

# Incremental Computation of Planar Maps

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## Abstract

A *planar map* is a figure formed by a set of intersecting lines and curves. Such an object captures both the geometrical and the topological information implicitly defined by the data. In the context of 2D drawing, it provides a new interaction paradigm, *map sketching*, for editing graphic shapes.

To build a planar map, one must compute curve intersections and deduce from them the map they define. The computed topology must be consistent with the underlying geometry. Robustness of geometric computations is a key issue in this process. We present a robust solution to Bézier curve intersection that uses exact forward differencing and bounded rational arithmetic. Then, we describe data structure and algorithms to support incremental insertion of Bézier curves in a planar map. A prototype illustration tool using this method is also discussed.

**CR Categories and Subject Descriptors:** I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling – Curve, surface, solid, and object representations; Geometric algorithms, languages, and systems; I.3.6 [Computer Graphics]: Methodology and Techniques – Interaction techniques; G.1.1 [Numerical Analysis]: Interpolation – Spline and piecewise polynomial interpolation; J.5 [Arts and Humanities]: Arts, fine and performing.

**Additional Keywords and Phrases:** Bézier curves, forward differences, curve intersection, planar maps, map sketching.

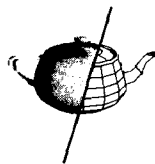
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## 1 Introduction

There is growing interest in the robustness of geometric computations [10,6,13]. Different graphics algorithms have different sensitivity to numerical errors. In some cases numerical errors are acceptable. In others, one can find ways around them. However, exact computation is sometimes mandatory. The following examples demonstrate the range of effects.

When scan-converting 3D polygons, rounding errors on face equations will not prevent the z-buffer method from rendering a scene. The few erroneous pixels may not even be visible. This is a case where numerical errors are innocuous. A second example is a function performing point location in a polygon with a parity test, using floating point arithmetic. If the result returned by this function is used for identification of the polygon and, say, modification of its color, then it is acceptable for the function to return an empty result when a reliable answer cannot be computed. Hence, in some 2D drawing programs, the user must click well inside a polygon to select it (which is better than selecting the wrong polygon). As a third example consider a program implementing an algorithm which presumes infinite precision. The Bentley–Ottmann algorithm [3,20] for reporting intersections of a set of non vertical line segments relies on the fact that two segments may intersect iff there exists a position of the vertical sweep-line where they are consecutive. If the implementation produces an error when inserting a new segment in the sweep-line then some intersections may be missed. In this case, it is imperative to provide an exact answer.

Methods involving topological decisions based on geometric computations are generally difficult to implement. We describe a robust solution to an intersection problem which arises in the context of a 2D drawing application. A set of lines and curves like in Fig. 1 dissects the plane into vertices, edges and faces. This type of geometric object is known in graph theory as a map of a planar multigraph [24], hence the name *planar map* we use below, and in computational geometry as an *arrangement* in the plane [7]. Data structures describing embeddings of planar graphs in the plane can be traced back to Baumgart's winged-edge data structure and have been studied by numerous researchers [20,12,9]. It is standard practice to distinguish between the geometry, the position of the vertices, geometric definition of the edges and the topology, the incidence and adjacency of the vertices,



edges, and faces.

The problem addressed here is building a data structure to support *incremental insertion* of new curves in a planar map, dynamically computing new intersections and updating the data structure. In this case, topological information has to be deduced from geometrical information. When two curves intersect at a new vertex, the ordering of the four edges around the vertex provides topological information used to follow the contour of a face incident to the vertex. If floating point arithmetic is used, it has been shown that the computed slopes can give the wrong order [10,18]. This is similar to the Bentley–Ottmann algorithm example above.

Our first implementation [19] used the Bentley–Ottmann algorithm and rational arithmetic to compute the planar map formed by a set of line segments, [11] is the description of a 2D illustration tool based on this first software. The method was not incremental and the map had to be recomputed each time a new segment was added. In [5], Greene and Yao solve the intersection problem for line segments by working directly in the discrete plane. In [8], Edelsbrunner et al. study arrangements of Jordan curves in the plane from a theoretical point of view.

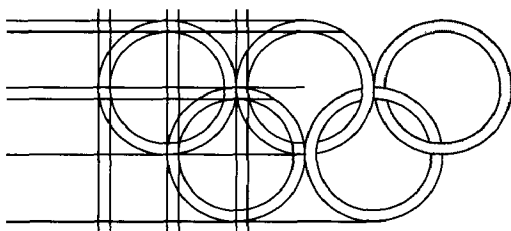


Figure 1: A planar map.

In the next section, the utility of planar maps for 2D drawing is briefly discussed. Section 3 details curve intersection. First, Bézier curves are interpolated by polylines using forward differencing. Then, the intersection between two interpolating polylines is computed with rational arithmetic, we show how it is possible to limit the number of bits in this process and how to control the quality of the interpolation. Section 4 describes the map data structure and the two main algorithms used in the planar map construction process: incremental insertion of a curve and point location in a map. The map topology is computed from the geometry of the polylines. Since exact arithmetic is used in this process, the map topology, although it may be different from the topology defined by the true curves, is always consistent with the geometry of the interpolating polylines.

## 2 Map Sketching

Our interest in planar maps is motivated by practical concerns: with traditional graphic arts media (pencil, eraser, ink,

etc.), it is common practice to build shapes by drawing lines and curves, erase some pieces thereof, and color or ink the areas they delimit (see [1] and Fig. 2).

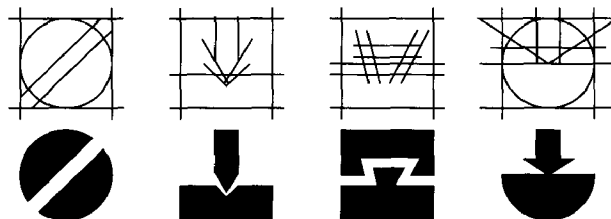


Figure 2: Graphic design by space division (B. Munari in [1]).

