

Generating formally certified bounds on values and round-off errors

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Introduction

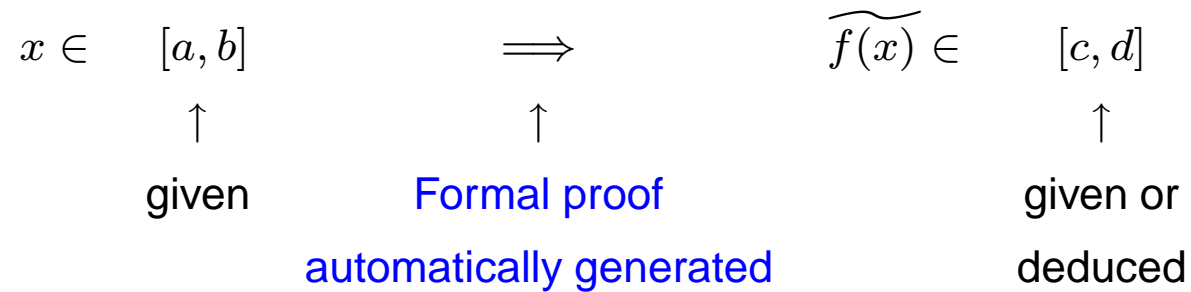
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- Certifying a numerical code is tedious and error-prone.
- **Tools** for naive users are necessary.

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- Certifying a numerical code is tedious and error-prone.
- Tools for naive users are necessary.
- Certification goals:
 - variables are bounded (e.g. no square root of a negative number, etc), so that no exceptional behavior is triggered,
 - round-off errors are contained, so that the final result of an algorithm is sufficiently accurate.

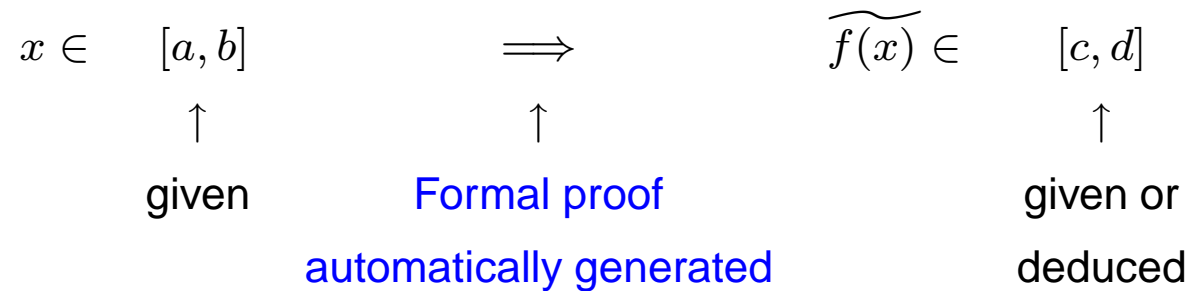
Introduction

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- Constraints on f :
 - loops are static,
 - no branching,
 - equivalent to single assignment.
- These constraints are handled by other classical certification tools.

Outline

- The example of an aeronautical application.
Robustness is critical, the implementation needs to be certified. Yet the algorithm is mathematically simple, its study should not require much human expertise.
- Methods used to analyze numerical programs.
Bounds on variable are computed through interval arithmetic, and round-off errors are propagated through forward error analysis.
- Formal proofs.
Why a formal proof? Conclusion and perspectives.

Safety distances between aircraft

- Conversion from geodetic to euclidean data.

$$\phi_0 \leftarrow (\phi_1 + \phi_2)/2$$

$$r_{p0} \leftarrow R_p(\phi_0)$$

$$s_{N1} \leftarrow v_{N1}/R_m(\phi_1)$$

$$s_{N2} \leftarrow v_{N2}/R_m(\phi_2)$$

$$p_x \leftarrow (\lambda_1 - \lambda_2) * r_{p0}$$

$$p_y \leftarrow (\phi_1 - \phi_2) * R_m(\phi_0)$$

$$v_{x1} \leftarrow v_{E1} * r_{p0}/R_p(\phi_1 + s_{N1} * t_r)$$

$$v_{x2} \leftarrow v_{E2} * r_{p0}/R_p(\phi_2 + s_{N2} * t_r)$$

Safety distances between aircraft

- Earth local radii (WGS84):

$$R_p(\phi) = \frac{a}{1 + (1 - f)^2 \tan^2 \phi}$$

$$R_m(\phi) = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{3/2}}$$

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- Use of **approximations** (introduces **truncation** errors):

$$x \leftarrow 511225 \times 2^{-18} - \phi^2$$

$$\hat{R}_p(\phi) \leftarrow 4439091 \cdot 2^{-2} + x \times (9023647 \cdot 2^{-2} + x \times (13868737 \cdot 2^{-6} + x \times (13233647 \cdot 2^{-11} + x \times (-1898597 \cdot 2^{-14} + x \times (-6661427 \cdot 2^{-17}))))))$$

Containing the error

- Taking into account measurement errors on input, truncation errors on algorithm, and **rounding** errors during computations.
- Numerical properties of the program are described by a formal proof, automatically certified with the **Coq** proof checker.

Containing the error

- Parts of the Coq script:

```
Variable V_vx_2: float.  
Hypothesis H_vx_2: (Div_float V_T8 V_Rp_2_c0 V_vx_2).  
Definition B_16 := (Eint_bound Cl_16 Cu_16 V_Rp_a3).  
...  
Definition Ce_195 := (Float `1` `-11`).  
Lemma E_195: ... -> B_16 -> ... ->  
              (Eint_error Ce_195 V_vx_2 R_vx_2).
```

- The last lemma can be read as follow.

