8275 | 8.11             | 58.87           | 7.35             | 105.04          | 27.85      |        |        |
| 14             | 10             | 10             | 10             | 10             | 22.6062          | 0.14    | 0       | -0.4029  | 126.35   | 25.0183   | 0.12            | 97      | -5.7553   | 25.1944          | 1.12    | 21.4052  | 11.45            | 10.67           | 5.31             | 125.46          | 101.78     |        |        |
| 15             | 10             | 10             | 15             | 10             | 33.6002          | 0.19    | 8       | 100.10   | 37.8076  | 37.8076   | 0.14            | 214     | 30.1942   | 36.7806          | 1.12    | 30.8546  | 9.47             | 12.52           | 8.17             | 10.14           | -          |        |        |
| 16             | 10             | 10             | 25             | 10             | 83.5487          | 0.34    | 16      | 100.04   | 98.8408  | 98.8408   | 0.20            | 1441    | 63.4642   | 94.5922          | 1.14    | 76.2523  | 13.22            | 18.30           | 8.73             | 24.04           | -          |        |        |
| 17             | 10             | 10             | 50             | 10             | 224.3872         | 2.62    | 128     | 100.04   | 293.6610 | 293.6610  | 1.28            | 7090    | 190.0849  | 260.2859         | 1.12    | 220.5750 | 16.00            | 30.87           | 1.70             | 15.29           | -          |        |        |
| 18             | 10             | 10             | 100            | 10             | 528.3890         | 3600    | 27446   | -        | 100.06   | 1115.1828 | 14.04           | 24070   | 18.2724   | 673.5000         | 3.34    | 505.6578 | 27.46            | 111.05          | 4.30             | 96.54           | -          |        |        |
| 19             | 10             | 10             | 15             | 15             | 26.3825          | 0.16    | 0       | 21.1404  | 118.05   | 30.3606   | 0.16            | 136     | 24.8885   | 28.6558          | 1.14    | 25.4341  | 8.62             | 15.08           | 3.59             | 5.66            | 19.87      |        |        |
| 20             | 10             | 10             | 20             | 20             | 31.7317          | 0.53    | 11      | 31.7317  | 111.74   | 36.3980   | 0.16            | 137     | 3.8001    | 33.1617          | 1.25    | 3.5893   | 4.51             | 14.71           | 88.69            | 88.02           | 0          |        |        |
| 21             | 10             | 10             | 30             | 30             | 72.1211          | 0.45    | 0       | 72.1211  | 104.01   | 74.4190   | 0.17            | 127     | 23.6518   | 75.0389          | 2.87    | 20.4985  | 4.05             | 3.19            | 71.58            | 67.21           | 0          |        |        |
| 22             | 10             | 10             | 50             | 50             | 136.4254         | 0.58    | 0       | 136.4254 | 160.91   | 138.6006  | 0.47            | 178     | 21.9668   | 138.6611         | 10.42   | 23.1642  | 1.64             | 1.64            | 83.02            | 83.90           | 0          |        |        |
| 23             | 10             | 10             | 75             | 75             | 252.9291         | 0.19    | 0       | 68.3765  | 267.24   | 81.4697   | 0.17            | 844     | 67.0361   | 82.7945          | 1.12    | 71.0009  | 9.62             | 7.87            | 6.00             | 11.24           | 9.47       |        |        |
| 24             | 10             | 10             | 100            | 100            | 449.3814         | 342.98  | 5397    | 449.3814 | 140.04   | 791.8533  | 6.91            | 17234   | -8.9498   | 587.1932         | 9.44    | 275.2894 | 8.85             | 11.45           | 14.87            | 74.95           | 10.01      |        |        |
| 25             | 10             | 10             | 40             | 40             | 277.3507         | 0.62    | 10      | 272.6447 | 197.17   | 284.4819  | 0.70            | 5434    | 233.7903  | 287.5607         | 3.87    | 236.2920 | 3.68             | 2.57            | 14.80            | 15.71           | 1.70       |        |        |
| 26             | 10             | 10             | 50             | 50             | 249.0903         | 1.33    | 3       | 207.9638 | 123.61   | 292.5315  | 1.23            | 7092    | 8.9215    | 274.4524         | 5.16    | 213.5404 | 10.18            | 17.44           | 14.27            | 96.42           | 16.51      |        |        |
| 27             | 10             | 10             | 60             | 60             | 348.7664         | 6.96    | 118     | 322.6085 | 418.69   | 456.8337  | 1.75            | 8436    | 9.7112    | 363.7417         | 9.44    | 275.2894 | 4.29             | 30.99           | 21.07            | 97.22           | 7.50       |        |        |
| 28             | 10             | 10             | 80             | 80             | 535.1680         | 34.98   | 5397    | 449.3814 | 140.04   | 791.8533  | 6.91            | 17234   | -8.9498   | 587.1932         | 9.44    | 275.2894 | 9.72             | 47.96           | 16.06            | 101.67          | 16.03      |        |        |