The design of logos and monograms, floor plan sketching by architects, cartoon cells drawing and inking are examples where this technique is used. In typical drawing software there is no way to mimic this method. If Fig. 3a is drawn by the user of a drawing application as four lines, it is impossible for him to color the rectangle (as in Fig. 3b) since no such rectangle exists. If the drawing were computed as a planar map, this dual interpretation would be possible.

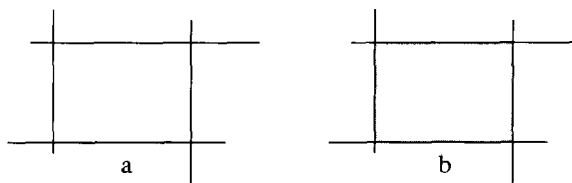


Figure 3: Four lines and a rectangle.

In [2], we have proposed two extensions to the 2D graphics drawing paradigm: a) objects are multicolor, multicontour shapes (i.e., planar maps), b) they are constructed by iteration of three basic steps: drawing, erasing, and coloring. We call this technique *map sketching* and have implemented it in prototype illustration software used to draw the figures in this paper. Fig. 4 illustrates map sketching. Strokes drawn by the user are incrementally added to the map describing the drawing. Two additional operations are allowed on a map: edge erasing and face coloring. These steps can be iterated in any order. Map sketching closely parallels the traditional pencil and eraser method and is more natural and more efficient for constructing certain classes of drawings. User interface design issues in map based illustration software are discussed in [2].

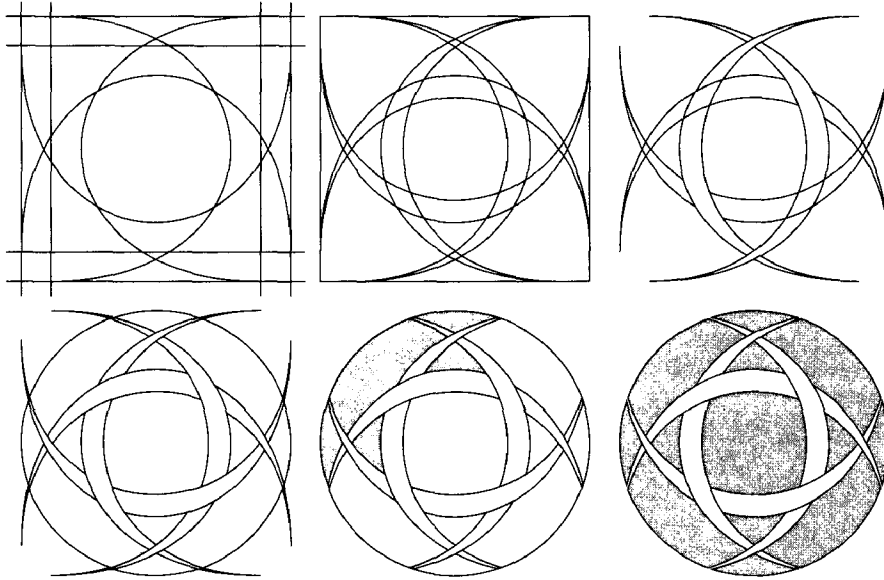


Figure 4: Map sketching.

### 3 Bézier Curve Interpolation and Intersection

#### 3.1 Overview

Curves to be inserted in a map are first converted to Bézier form [21,4]. The incremental insertion algorithm (Sec. 4) has two requirements. First, intersection points must be ordered without error along a curve by their parameter values, including the case of self-intersection. Second, if two or more curves intersect at one point, they must be ordered without error around the point. To meet these requirements, we use the following strategy:

1. The control points of the Bézier curves have integer coordinates on a grid large enough for 2D graphics applications. Grid size is discussed in Sec. 3.4.
2. The curve is replaced with an interpolating *polyline*. We compute an *exact* interpolation of the curve by exact forward differencing (FD). It is necessary that enough bits are available to perform FD without a loss of precision (Sec. 3.2). Rather than storing polylines in the data structure, they are computed as needed.
3. Computing the intersection of two exact polylines causes an explosion in the number of bits. Thus, we round the points of an exact polyline to the grid. This reduces the intersection of two *rounded polylines* to the intersection of line segments whose endpoints have integer coordinates. Ordering two intersection points along the same line segment and ordering two intersecting line segments around their intersection point is done with rational arithmetic. Note that the intersection points are

*not rounded* since this could modify the map topology.

4. Finally, it is natural with the map sketching technique to use an existing intersection point as a new curve endpoint. We will show how to achieve this without increasing the bit length of the arithmetic (Sec. 3.4).

The map deduced from the intersection process is the one defined by the *rounded polylines*. No other rounding occurs. The map topology, although it may be different from the topology defined by the true curves, is always consistent with the geometry of the rounded polylines.

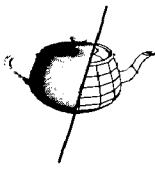
#### 3.2 Interpolation Method

Wang [25,23] gives the following result. If the de Casteljau subdivision algorithm (midpoint case) is applied down to depth  $k$  to a polynomial Bézier curve of degree  $d \geq 2$  with control points  $V_r$ , where:

$$k = \left\lceil \log_4 \frac{d(d-1)}{8} \frac{D}{\epsilon} \right\rceil, \quad (1)$$

then, all the chords (straight line segments) joining the endpoints of the  $2^k$  control polygons which are the leaves of the subdivision tree are closer to the curve than the threshold  $\epsilon$ . In (1),  $D = \max_{0 \leq r \leq d-2} \|V_{r+2} - 2V_{r+1} + V_r\|$  and  $\|v\| = \max(|x_v|, |y_v|)$  for a vector  $v$ .  $D$  can be called the *diagonal* of the curve. Since reference [25] is not available to us, an independent proof of this result is given in appendix A, together with a bound on the chord length.