- Hypotheses:

$$H_{vx_2} : \tilde{v}_{x_2} \leftarrow \tilde{t}_8 \odot R_{p2c},$$
$$B_{16} : R_{p3} = 13233647 \times 2^{-11},$$

...

- Conclusion: $|v_{x_2} - \tilde{v}_{x_2}| \leq 2^{-11}$.

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Bounding the variables

- Compute the bounds by **interval arithmetic**.
For each floating point operator \square , define an interval operator such that

$$\forall A, B \in \mathbb{IF}, A \square B \supseteq \{ \tilde{c} \in \mathbb{F} \mid \tilde{a} \in A, \tilde{b} \in B, c = \tilde{a} \square \tilde{b} \}$$

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- Floating point operators are **monotone**:
if $A = [\underline{a}, \bar{a}]$ and $B = [\underline{b}, \bar{b}]$, then we define

- $A \oplus B = [\underline{a} \oplus \underline{b}, \bar{a} \oplus \bar{b}],$

- $A \ominus B = [\underline{a} \ominus \bar{b}, \bar{a} \ominus \underline{b}],$

- $A \otimes B = [\min(\underline{a} \otimes \underline{b}, \underline{a} \otimes \bar{b}, \bar{a} \otimes \underline{b}, \bar{a} \otimes \bar{b}), \max(\underline{a} \otimes \underline{b}, \underline{a} \otimes \bar{b}, \bar{a} \otimes \underline{b}, \bar{a} \otimes \bar{b})],$

- etc.

Bounding the variables

- Floating-point interval operators:
 - $A \oplus B = [\underline{a} \oplus \underline{b}, \bar{a} \oplus \bar{b}]$,
 - $A \ominus B = [\underline{a} \ominus \bar{b}, \bar{a} \ominus \underline{b}]$,
 - $A \otimes B = [\min(\underline{a} \otimes \underline{b}, \underline{a} \otimes \bar{b}, \bar{a} \otimes \underline{b}, \bar{a} \otimes \bar{b}), \max(\underline{a} \otimes \underline{b}, \underline{a} \otimes \bar{b}, \bar{a} \otimes \underline{b}, \bar{a} \otimes \bar{b})]$,
- This is an **exotic** interval arithmetic implementation: the computed bounds are not to be rounded up or down.
 - Hence we use the **Boost**^a interval arithmetic library.
 - It is combined with the **SoftFloat** library to provide support for all floating point formats.

^aBrönnimann, Melquiond, Pion, <http://www.boost.org/libs/numeric/interval/>

Bounding the errors

- Errors are also bounded by intervals ($\tilde{v} \in \mathbb{F}$, $v \in \mathbb{R}$):
 - **absolute** error: $v - \tilde{v} \in A_{\tilde{v},v}$,
 - **relative** error: $v/\tilde{v} - 1 \in R_{\tilde{v},v}$ for v and \tilde{v} of same sign.

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 - **relative** error: $v/\tilde{v} - 1 \in R_{\tilde{v},v}$ for v and \tilde{v} of same sign.
- Absolute error of the multiplication:

$$\begin{aligned}x \times y - \tilde{x} \otimes \tilde{y} &= (x \times y - \tilde{x} \times \tilde{y}) + (\tilde{x} \times \tilde{y} - \tilde{x} \otimes \tilde{y}) \\ &= (x - \tilde{x})\tilde{y} + (y - \tilde{y})\tilde{x} + (x - \tilde{x})(y - \tilde{y}) + \epsilon_0\end{aligned}$$

$$A_{x \times y, \tilde{x} \otimes \tilde{y}} \subseteq A_{x, \tilde{x}}\tilde{Y} + A_{y, \tilde{y}}\tilde{X} + A_{x, \tilde{x}}A_{y, \tilde{y}} + A_{\tilde{x} \otimes \tilde{y}}^0$$

$\epsilon_0 \in A_{\tilde{x} \otimes \tilde{y}}^0$ is the **rounding** error.

Bounding the errors

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- This is **traditional** interval arithmetic:
 - interval operators deal with **real** numbers,
 - interval bounds are **rounded** up and down through **MPFR**.

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Formal proof

- Automatically computing bounds on variables and errors may suffer from **limitations** or **bugs** from the tools. Examples:
 - no support for subnormal numbers,
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Formal proof

- Automatically computing bounds on variables and errors may suffer from **limitations** or **bugs** from the tools. Examples:
 - no support for subnormal numbers,
 - a problem in the underlying arithmetic libraries.
- Two solutions:
 - 1. certify the tools and their libraries,
 - 2. generate a **formal proof** along the computations.

Oracles

- The tool is external, not tied to any proof checker. It can act as an oracle and simplify the proof.
 - If the user just wants the tool to prove $|x - \tilde{x}| < 1.5$, no need to generate the complex proof of the optimal bound $|x - \tilde{x}| < 1.43569726$.
 - The shorter the numbers are, the faster multi-precision floating point arithmetic is. Simpler proofs are validated faster.

Conclusion

- Certifying the numerical behavior of a program is a tedious task:
 - tools are a necessity,
 - they should not require extensive knowledge on the domain.
- Advantage of our approach to formal proof:
 - no need for a blind faith in the tools,
 - results are usable in an extensive formal certification of a program.

Perspectives

- Interface our tools with **Why**:
 - a software verification tool,
 - it generates proof obligations for **Coq**, **PVS**, etc.
- Handle alternate computer arithmetics:
 - floating point, **double-double**,
 - **fixed point** arithmetic.
- Generate formal proofs for other proof checkers.

Questions?

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