| 29             | 10             | 10             | 100            | 100            | 635.8799         | 61.63   | 607     | 481.5597 | 143.05   | 1127.7385 | 12.54           | 17981   | -128.7959 | 702.7596         | 40.23   | 561.8533 | 10.52            | 77.35           | 11.64            | 120.25          | 24.27      |        |        |
| 30             | 5              | 10             | 20             | 20             | 78.9154          | 0.20    | 0       | 78.9232  | 299.23   | 88.9431   | 0.19            | 547     | 69.5909   | 87.5558          | 1.12    | 55.1596  | 10.95            | 12.71           | 30.10            | 11.82           | 0.37       |        |        |
| 31             | 5              | 10             | 30             | 30             | 107.7690         | 0.33    | 14      | 107.7690 | 287.46   | 124.9927  | 0.27            | 1806    | 81.5047   | 125.6967         | 1.45    | 83.8296  | 16.64            | 15.98           | 22.21            | 24.37           | 0          |        |        |
| 32             | 5              | 10             | 50             | 50             | 261.2599         | 2.28    | 76      | 218.2693 | 189.77   | 327.7845  | 1.37            | 6654    | 17.4233   | 281.1305         | 4.62    | 208.9514 | 7.61             | 25.46           | 20.02            | 93.33           | 16.46      |        |        |
| 33             | 5              | 10             | 100            | 100            | 718.3253         | 3600    | 60108   | 592.5332 | 156.26   | 1223.9257 | 10.11           | 21909   | -5.3676   | 778.6186         | 30.06   | 641.4299 | 8.39             | 70.39           | 10.70            | 100.75          | 17.51      |        |        |
| 34             | 10             | 5              | 20             | 20             | 82.5543          | 0.22    | 6       | -        | 100.14   | 95.4767   | 0.19            | 877     | 65.9505   | 96.2109          | 1.25    | 92.5293  | 16.54            | 15.65           | 36.37            | 20.11           | -          |        |        |
| 35             | 10             | 5              | 30             | 30             | 139.8913         | 0.25    | 0       | 64.7864  | 108.05   | 153.7862  | 0.23            | 1468    | 73.5024   | 148.7222         | 1.34    | 117.1745 | 6.31             | 9.93            | 16.24            | 47.46           | 53.69      |        |        |
| 36             | 10             | 5              | 50             | 50             | 264.8352         | 3.40    | 223     | 163.2211 | 169.41   | 336.7424  | 1.33            | 6238    | -2.5999   | 293.6446         | 3.87    | 242.5335 | 10.88            | 27.15           | 8.42             | 100.98          | 38.37      |        |        |
| 37             | 10             | 5              | 100            | 100            | 697.2383         | 54.62   | 458     | 508.3070 | 236.43   | 1189.5655 | 15.43           | 23778   | -23.5572  | 793.4361         | 27.89   | 657.2326 | 13.80            | 70.61           | 5.74             | 103.38          | 27.10      |        |        |
| 38             | 20             | 20             | 50             | 50             | 253.2034         | 3.28    | 102     | 223.5181 | 124.83   | 341.8596  | 1.42            | 6701    | 13.6176   | 286.0169         | 10.55   | 210.2594 | 12.96            | 35.01           | 16.96            | 94.62           | 11.72      |        |        |
| 39             | 20             | 20             | 80             | 80             | 408.3108         | 2078.71 | 45814   | 310.9974 | 123.07   | 695.9120  | 6.02            | 13626   | 0.7315    | 448.6559         | 42.10   | 336.2831 | 9.88             | 70.44           | 17.64            | 99.82           | 23.83      |        |        |
| 40             | 20             | 20             | 100            | 100            | 714.4549         | 56.47   | 524     | 634.0884 | 209.65   | 1145.7748 | 21.20           | 30185   | -5.4699   | 832.7352         | 74.51   | 694.2285 | 16.56            | 60.37           | 2.83             | 100.77          | 11.25      |        |        |
| Minimum values |                |                |                |                | 22.6062          | 0.14    | 0       | -0.4029  | 100.03   | 25.0183   | 0.12            | 97      | -128.7959 | 25.1944          | 1.12    | 3.5893   | 1.64             | 1.64            | -5.64            | -5.64           | 5.66       | 0      |        |
| Maximum values |                |                |                |                | 777.9995         | 3600    | 60108   | 634.0884 | 418.69   | 1236.0251 | 21.20           | 30185   | 233.7903  | 841.0861         | 74.51   | 720.8275 | 27.46            | 111.05          | 27.46            | 29.83           | 88.69      | 125.46 | 101.78 |
| Average values |                |                |                |                | 256.37           | -       | 4307.97 | -        | -        | 385.19    | 3.35            | 7321.42 | 34.44     | 287.03           | 9.10    | 222.30   | 10.86            | 19.93           | 22.06            | 22.06           | 65.14      | 65.14  | -      |