Consider the chord endpoints  $E_i, 0 \leq i \leq 2^k$ . They form a polyline  $E$ . It is faster to use ordinary FD [16] than subdivision to compute  $E$ . Since *a priori* subdivision computes



the complete tree to depth  $k$ , FD with fixed step size  $2^{-k}$  will generate the same polyline, provided that exact computations are done in both cases. We now show that the number of bits needed to perform exact FD is bounded. Suppose that the control points of the curve have coordinates coded into  $b$  bits. Then, computing the subdivision tree down to depth  $k$  requires at most  $b + kd$  bits for the coordinates of the  $E_i$ . In the FD loop, the only values involved in the  $i$ th iteration are the forward differences  $\Delta^j E_i$ ,  $0 \leq j \leq d$ . Since we know from subdivision that, for all  $i$ , the computation of  $E_i = \Delta^0 E_i$  requires at most  $b + kd$  bits,  $\Delta^j E_i$  requires at most  $b + kd + j$  bits. Thus, exact FD with step size  $2^{-k}$  can be performed on the curve if  $b + (k + 1)d$  bits are available.

To limit the total number of bits needed when updating a planar map, the intersection algorithm uses *rounded* polylines. FD computes the exact coordinates of the  $E_i$  which are then rounded to  $b$  bits.

### 3.3 Intersection Algorithm

Bézier curve intersection is studied by Sederberg and Parry [23]. In two of the algorithms they consider, rejection of non-intersecting pieces of two curves is done by bounding box comparison. FD is not convenient for successive midpoint evaluations of a curve. To take advantage of bounding boxes, a preprocessing step breaks the rounded polylines into monotonic pieces. For such a piece, the box of any subpiece is given by its endpoints coordinates. This method is also used by Koparkar and Mudur [14] with another curve evaluation method. During the planar map construction process, a new curve is intersected with a subset of the curves already inserted in the map. The preprocessing of the new curve finds the monotonic pieces, saving data to be used in later computations. The new curve is then immediately inserted in the map.

**Preprocessing.** Let  $C$  be the new curve. a) use FD to compute the exact polyline  $E$  and the rounded polyline  $P$  of  $C$ , b) store  $P$  in an array, to be discarded after the insertion of  $C$ , c) find the monotonic pieces of  $P$ , d) at the end of a monotonic piece, save the permanent data associated with it, that is, its first and last indices  $(i_f, i_t)$ , its bounding box, its quadrant, and the FD context at  $i_f$  (i.e.,  $\Delta^j E_{i_f}$ , for all  $j$ ). All these steps can be performed in one single FD loop.

**Intersection.** It is enough to consider the intersection of a monotonic piece of  $P$ , with indices  $(i_f, i_t)$ , with a monotonic piece of an existing curve  $G$ , with indices  $(j_f, j_t)$ , whose bounding boxes overlap. First, compute  $Q$ , the rounded polyline of  $G$ , between  $j_f$  and  $j_t$  using the FD context at  $j_f$  which has been saved when preprocessing  $G$ , and store the result into an array. Then, search the intersecting chords using binary subdivision on the respective arrays (the box of any subset of points considered in this subdivision is given by its two endpoints and the quadrant information). In the map sketch-

ing application, the existing curve  $G$  may be partially erased. In this case only the monotonic pieces containing a non-erased part of  $G$  are intersected with  $C$ .

Two special cases must be handled: rounded chords with a null length, and partially overlapping polylines (i.e., non transverse intersections). After preprocessing, a new curve is intersected with itself to detect multiple points and self-overlapping. Naturally, line segments are not subdivided since they are ready for intersection. It is worthwhile to cache partially or totally generated curves. Two cases are frequent: a) the same monotonic piece of  $G$  intersects different pieces of  $C$ , b) successive new curves intersect the same existing curve.

### 3.4 Topology Consistency

This section describes how a consistent topology is obtained from the geometrical data given by the intersection process. For illustration software, the input can be rounded to an integer grid if the grid size is large enough and if the scaling factors are limited accordingly. A typical case is to output the results on a 24"  $\times$  24" page at 300 dpi. Then, input control points may be defined on twice as large an area, to permit clipped curves. We must also choose a maximum zoom factor: a reasonable value is 8. Since the rounded chords must have even coordinates (see below), the input is scaled up by a factor of two. The control points coordinates are thus coded on  $b = 18$  bits. Setting  $\epsilon = 1$  in equation (1) gives  $k = 10$  for a degree 4 curve with the maximum diagonal, which is twice the grid size. Thus, 62 bits are needed for the exact FD of this curve, this goes up to 102 bits for a degree 7 curve with the same diagonal. The much more usual case of a cubic with a 4" diagonal is 45 bits.

If chord intersection is performed on the exact polylines the number of bits grows very rapidly. When two chords  $AB$  and  $CD$  intersect at  $I$ , the coordinates of  $I$  and the values of the two parameters  $u$  and  $v$  such that  $AI = uAB$  and  $CI = vCD$  must be computed exactly. All of these can be expressed as rational numbers; for example,  $u = (AC \times CD) \div (AB \times CD)$  where  $\times$  is the cross product. With endpoints coded on  $b + kd$  bits, this is  $2(b + kd) + 3$  bits for the numerator and the denominator of the rationals. Since different intersections along the same chord are ordered by comparing their rational parameter values, the final number of bits is  $4(b + kd) + 6$ . For the first curve in the example above, this is 238 bits. The situation is worse if we want to use an existing intersection as the endpoint of a new curve. Setting  $b = 238$  in the above computations gives 1118 bits. As noted by Forrest and Newell [10], the major drawback in the rational arithmetic approach is the blow-up in the number of bits.

To limit this number, the chord endpoints of the exact polylines are rounded to even integer values. Chord intersection is done on the rounded chords and the intersection points are exactly represented as rational numbers. To use an exist-

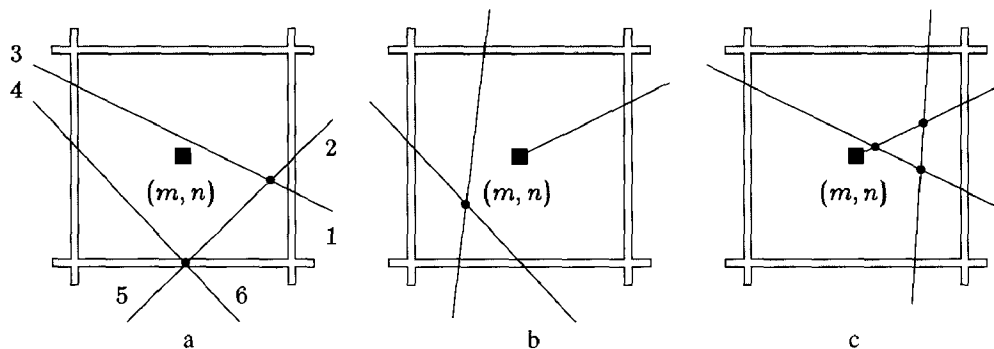


Figure 5: Vertex  $(m, n)$  and chord ordering.

ing intersection point as the endpoint of a new curve without increasing the bit length, we consider the semi-open rectangles  $R_{m,n} = [m-1, m+1] \times [n-1, n+1]$ , where  $m$  and  $n$  are even. Since the vertical and horizontal lines limiting the rectangles have odd coordinates, there are no rounded chords collinear with these lines. So, it is always possible, if two or more chords intersect inside  $R_{m,n}$ , to order them along the boundary of  $R_{m,n}$  by using either the coordinates of their intersections with the lines limiting the rectangle or their slopes if they leave  $R_{m,n}$  at exactly the same point (Fig. 5a). We define the center of  $R_{m,n}$  as the *vertex* of the intersection points lying inside  $R_{m,n}$ . This associates intersection points with vertices but does not round their coordinates. To use a chord intersection point as the endpoint of a new curve, we do not use the point itself but the coordinates of the associated vertex (Fig. 5b). Therefore, small faces lying inside a single rectangle will not be represented in the map data structure (Fig. 5c).

On a curve, an intersection point is represented as a *parameter value*  $p = (i, u)$  where  $i$  is the chord index on the polyline and  $u$  a rational number giving the exact position of the point on the chord. Since all chords have now rounded endpoints, ordering two intersection points along one curve requires at most  $4b + 6$  bits. We need also to order the intersections of the chords with the lines limiting the rectangles  $R_{m,n}$ . These are Bézier curves of degree 1, thus the stated bound is valid. In the common case where only one intersection point is associated with a vertex, the slopes are used to order the chords, requiring at most  $2b + 3$  bits. In addition, the method must support the erasing of curve pieces limited by intersection points. It is therefore necessary to keep the initial data defining the curve and to mark as erased or non-erased the corresponding pieces. As noted above, the intersection algorithm uses this information to return only actual intersections. A curve is removed from the map data structure iff it has been totally erased.

The method has two limitations. First, intersection is performed on the rounded polylines. Thus, there are situations

(e.g., tangencies) where intersections between the true curves are ignored. Likewise, polylines may intersect even if true curves do not (e.g., two concentric circles with very close radii interpolated by regular polygons whose sides intersect pairwise). Second, the topology of a map computed in this way is not invariant under general affine transforms. Thus, the map has to be recomputed from the original data whenever it is rotated or scaled. The first limitation is inherent in any linear interpolation process. However, for 2D graphics applications, it is always possible to prevent any visible effects by choosing an appropriate grid size. The second limitation can only be solved by using exact arithmetic on real numbers or symbolic computation on algebraic curves, which are currently too slow for interactive applications. Without recomputing the map, it is possible to perform integer translation (i.e., dragging) and scaling by a power of 2 (i.e., zooming), if we remain inside the grid.

In this section, we have shown that a robust method for the computation of planar maps with linearly interpolated Bézier curves requires at most  $b + (k+1)d$  bits for the FD step and  $4b + 6$  bits for the intersection and sorting steps. Our implementation uses a variable length integer arithmetic package coded in assembly language. In practice, the average size of the numbers involved in the process is much smaller than the above bounds. The only operation we must perform on rational numbers is comparison, which is two integer multiplications and a test. The value of  $b$  is a parameter of the program, allowing the grid size to be adapted to the resolution of the display.

## 4 Data Structure and Algorithms

After describing the planar map data structure, we detail below the two main algorithms. Curve insertion uses point location in a map to find the face containing the first endpoint of a curve. However, since point location is equivalent to the insertion of a dummy line segment, curve insertion will come first.