“-”: No solution found.

Table 5: Upper and lower bounds for QLBP using Sparse Data

| #                 | Instance Size |    |     | MILP <sub>2</sub> |           | B&Bn     |           | VNS       |           | LP <sub>2</sub> |         |          | SDP <sub>2</sub> |           | Gaps Ubs % |           | Gaps NFS %       |                 | Gapvns % |                  |                 |
|-------------------|---------------|----|-----|-------------------|-----------|----------|-----------|-----------|-----------|-----------------|---------|----------|------------------|-----------|------------|-----------|------------------|-----------------|----------|------------------|-----------------|
|                   | m1            | m2 | n1  | n2                | Bf        | Time(s)  | B&Bn      | B.Sol.    | Time(s)   | Ub              | Time(s) | Iter.    | F(LP)            | Ub        | Time(s)    | F(SDP)    | SDP <sub>2</sub> | LP <sub>2</sub> |          | SDP <sub>2</sub> | LP <sub>2</sub> |
| Using dense data  |               |    |     |                   |           |          |           |           |           |                 |         |          |                  |           |            |           |                  |                 |          |                  |                 |
| 1                 | 5             | 5  | 100 | 10                | 1141.6494 | 1206.41  | 489       | 600.3199  | 269.88    | 4245.7666       | 5.68    | 10788    | 12.1928          | 1302.5021 | 2.11       | 1117.0740 | 14.09            | 271.90          | 2.15     | 98.93            | 47.42           |
| 2                 | 5             | 5  | 100 | 10                | 1237.7913 | 43200    | 24739     | 100.14    | 4275.7176 | 4.84            | 10484   | 41.6538  | 1384.5175        | 2.23      | 1217.7236  | 11.85     | 245.43           | 1.62            | 96.63    | -                | -               |
| 3                 | 5             | 5  | 100 | 10                | 1141.6494 | 1212.10  | 489       | 100.57    | 4245.7666 | 5.77            | 10788   | 12.1928  | 1302.5021        | 2.56      | 1117.0740  | 14.09     | 271.90           | 2.15            | 98.93    | -                | -               |
| 4                 | 5             | 5  | 100 | 10                | 1296.8547 | 43200    | 23970     | 100.79    | 4250.5737 | 4.85            | 10455   | 45.9600  | 1377.6426        | 3.12      | 1354.8521  | 10.86     | 227.76           | -4.47           | 96.46    | -                | -               |
| 5                 | 10            | 10 | 100 | 10                | 1418.4456 | 2002.72  | 1132      | 100.01    | 4371.2336 | 5.45            | 10948   | 117.0846 | 1572.7904        | 3.21      | 1414.1212  | 10.88     | 208.17           | 0.30            | 91.75    | -                | -               |
| 6                 | 10            | 10 | 100 | 10                | 1144.7209 | 1162.31  | 519       | 100.04    | 4217.5311 | 6.76            | 11283   | 9.2065   | 1401.7868        | 3.09      | 1099.6248  | 22.46     | 268.43           | 3.93            | 99.20    | -                | -               |
| 7                 | 10            | 10 | 100 | 10                | 1292.7546 | 1198.29  | 573       | 100.52    | 4314.7101 | 5.82            | 10869   | -19.7683 | 1507.3185        | 3.45      | 1154.2973  | 16.60     | 233.76           | 10.71           | 101.53   | -                | -               |
| 8                 | 10            | 10 | 100 | 10                | 1019.9108 | 1863.91  | 1075      | 100.99    | 4110.4849 | 5.26            | 10903   | -37.3888 | 1199.4160        | 1.89      | 956.6657   | 17.60     | 303.02           | 6.19            | 103.67   | -                | -               |
| 9                 | 10            | 10 | 60  | 60                | 625.8226  | 153.57   | 1007      | 533.2816  | 187.90    | 1550.4507       | 0.97    | 3796     | 9.2865           | 672.3082  | 9.55       | 571.9713  | 7.43             | 147.75          | 8.60     | 98.52            | 14.79           |
| 10                | 10            | 10 | 60  | 60                | 752.8709  | 92.51    | 566       | 671.6018  | 245.89    | 1660.4827       | 0.94    | 4394     | 4.2149           | 788.2723  | 9.53       | 667.3929  | 4.70             | 120.55          | 11.35    | 99.44            | 10.79           |
| 11                | 10            | 10 | 60  | 60                | 658.3569  | 193.21   | 1590      | 646.4093  | 345.11    | 1579.0573       | 1.20    | 3927     | -13.0646         | 710.0734  | 9.98       | 593.0744  | 7.86             | 139.85          | 9.92     | 101.98           | 1.81            |
| 12                | 10            | 10 | 60  | 60                | 487.9544  | 3164.13  | 23969     | 421.6188  | 301.30    | 1513.8743       | 1.14    | 3661     | 1.0076           | 553.2618  | 9.56       | 366.8582  | 13.38            | 210.25          | 24.82    | 99.79            | 13.59           |
| 13                | 10            | 10 | 100 | 100               | 1338.9288 | 1302.65  | 504       | 1120.6216 | 129.90    | 4380.7599       | 3.76    | 11457    | 57.9483          | 1507.4583 | 34.88      | 1295.9706 | 12.59            | 227.18          | 3.21     | 95.67            | 16.30           |
| 14                | 10            | 10 | 100 | 100               | 1188.1086 | 1167.41  | 544       | 975.7671  | 154.82    | 4335.8977       | 6.10    | 10257    | -13.2206         | 1383.7955 | 34.12      | 1150.6494 | 16.47            | 264.94          | 3.15     | 101.11           | 17.87           |
| 15                | 10            | 10 | 100 | 100               | 1331.5579 | 1122.41  | 504       | 1035.0697 | 147.70    | 4389.8896       | 5.24    | 10837    | -10.0054         | 1501.1034 | 33.45      | 1263.7394 | 12.73            | 229.68          | 5.09     | 100.75           | 22.27           |
| 16                | 10            | 10 | 100 | 100               | 1175.6087 | 2144.57  | 1008      | 970.1638  | 171.46    | 4262.3415       | 3.04    | 11208    | -44.0783         | 1333.5829 | 33.66      | 1104.0975 | 13.44            | 262.56          | 6.08     | 103.75           | 17.48           |