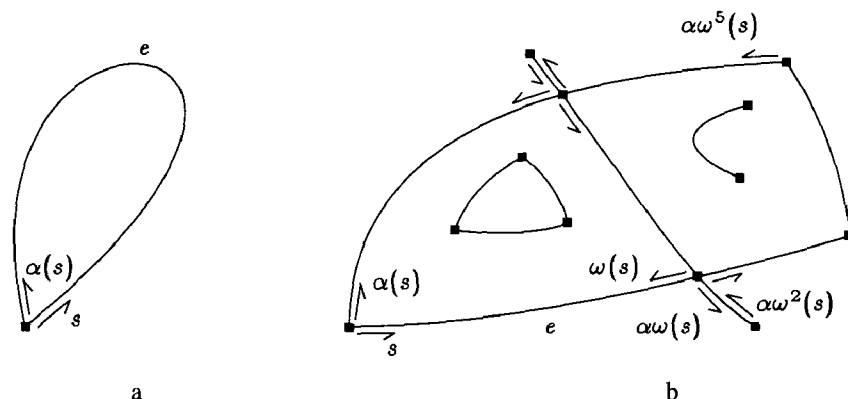
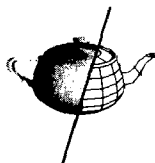


Figure 6: Map topology.

#### 4.1 Planar Map Description

A map contains two different sets of data. The first one describes the geometry of the curves and their intersections, and the second contains the topological data. In what follows, the word curve should be understood as the rounded polyline associated with the curve.

**Geometry.** When inserted, a curve is cut into *arcs* by the other curves. An arc is described by its endpoints on the curve. Each *point* (i.e., an intersection or a curve endpoint) is known by its parameter value  $p = (i, u)$  on the curve, as in Sec. 3.4. An intersection yields two points, one on each curve. As parameters are totally ordered along a curve, an arc is noted below as a parameter interval  $[p_1, p_2]$ . Arcs are marked as either erased or non-erased.

**Topology.** The mapping defined in Sec. 3.4 associates with each point a unique vertex. To support arc overlap, we attach to an arc an *edge* connecting its vertices. Arcs lying entirely in one rectangle  $R_{mn}$  are not considered. Overlapping arcs share the same edge. The geometry of an edge is the geometry of one of the arcs it *supports*. Different edges can connect the same pair of vertices. The ordering of the edges around a vertex is the chord ordering defined by the rectangles  $R_{mn}$ .

To access the faces of a planar map, it is convenient to consider an edge as two directed edges, called *sides*. If an edge  $e$  is a loop incident to the vertex  $v$ , then the clockwise (cw) and counterclockwise (ccw) orientations along  $e$  define the two sides associated with  $e$  (Fig. 6a). Two mappings are defined on the sides of a map:  $\alpha(s)$  is the side next to  $s$  in the ccw order around the vertex incident to  $s$ , and  $\omega(s)$  is the other side of the edge [17]. We note the ordering of the sides around a vertex,  $\alpha$ -order. To follow the boundary containing a side  $s$ , the compound mapping  $\alpha\omega$  is applied repeatedly until back in  $s$  (Fig. 6b). The result is a face boundary

called a *contour*. Contours with a ccw orientation are *outer contours*, others are *inner contours*. Adding a virtual inner contour located at infinity, there is exactly one inner contour for each face of a map.

The edges may form several connected components which are partially ordered by inclusion in the plane. This partial ordering is described by an *inclusion tree* whose nodes are the contours. The root is the virtual inner contour at infinity. The leaves are either inner contours with no connected component included or outer contours with an empty interior. This tree is stored in the data structure and used by the curve insertion and point location algorithms.

#### 4.2 Curve Insertion

We say that an arc is *visible* in a face if this arc is supported by an edge of which at least one side is in a contour bounding the face. Curve insertion uses the method described in Sec. 3 to compute the intersections between a new curve  $C$  and all the arcs visible in the faces where  $C$  is lying (Fig. 7). Along  $C$ ,  $p$  is the current parameter value and  $next(p)$  the parameter value of the next intersection point,  $p < next(p)$ .

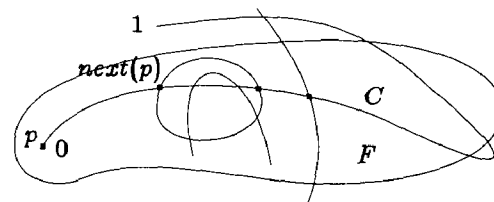


Figure 7: Curve insertion (first iteration).

**Step 1.** Using point location (Sec. 4.3), find the face  $F$  containing the first point of  $C$ . Set parameter  $p$  to 0.

**Step 2.** If  $F$  has already been processed, jump to step 3. Otherwise, compute the intersections between arc  $[p, 1]$  and all the arcs visible in  $F$ . Cut the arc  $[p, 1]$  and the intersected arcs of  $F$  at each intersection point and create the corresponding vertices and sides. At the end of this step, there are no more intersections between  $p$  and  $next(p)$ , and the  $\alpha$ -order around the vertices along  $C$  has been updated.

**Step 3.** If there is no overlapping, create an edge between the vertices associated with  $p$  and  $next(p)$ , link it with the arc  $[p, next(p)]$  in the data structure, and update the inclusion tree accordingly (see appendix B). Otherwise, the edge already exists, link it with the arc  $[p, next(p)]$ .

**Step 4.** If  $next(p) = 1$  then stop. Otherwise, let  $s$  be the side of arc  $[next(p), next(next(p))]$  associated with  $next(p)$ . Since the  $\alpha$ -order around the two corresponding points is known,  $s$  is known. Set  $F$  to the face incident to  $\alpha(s)$ , and  $p$  to  $next(p)$ . Repeat step 2.

An arc is visible in at most two faces but an existing curve  $G$  can be visited several times. However, the intersection between the arc  $[p, 1]$  of  $C$  and  $G$  is done only once: the first time an arc of  $G$  becomes visible in the current face  $F$ . The intersection points located outside  $F$  are stored for further use.