| Average values    |               |    |     |                   |           |          |           |           |           |                 |         |          |                  |           |            |           |                  |                 |          |                  |                 |
|                   |               |    |     |                   | 1078.31   | -        | 5167.37   | -         | -         | 3606.53         | 4.17    | 9134.68  | 10.82            | 1222.39   | 12.21      | 1027.82   | 12.93            | 227.07          | 5.92     | 99.25            | -               |
| Using sparse data |               |    |     |                   |           |          |           |           |           |                 |         |          |                  |           |            |           |                  |                 |          |                  |                 |
| 1                 | 5             | 5  | 100 | 10                | 607.7909  | 43200    | 490475    | 100.04    | 1081.1269 | 12.39           | 22027   | 19.9100  | 685.5295         | 2.01      | 597.9571   | 12.79     | 77.88            | 1.62            | 96.72    | -                | -               |
| 2                 | 5             | 5  | 100 | 10                | 530.0028  | 43200    | 524125    | 100.77    | 1077.1140 | 12.25           | 21144   | -30.8247 | 674.1771         | 2.00      | 509.0741   | 27.20     | 103.23           | 3.83            | 105.82   | -                | -               |
| 3                 | 5             | 5  | 100 | 10                | 662.9084  | 104.56   | 779       | 100.60    | 1036.5351 | 23.78           | 34784   | 0.3956   | 727.7409         | 2.01      | 641.3587   | 9.78      | 56.36            | 3.25            | 99.94    | -                | -               |
| 4                 | 5             | 5  | 100 | 10                | 615.5096  | 33800.13 | 364843    | 100.52    | 1086.5915 | 12.82           | 27290   | -79.1913 | 688.2085         | 2.00      | 572.9640   | 11.81     | 76.54            | 6.91            | 112.87   | -                | -               |
| 5                 | 10            | 10 | 100 | 10                | 590.5492  | 43200    | 326464    | 100.32    | 1127.0908 | 16.29           | 23170   | 8.2361   | 699.2513         | 3.32      | 586.1238   | 18.41     | 90.85            | 0.75            | 98.61    | -                | -               |
| 6                 | 10            | 10 | 100 | 10                | 510.5575  | 43200    | 421298    | 100.70    | 1048.5178 | 16.02           | 26910   | 21.5741  | 595.8553         | 3.21      | 500.3244   | 16.71     | 105.37           | 2.00            | 95.77    | -                | -               |
| 7                 | 10            | 10 | 100 | 10                | 643.9213  | 145.29   | 746       | 100.76    | 1097.0511 | 20.33           | 29310   | 78.8401  | 722.7498         | 3.23      | 602.2587   | 12.24     | 70.37            | 6.47            | 87.76    | -                | -               |
| 8                 | 10            | 10 | 100 | 10                | 516.1985  | 43200    | 368276    | 100.59    | 1053.4254 | 14.31           | 24097   | -1.6314  | 612.9446         | 3.31      | 0          | 18.74     | 104.07           | 100.00          | 100.32   | -                | -               |
| 9                 | 10            | 10 | 60  | 60                | 292.6539  | 10.83    | 365       | 273.3896  | 382.48    | 427.3721        | 2.09    | 9332     | 0.1735           | 329.8581  | 9.53       | 213.4537  | 12.71            | 46.03           | 27.06    | 99.94            | 6.58            |
| 10                | 10            | 10 | 60  | 60                | 352.6367  | 5.60     | 142       | 321.6301  | 206.27    | 465.6281        | 2.59    | 11738    | 12.9288          | 373.3216  | 9.44       | 276.2452  | 5.87             | 32.04           | 21.66    | 96.33            | 8.79            |
| 11                | 10            | 10 | 60  | 60                | 335.3551  | 10.22    | 451       | 298.3066  | 124.52    | 468.6742        | 2.53    | 10644    | -11.8498         | 369.8965  | 9.33       | 300.7020  | 10.30            | 39.75           | 10.33    | 103.53           | 11.05           |
| 12                | 10            | 10 | 60  | 60                | 391.1984  | 4.43     | 49        | 357.9167  | 214.91    | 473.6583        | 2.37    | 11046    | 3.0652           | 419.4563  | 9.42       | 326.2233  | 7.22             | 21.08           | 16.61    | 99.22            | 8.51            |
| 13                | 10            | 10 | 100 | 100               | 611.2067  | 200.58   | 2575      | 416.1907  | 93.27     | 1167.3997       | 14.73   | 19934    | 2.4490           | 704.1117  | 36.72      | 496.9076  | 15.19            | 90.98           | 18.71    | 99.60            | 31.91           |
| 14                | 10            | 10 | 100 | 100               | 644.3821  | 90.59    | 1024      | 480.3466  | 157.28    | 1176.4985       | 10.11   | 22102    | -26.0591         | 750.6684  | 36.74      | 536.3100  | 16.49            | 82.65           | 16.77    | 104.04           | 25.46           |
| 15                | 10            | 10 | 100 | 100               | 817.2638  | 103.32   | 904       | 660.7204  | 58.78     | 1236.2411       | 13.18   | 23830    | 18.4616          | 863.3156  | 36.08      | 709.7394  | 5.63             | 51.27           | 13.16    | 97.74            | 19.15           |
| 16                | 10            | 10 | 100 | 100               | 674.7793  | 55.30    | 463       | 525.0709  | 141.92    | 1175.7831       | 10.01   | 19419    | 44.5088          | 767.7439  | 38.37      | 598.1274  | 13.78            | 74.25           | 11.36    | 93.40            | 22.19           |
| Average values    |               |    |     |                   |           |          |           |           |           |                 |         |          |                  |           |            |           |                  |                 |          |                  |                 |
|                   |               |    |     |                   | 549.81    | -        | 156436.18 | -         | -         | 949.94          | 11.61   | 21048.06 | 3.81             | 624.05    | 12.92      | 466.77    | 13.42            | 70.17           | 16.28    | 99.47            | -               |

“-”: No solution found.

Table 6: Upper and lower bounds for QLBP instances



| #              | Instance Size |       |       |       | LP <sub>1</sub> |         |          |          | SDP <sub>1</sub> |         |           | Gaps Feas %     |                  |   |
|----------------|---------------|-------|-------|-------|-----------------|---------|----------|----------|------------------|---------|-----------|-----------------|------------------|---|
|                | $m_1$         | $m_2$ | $n_1$ | $n_2$ | Ub              | Time(s) | Iter.    | F(LP)    | Ub               | Time(s) | F(SDP)    | LP <sub>1</sub> | SDP <sub>1</sub> |   |
| 1              | 5             | 5     | 100   | 10    | 4194.9434       | 4.98    | 10395    | 4.5345   | 1238.8025        | 2.32    | 1039.1279 | 99.89           | <b>16.16</b>     |   |
| 2              |               |       | 150   | 10    | 9654.1857       | 23.81   | 23422    | 2.0463   | 2670.3033        | 3.37    | 2375.2888 | 99.98           | <b>11.28</b>     |   |
| 3              |               |       | 200   | 10    | 16701.8323      | 73.43   | 44403    | -38.7009 | 3714.2002        | 5.51    | 3266.1342 | 100.23          | <b>12.12</b>     |   |
| 4              |               |       | 300   | 10    | 37503.7317      | 836.45  | 97283    | -5.8466  | 6767.6996        | 16.25   | 5857.4926 | 100.02          | <b>13.45</b>     |   |
| 5              |               |       | 500   | 10    | 177800.7033     | 1007.03 | 75857    | -        | 14958.2734       | 56.04   | -         | -               | -                | - |
| 6              |               |       | 700   | 10    | 378737.3895     | 1015.91 | 74330    | -        | 25467.8561       | 132.91  | -         | -               | -                | - |
| 7              |               |       | 100   | 50    | 4355.3638       | 5.41    | 10536    | 26.5584  | 1449.7933        | 15.05   | 1264.2912 | 99.39           | <b>12.81</b>     |   |
| 8              |               |       | 150   | 50    | 9575.2988       | 20.74   | 23593    | 18.5367  | 2610.9838        | 18.05   | 2343.7303 | 99.81           | <b>10.30</b>     |   |
| 9              |               |       | 200   | 50    | 16909.6497      | 75.32   | 46118    | -25.1861 | 3977.7075        | 40.12   | 3507.2289 | 100.15          | <b>11.81</b>     |   |
| 10             |               |       | 300   | 50    | 37945.5939      | 360.29  | 96831    | 34.0233  | 6951.5177        | 56.58   | 6081.9603 | 99.91           | <b>12.51</b>     |   |
| 11             |               |       | 500   | 50    | 174973.7301     | 1006.17 | 84807    | -        | 15499.4804       | 158.28  | -         | -               | -                | - |
| 12             |               |       | 700   | 50    | 390029.3989     | 1011.60 | 45045    | -        | 25816.0354       | 383.34  | -         | -               | -                | - |
| 13             |               |       | 120   | 120   | 6297.2916       | 10.71   | 14982    | 46.0719  | 2075.0745        | 767.31  | 1795.5508 | 99.27           | <b>13.71</b>     |   |
| 14             |               |       | 150   | 150   | 9464.9006       | 21.71   | 24354    | -31.8676 | 2400.0379        | 1652.21 | 1990.5944 | 100.34          | <b>17.09</b>     |   |
| Minimum values |               |       |       |       | 4194.9434       | 4.98    | 10395    | -38.7009 | 1238.8025        | 2.32    | 1039.1279 | 99.27           | <b>10.30</b>     |   |
| Maximum values |               |       |       |       | 390029.3989     | 1015.91 | 97283    | 46.0719  | 25816.0354       | 1652.21 | 6081.9603 | 100.34          | <b>17.09</b>     |   |
| Average values |               |       |       |       | 91010.28        | 390.96  | 47996.85 | -        | 8256.98          | 236.23  | -         | -               | -                |   |