### 4.3 Point Location

Given a query point with integer coordinates, the point location algorithm returns either a face, an edge, or a vertex. In map sketching, all selections are done through this algorithm (e.g., coloring a face or selecting an existing intersection as the endpoint of a new curve). A first method is to intersect a line segment  $S$  with all curves.  $S$  is defined by the query point  $M$ , with parameter 0, and a point outside the bounding box of the map, with parameter 1. If no intersection is found,  $M$  is inside the infinite face. Else, retain the curve  $G$  whose intersection is closest to  $M$ . This intersection is known by its parameter values  $p$  on  $S$  and  $q$  on  $G$ . The parameter  $q$  gives the arc of  $G$  containing the intersection. If  $p = 0$ , then  $M$  is exactly on the edge supporting this arc. Otherwise,  $M$  is inside one of the two faces incident to the edge. The side which sees  $M$  to its right gives the answer (a side defines a unique orientation on a curve).

This method does not take advantage of the partition of the plane defined by the faces of the map. To reduce the average number of visited curves, the following algorithm uses face adjacency (Fig. 8). This algorithm is similar to the curve insertion algorithm, but it uses curve intersection instead of arc intersection. We say that a curve is *visible* in a face if one of its arcs is visible in the face.

**Step 1.** Set  $F$  to the infinite face and  $S$  to  $[0, 1]$ .  $S$  is the line segment defined above.



Figure 8: Point location.

**Step 2.** If  $F$  has no outer contours then return  $F$ . Otherwise, intersect  $S$  with the curves visible in the outer contours of  $F$  (if two or more curves are overlapping, it is enough to consider one of them). If there is no intersection, return  $F$ . Otherwise, let  $e$  be the edge which gives the smallest parameter  $p$  on  $S$ , and  $s$  the side of  $e$  which sees  $M$  to its right. If  $s$  is part of an outer contour, return  $F$ . Otherwise, set  $S$  to  $[0, p]$  and set  $F$  to the face incident to  $s$ .

**Step 3.** Intersect  $S$  with the curves visible in the inner contour of  $F$ . If there is no intersection, call recursively step 2. Otherwise, let  $e$  be the edge which gives the smallest parameter  $p$  on  $S$ , and  $s$  the side of  $e$  which sees  $M$  to its right,  $s$  is necessarily part of an inner contour. Set  $F$  to the face incident to this contour and  $S$  to  $[0, p]$ . Repeat step 3.

As curve insertion does, point location may visit a curve several times, but only one intersection with  $S$  is performed. The geometric tests are performed on rational numbers, they are thus exact. Indications on the complexity of both algorithms are given by the horizon theorem for Jordan curves included in [8]. However, this last result cannot be applied in a straightforward way as the number of intersections between two polylines may be greater than the number of intersections between the true curves, and the polylines may be partially overlapping.

## 5 Conclusion

A method has been presented which allows for incremental construction of planar maps. Robustness of the computation and consistency between geometry and topology are achieved through linear interpolation of Bézier curves and exact intersection of the rounded resulting polylines. Our main goal was to produce fast and reliable code to be used in the context of 2D drawing.

Though it has some limitations, the method described in this paper gives a powerful tool for constructing illustrations. The planar map data structure allows also for automated compound operations, such as the ones described by Fig. 9 and 10. Fig. 11 and 12 show illustrations produced with the map sketching technique.

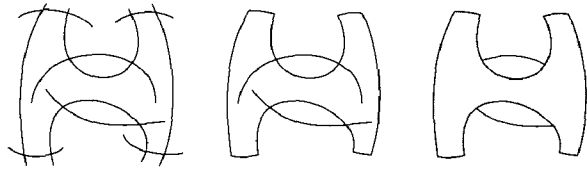
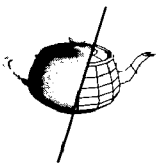


Figure 9: Cleaning a face removes its dangling edges.

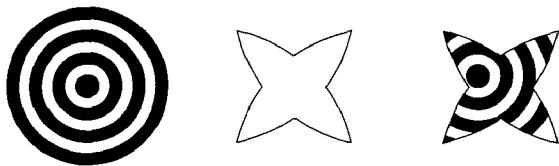


Figure 10: Cookie-cutter.

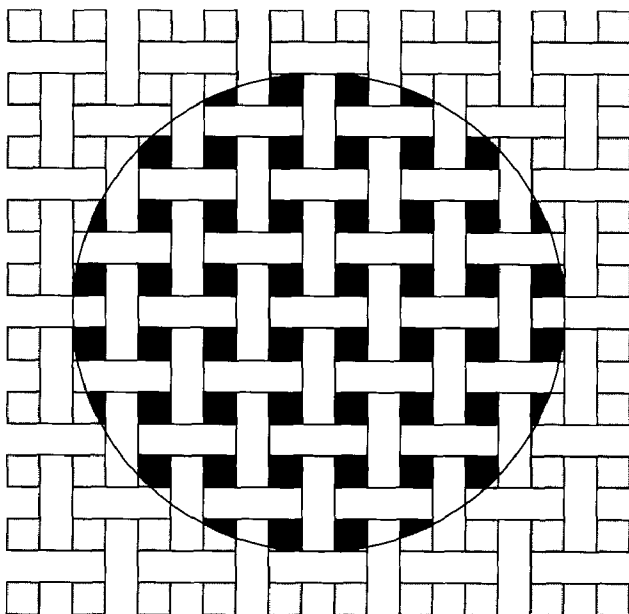


Figure 11: Wickerwork.