“-”: No solution found.

Table 7: Average bounds for QGBP on large instances using Dense Data

| #              | Instance Size |       |       |       | LP <sub>1</sub> |         |          |          | SDP <sub>1</sub> |         |           | Gaps Feas %     |                  |   |
|----------------|---------------|-------|-------|-------|-----------------|---------|----------|----------|------------------|---------|-----------|-----------------|------------------|---|
|                | $m_1$         | $m_2$ | $n_1$ | $n_2$ | Ub              | Time(s) | Iter.    | F(LP)    | Ub               | Time(s) | F(SDP)    | LP <sub>1</sub> | SDP <sub>1</sub> |   |
| 1              | 5             | 5     | 100   | 10    | 1095.6861       | 17.27   | 26879    | 20.6451  | 731.5054         | 2.29    | 659.6395  | 98.09           | <b>9.82</b>      |   |
| 2              |               |       | 150   | 10    | 2280.5856       | 59.52   | 44755    | -56.0256 | 1051.1653        | 3.36    | 882.4457  | 102.56          | <b>16.00</b>     |   |
| 3              |               |       | 200   | 10    | 4225.6556       | 249.25  | 67026    | 16.5409  | 1887.5297        | 5.51    | 1594.3054 | 99.59           | <b>15.52</b>     |   |
| 4              |               |       | 300   | 10    | 13860.4077      | 1002.15 | 68182    | 315.9628 | 3430.2302        | 15.98   | 3074.0659 | 97.83           | <b>10.38</b>     |   |
| 5              |               |       | 500   | 10    | 48942.0436      | 1005.91 | 34492    | -        | 7335.3501        | 53.00   | -         | -               | -                | - |
| 6              |               |       | 700   | 10    | 100248.4777     | 1012.85 | 19868    | -        | 12132.2680       | 132.51  | -         | -               | -                | - |
| 7              |               |       | 100   | 50    | 1118.4788       | 12.64   | 21334    | -6.4210  | 653.6022         | 12.48   | 503.0565  | 100.55          | <b>23.62</b>     |   |
| 8              |               |       | 150   | 50    | 2484.7023       | 77.70   | 44284    | -23.4666 | 1247.2033        | 16.58   | 1071.4666 | 100.96          | <b>14.23</b>     |   |
| 9              |               |       | 200   | 50    | 4177.2374       | 252.30  | 72914    | -14.0865 | 1890.7834        | 30.49   | 1619.8600 | 100.34          | <b>14.32</b>     |   |
| 10             |               |       | 300   | 50    | 12534.5193      | 1002.15 | 79430    | 144.4320 | 3615.3442        | 52.95   | 3175.6822 | 98.88           | <b>12.26</b>     |   |
| 11             |               |       | 500   | 50    | 49729.9336      | 1005.17 | 28753    | -        | 7371.3501        | 157.56  | 6487.6917 | -               | <b>12.03</b>     |   |
| 12             |               |       | 700   | 50    | 100839.1351     | 1011.32 | 19293    | -        | 12706.9392       | 400.54  | -         | -               | -                | - |
| 13             |               |       | 120   | 120   | 1709.3305       | 29.63   | 30070    | -7.1515  | 1045.4132        | 566.40  | 817.0890  | 100.46          | <b>21.78</b>     |   |
| 14             |               |       | 150   | 150   | 2642.9506       | 70.14   | 43902    | 1.6827   | 1412.1566        | 1571.11 | 1101.6331 | 99.93           | <b>22.42</b>     |   |
| Minimum values |               |       |       |       | 1095.6861       | 12.64   | 19293    | -56.0256 | 653.6022         | 2.29    | 503.0565  | 97.83           | <b>9.82</b>      |   |
| Maximum values |               |       |       |       | 100839.1351     | 1012.85 | 79430    | 315.9628 | 12706.9392       | 1571.11 | 6487.6917 | 102.56          | <b>23.62</b>     |   |
| Average values |               |       |       |       | 24706.36        | 486.28  | 42941.57 | -        | 4036.48          | 215.76  | -         | -               | -                |   |