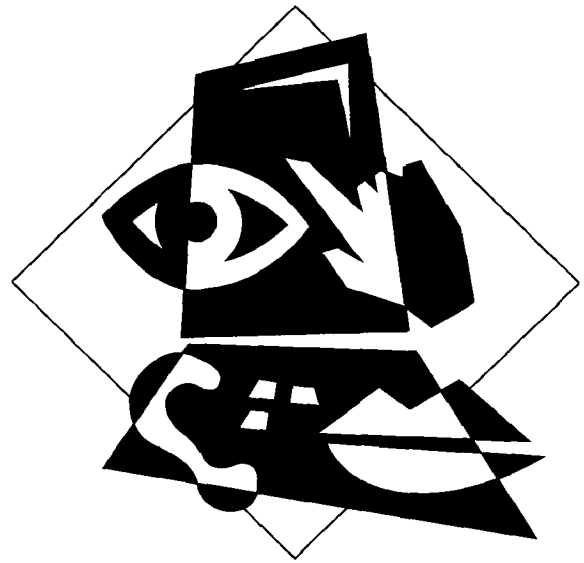


Figure 12: CHI'88 logo.

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### A A priori Subdivision

A polynomial Bézier curve of degree  $d$  is defined by:

$$V(t) = \sum_{r=0}^d V_r B_r^d(t), \quad 0 \leq t \leq 1,$$

where the  $V_r$  are the  $d+1$  control points that form the control polygon of  $V(t)$ , and  $B_r^d(t) = \binom{d}{r} t^r (1-t)^{d-r}$  is the  $r$ th Bernstein polynomial of degree  $d$  [21,4].

If  $v$  is a vector in the Euclidean plane, we note  $\|v\|$  the quantity  $\max(|x_v|, |y_v|)$ . For  $d \geq 2$ , the *diagonal*  $D$  and the *length*  $L$  of the control polygon of  $V(t)$  are defined as:

$$D = \max_{0 \leq r \leq d-2} \|V_{r+2} - 2V_{r+1} + V_r\|,$$

$$L = \max_{0 \leq r \leq d-1} \|V_{r+1} - V_r\|.$$

$D$  is considered by Wang [25,23] as a subdivision criterion.

Computing the first and second derivatives of  $V(t)$  and using the properties of the Bernstein polynomials gives, for all  $t$  in  $[0, 1]$ :

$$\|V^{(2)}(t)\| \leq d(d-1) D, \quad (2)$$

$$\|V^{(1)}(t)\| \leq d L. \quad (3)$$



To find the number of subdivisions, we use a chord interpolation theorem [22] which states that: if  $f(t)$  is a real valued function of class  $C^\infty$  on  $[a, b]$ , then, for all  $t$  in  $[a, b]$ :

$$|f(t) - c(t)| \leq \frac{(b-a)^2}{8} \max_{a \leq t \leq b} |f^{(2)}(t)|, \quad (4)$$

where  $c(t)$  is the chord (straight line segment) between  $(a, f(a))$  and  $(b, f(b))$ . This result is used by Lane [15] in the context of curve rendering.

Let  $\sigma$ ,  $0 < \sigma \leq 1$ , be a step size on the interval  $[0, 1]$  and  $n$  the integer such that  $n\sigma = 1$ . This defines  $n$  intervals  $I_i = [i\sigma, (i+1)\sigma]$ , and  $n$  chords  $C_i(t)$  with endpoints  $E_i = V(i\sigma)$  and  $E_{i+1} = V((i+1)\sigma)$ . Let  $\epsilon$  be a given threshold measuring the maximum allowed deviation between the curve and the chords  $C_i$ . We want to ensure that:

$$\|V(t) - C_i(t)\| \leq \epsilon \quad (5)$$

holds for all  $i$ ,  $0 \leq i < n$ , and all  $t$  in  $I_i$ .

Applying (4) to the coordinates of  $V(t)$  on the interval  $I_i$  and using the bound (2), it is straightforward to see that any  $\sigma \leq 1$  such that:

$$0 < \sigma^2 \leq \frac{8\epsilon}{d(d-1)D} \quad (6)$$

will satisfy (5).

Let  $k$  be the smallest integer such that  $\sigma = 2^{-k}$  satisfies (6), then:

$$k = \left\lceil \log_4 \frac{d(d-1)D}{8\epsilon} \right\rceil \quad (7)$$

Equation (7) is cited in [23] as a result derived by Wang.

We can now bound the length of the chords given by a *a priori* subdivision of depth  $k$ . The mean value theorem applied to  $V(t)$  on interval  $I_i$  gives:

$$\|E_{i+1} - E_i\| \leq \sigma \max_{t \in I_i} \|V^{(1)}(t)\|.$$

The maximum of  $\|V^{(1)}(t)\|$  over  $I_i$  is less than or equal to its maximum over  $[0, 1]$ . Using (3), one gets:

$$\|E_{i+1} - E_i\| \leq \sigma dL.$$

The bound on  $\sigma$  from (6) gives for all  $i$ ,  $0 \leq i < 2^k$ :

$$\|E_{i+1} - E_i\| \leq \sqrt{\frac{8d}{d-1}} \frac{L}{\sqrt{D}} \sqrt{\epsilon}.$$

## B Updating the Inclusion Tree

The degree  $d$  of a vertex  $v$  is the number of sides incident to  $v$ ,  $v$  is a *dangling vertex* if  $d(v) = 1$ . If both sides of an edge are in the same contour, it is a *dangling edge*, otherwise it is a *border edge*. A dangling edge is *connecting* if it has no dangling vertex, *terminal* if it has exactly one dangling vertex, and *isolated* if it has two dangling vertices. A loop incident to a vertex  $v$  with  $d(v) = 2$  is an *isolated border edge*.

An edge falls into one of the following types:

1. a terminal edge with both sides in an inner contour,
2. a connecting edge with both sides in an inner contour,
3. a border edge with both sides in two distinct inner contours,
4. a border edge with one side in an inner contour and the other side in an outer contour,
5. an isolated edge,
6. a terminal edge with both sides in an outer contour,
7. a connecting edge with both sides in an outer contour,
8. an isolated border edge.