“-”: No solution found.

Table 8: Average bounds for QGBP on large instances using Sparse Data

| #              | Instance Size |       |       |       | LP <sub>2</sub> |         |          |         | SDP <sub>2</sub> |         |           | Gaps Feas %     |                  |   |
|----------------|---------------|-------|-------|-------|-----------------|---------|----------|---------|------------------|---------|-----------|-----------------|------------------|---|
|                | $m_1$         | $m_2$ | $n_1$ | $n_2$ | Ub              | Time(s) | Iter.    | F(LP)   | Ub               | Time(s) | F(SDP)    | LP <sub>2</sub> | SDP <sub>2</sub> |   |
| 1              | 5             | 5     | 100   | 10    | 4299.1272       | 3.91    | 10587    | 14.3569 | 1373.0719        | 1.85    | 1254.4880 | 99.68           | <b>8.61</b>      |   |
| 2              |               |       | 150   | 10    | 9536.3426       | 14.46   | 24780    | 41.4240 | 2582.0491        | 3.31    | 2301.2810 | 99.57           | <b>10.90</b>     |   |
| 3              |               |       | 200   | 10    | 16599.8119      | 56.76   | 39653    | 1.5657  | 3502.2409        | 4.85    | 3001.3221 | 99.99           | <b>14.44</b>     |   |
| 4              |               |       | 300   | 10    | 37613.8900      | 651.62  | 92408    | 2.0087  | 7020.5496        | 14.07   | 6231.8535 | 99.99           | <b>11.24</b>     |   |
| 5              |               |       | 500   | 10    | 181659.6922     | 1006.37 | 71906    | -       | 16011.7326       | 50.50   | -         | -               | -                | - |
| 6              |               |       | 700   | 10    | 384009.5810     | 1016.01 | 57441    | -       | 24889.2887       | 127.81  | -         | -               | -                | - |
| 7              |               |       | 100   | 50    | 4297.4742       | 3.04    | 11144    | 14.0110 | 1429.4363        | 7.68    | 1264.9537 | 99.67           | <b>11.51</b>     |   |
| 8              |               |       | 150   | 50    | 9518.4933       | 9.96    | 24545    | 33.2311 | 2395.7141        | 12.55   | 2026.0953 | 99.65           | <b>15.43</b>     |   |
| 9              |               |       | 200   | 50    | 16845.2865      | 36.71   | 43580    | 19.6678 | 3849.7035        | 17.24   | 3440.4623 | 99.88           | <b>10.65</b>     |   |
| 10             |               |       | 300   | 50    | 37662.8023      | 358.12  | 97514    | 15.7980 | 7036.6020        | 42.08   | 6193.1584 | 99.96           | <b>12.04</b>     |   |
| 11             |               |       | 500   | 50    | 110628.0584     | 1006.09 | 250248   | -       | 15256.4846       | 130.63  | -         | -               | -                | - |
| 12             |               |       | 700   | 50    | 394217.5570     | 1015.40 | 31093    | -       | 23077.1148       | 291.74  | -         | -               | -                | - |
| 13             |               |       | 120   | 120   | 6122.2700       | 4.68    | 15817    | -6.3752 | 1884.7035        | 36.07   | 1571.6368 | 100.11          | <b>16.70</b>     |   |
| 14             |               |       | 150   | 150   | 9717.0871       | 16.76   | 24460    | -8.9348 | 2647.2006        | 80.25   | 2219.6493 | 100.09          | <b>16.36</b>     |   |
| Minimum values |               |       |       |       | 4297.4742       | 3.04    | 10587    | -8.9348 | 1373.0719        | 1.85    | 1254.4880 | 99.57           | <b>8.61</b>      |   |
| Maximum values |               |       |       |       | 394217.5570     | 1016.01 | 250248   | 41.4240 | 24889.2887       | 291.74  | 6231.8535 | 100.11          | <b>16.70</b>     |   |
| Average values |               |       |       |       | 87337.67        | 371.42  | 56798.28 | -       | 8068.27          | 58.61   | -         | -               | -                |   |