When inserting a curve, new edges may be added to the map. Adding an edge implies creating its sides, vertices, and updating the  $\alpha$ -order around the latter. The updated contours are thus available through the mapping  $\alpha\omega$ . The inclusion tree is updated *after* each new edge addition, it is therefore possible to find the type of a new edge by counting its dangling vertices and checking the updated contours. The inclusion tree is then updated by performing the following actions, indexed by edge type:

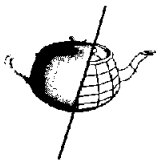
2. merge: inner & outer  $\rightarrow$  inner
3. split: inner  $\rightarrow$  inner & inner
4. split: outer  $\rightarrow$  outer & inner
5. create: outer
7. merge: outer & outer  $\rightarrow$  outer
8. create: outer & inner

For example, if an edge of type 2 is added to a map, one inner contour and one outer contour are merged to give a single inner contour.

When erasing a curve or an arc, old edges may be removed from the map. The inclusion tree is updated *before* each old edge removal, using the same tests as above. When the type of the edge has been found, the inclusion tree is updated by performing the following actions, indexed by edge type:

2. split: inner  $\rightarrow$  inner & outer
3. merge: inner & inner  $\rightarrow$  inner
4. merge: outer & inner  $\rightarrow$  outer
5. delete: outer
7. split: outer  $\rightarrow$  outer & outer
8. delete: outer & inner

Adding or removing an edge of type 1 or 6 (i.e., terminal edges) does not modify the inclusion tree.



## References

- [1] D. Baroni. *Art Graphique Design*. Editions du Chêne, Paris, 1987.
- [2] P. Baudelaire and M. Gangnet. *Planar Maps: an Interaction Paradigm for Graphic Design*. In *CHI'89 Proceedings*, Addison-Wesley, 1989.
- [3] J.L. Bentley and T.A. Ottmann. Algorithms for Reporting and Counting Geometric Intersections. *IEEE Trans. on Comput.*, 28(9), 1979.
- [4] T.D. DeRose and B.A. Barsky. Geometric Continuity for Catmull-Rom Splines. *ACM Transactions on Graphics*, 7(1), 1988.
- [5] D.H. Greene and F.F. Yao. Finite-Resolution Computational Geometry. In *Proc. 27th IEEE Symp. on Found. Comp. Sci.*, Toronto, 1986.
- [6] D. Dobkin and D. Silver. Recipes for Geometry and Numerical Analysis. In *Proceedings of the Fourth Annual ACM Symposium on Computational Geometry*, ACM Press, New York, 1988.
- [7] H. Edelsbrunner. *Algorithms in Combinatorial Geometry*. Springer-Verlag, New York, 1987.
- [8] H. Edelsbrunner, L. Guibas, J. Pach, R. Pollack, R. Seidel, and M. Sharir. *Calculating Arrangements of Segments, Circles, or Other Curves in the Plane*. 1988. Submitted for publication.
- [9] H. Edelsbrunner and L.J. Guibas. *Topologically Sweeping an Arrangement*. Research Report #9, Digital Equipment Systems Research Center, Palo Alto, 1986.
- [10] A. R. Forrest. Geometric Computing Environments: Some Tentative Thoughts. In *Theoretical Foundations of Computer Graphics and CAD*, Springer-Verlag, 1988.
- [11] M. Gangnet and J.C. Hervé. D2: un éditeur graphique interactif. In *Actes des Journées SM90*, Eyrolles, Paris, 1985.
- [12] L. Guibas and J. Stolfi. Primitives for the Manipulation of General Subdivisions and the Computation of Voronoi Diagrams. *ACM Transactions on Graphics*, 4(2), 1985.
- [13] C.M. Hoffmann, J.E. Hopcroft, and M.S. Karasick. Towards Implementing Robust Geometric Computations. In *Proceedings of the Fourth Annual ACM Symposium on Computational Geometry*, ACM Press, New York, 1988.
- [14] P.A. Koparkar and S.P. Mudur. A new class of algorithms for the processing of parametric curves. *Computer-Aided Design*, Vol. 15, 1983.
- [15] J.F. Lane. *Curve and Surface Display Techniques*. Tutorial, ACM SIGGRAPH'81, 1981.
- [16] S.L. Lien, M. Shantz, and V. Pratt. Adaptive Forward Differencing for Rendering Curves and Surfaces. *ACM Computer Graphics*, Vol. 21(4):111-118, 1987.
- [17] P. Lienhardt. Extensions of the Notion of Map and Subdivision of a Three-Dimensional Space. In *STACS'88*, Lecture Notes in Computer Science 294, 1988.
- [18] D. Michelucci. *Thèse*. Ecole Nationale Supérieure des Mines de Saint-Etienne, Saint-Etienne, 1987.
- [19] D. Michelucci and M. Gangnet. Saisie de plans à partir de tracés à main-levée. In *Actes de MICAD 84*, Hermès, Paris, 1984.
- [20] F.P. Preparata and M.I. Shamos. *Computational Geometry: an Introduction*. Springer-Verlag, New York, 1985.
- [21] L. Ramshaw. *Blossoming: A Connect-the-Dots Approach to Splines*. Research Report #19, Digital Equipment Systems Research Center, Palo Alto, 1987.
- [22] M. H. Schultz. *Spline Analysis*. Prentice Hall, 1973.
- [23] T.W. Sederberg and S.R. Parry. Comparison of three curve intersection algorithms. *Computer-Aided Design*, Vol. 18, 1986.
- [24] W.T. Tutte. *Graph Theory*. Addison-Wesley, 1984.
- [25] G. Wang. The Subdivision Method for Finding the Intersection between two Bézier Curves or Surfaces. *Zhejiang University Journal*, 1984. Cited in reference [23].