“-”: No solution found.

Table 9: Average bounds for QLBP on large instances using Dense Data

| #              | Instance Size |       |       |       | LP <sub>2</sub> |         |       |          | SDP <sub>2</sub> |         |           | Gaps Feas %     |                  |   |
|----------------|---------------|-------|-------|-------|-----------------|---------|-------|----------|------------------|---------|-----------|-----------------|------------------|---|
|                | $m_1$         | $m_2$ | $n_1$ | $n_2$ | Ub              | Time(s) | Iter. | F(LP)    | Ub               | Time(s) | F(SDP)    | LP <sub>2</sub> | SDP <sub>2</sub> |   |
| 1              | 5             | 5     | 100   | 10    | 1041.0711       | 13.99   | 22677 | -16.0797 | 631.5147         | 1.90    | 480.1761  | 101.57          | <b>23.86</b>     |   |
| 2              |               |       | 150   | 10    | 2326.4294       | 69.90   | 48084 | 5.0246   | 1097.5442        | 3.25    | 934.5643  | 99.76           | <b>14.85</b>     |   |
| 3              |               |       | 200   | 10    | 4215.1853       | 268.66  | 74428 | -7.1795  | 1926.1244        | 4.85    | 1730.7035 | 100.17          | <b>10.18</b>     |   |
| 4              |               |       | 300   | 10    | 12156.1537      | 1002.20 | 86235 | 460.1654 | 3447.0538        | 14.50   | 3032.2216 | 96.21           | <b>12.04</b>     |   |
| 5              |               |       | 500   | 10    | 50288.2956      | 1005.61 | 19271 | -        | 7459.4719        | 50.15   | -         | -               | -                | - |
| 6              |               |       | 700   | 10    | 99906.0443      | 1011.56 | 29058 | -        | 12111.7254       | 122.46  | -         | -               | -                | - |
| 7              |               |       | 100   | 50    | 1122.9176       | 9.56    | 20464 | -10.2867 | 657.9965         | 7.85    | 527.8076  | 100.95          | <b>19.67</b>     |   |
| 8              |               |       | 150   | 50    | 2379.0925       | 69.52   | 50889 | 0.2798   | 1277.2223        | 12.32   | 1073.7814 | 99.99           | <b>15.97</b>     |   |
| 9              |               |       | 200   | 50    | 4182.3160       | 172.27  | 65529 | -2.0442  | 1841.1944        | 17.07   | 1551.3813 | 100.05          | <b>15.88</b>     |   |
| 10             |               |       | 300   | 50    | 12361.0939      | 1002.08 | 87061 | 378.6008 | 3404.0488        | 42.29   | 2949.8167 | 96.89           | <b>13.47</b>     |   |
| 11             |               |       | 500   | 50    | 48877.4178      | 1006.64 | 40367 | -        | 7465.2323        | 146.17  | -         | -               | -                | - |
| 12             |               |       | 700   | 50    | 99656.3567      | 1010.58 | 20126 | -        | 12428.5810       | 328.52  | -         | -               | -                | - |
| 13             |               |       | 120   | 120   | 1661.4053       | 29.34   | 33074 | 8.9492   | 957.5524         | 37.39   | 771.6666  | 99.60           | <b>19.46</b>     |   |
| 14             |               |       | 150   | 150   | 2462.3484       | 61.14   | 46639 | 25.7236  | 1306.7844        | 80.22   | 995.2434  | 98.99           | <b>23.96</b>     |   |
| Minimum values |               |       |       |       | 1041.0711       | 9.56    | 19271 | -16.0797 | 631.5147         | 1.90    | 480.1761  | 96.21           | <b>10.18</b>     |   |
| Maximum values |               |       |       |       | 99906.0442      | 1011.56 | 87061 | 460.1654 | 12428.5810       | 328.52  | 3032.2216 | 101.57          | <b>23.96</b>     |   |
| Average values |               |       |       |       | 24474.00        | 480.93  | 45993 | -        | 4000.86          | 62.06   | -         | -               | -                |   |

“-”: No solution found.

Table 10: Average bounds for QLBP on large instances using Sparse Data