Proof and certification for an accurate discriminant

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Introduction

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Given a, b, and c, three floating-point numbers (precision t), the roots of $a \cdot x^2 + b \cdot x + c = 0$ are given by

$$\frac{-b + s \cdot \sqrt{D}}{a} \quad \text{and} \quad \frac{c}{-b + s \cdot \sqrt{D}}$$

with $D = b^2 - a \cdot c$ and $s \in \{-1, +1\}$.

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> *The shortest proofs found so far for the correctness of the algorithms are, as usual for floating-point, far longer and trickier than the algorithms in question.*

Requirement: an accurate discriminant

The tricky part lies in the computation of the discriminant $D = b^2 - a \cdot c$.

Goal: Assuming there is neither overflow nor underflow, compute a floating-point number d such that the error $\delta = |d - D|$ is relatively small: $\delta \leq 2 \cdot \mathrm{ulp}(d)$.

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Difficulty: When $b \otimes b$ and $a \otimes c$ are close, their subtraction is computed exactly. Hence the rounding errors from $b \otimes b$ and $a \otimes c$ are amplified.

Discriminant algorithm

Originally some Matlab vector code. Here is a C version:

```c
double discr(double a, double b, double c) {
  double p, q, d;
  p = b * b;
  q = a * c;
  d = p - q;
  if (3 * fabs(d) < p + q) {
    // slow path, d is not accurate enough
    double dp, dq;
    dp = fma(b, b, -p); // p + dp = b^2
    dq = fma(a, c, -q); // q + dq = a · c
    d = (p - q) + (dp - dq);
  }
  return d;
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```

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Two execution paths:

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2. $3 \otimes |d| < p \oplus q$ (slow path).
   Assuming $p \ominus q = p - q$.
   - Returned value: $d = (p - q) \oplus (dp \ominus dq)$.
     Mathematical value: $D = (p - q) + (dp - dq)$.

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Motivation for a formal proof

The proof relies on numerous case studies and constantly goes forth and back between real arithmetic and floating-point arithmetic.

A formal certification guarantees no case was forgotten and each step of the proof is sound.

The Coq proof assistant was used with the formalization by Daumas, Rideau, and Théry.

Formalization of floating-point arithmetic

A pair of integers (m, e) is associated to the real value $m \cdot 2^e$.
This is a representable floating-point number if $|m| < 2^t$ holds.

IEEE-754 specifies that the results of floating-point operations are the same as if they were first computed in infinite precision and then rounded to precision t.

Execution paths

Let us suppose that the hypotheses for the execution paths are:

1. $q \le \frac{p}{2}$ or $q \ge 2 \cdot p$ (fast path).
   - Equivalent to $3 \cdot |p - q| \ge p + q$.
   - Not equivalent to $3 \otimes |d| \ge p \oplus q$.

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   - Equivalent to $3 \cdot |p - q| \geq p + q$.
   - Not equivalent to $3 \otimes |d| \geq p \oplus q$.

2. $\frac{p}{2} \leq q \leq 2 \cdot p$ (slow path).
   - The assumption $p \ominus q = p - q$ now holds.
   - It implies $3 \otimes |d| \leq p \oplus q$.

The fast path: $q \leq \frac{p}{2}$ or $q \geq 2 \cdot p$

Returned value: $d = p \ominus d$ with $p = b \otimes b$ and $q = a \otimes c$.
Proof goal: $\mathrm{ulp}(p) + \mathrm{ulp}(q) \leq 3 \cdot \mathrm{ulp}(d)$.
Trivial case: $q = 0$; assuming $q \neq 0$.

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   - $d \ge p$ and $d \ge |q|$,
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   - $p - q \ge \frac{p}{2} \ge q \ge 0$,
   - $\mathrm{ulp}(d) \ge \frac{1}{2}\mathrm{ulp}(p) \ge \mathrm{ulp}(q)$,
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3. When q is positive and $q \ge 2 \cdot p$,
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The slow path: $\frac{p}{2} \le q \le 2 \cdot p$

Returned value: $d = (p - q) \oplus (dp \ominus dq)$
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Why $3 \cdot \min(\text{ulp}(p), \text{ulp}(q))$?

- $\max(\text{ulp}(p), \text{ulp}(q)) = 2 \cdot \min(\text{ulp}(p), \text{ulp}(q))$, since $\frac{p}{2} \leq q \leq 2 \cdot p$.
- $|dp - dq| \leq |dp| + |dq| \leq \frac{1}{2}\text{ulp}(p) + \frac{1}{2}\text{ulp}(q) \leq \frac{3}{2}\min(\text{ulp}(p), \text{ulp}(q))$.

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   - $|(p - q) + (dp \ominus dq)| \geq \frac{1}{2}|p - q| \geq |dp - dq|$,
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3. When $\text{ulp}(p) \neq \text{ulp}(q)$ and $|p - q| < 3 \cdot \min(\text{ulp}(p), \text{ulp}(q))$,
   - $|p - q|$ is a multiple of $\min(\text{ulp}(p), \text{ulp}(q))$, so either $2\times$ or $1\times$.
   - p and q have to be close to a power of 2.

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   - p and q have to be close to a power of 2.

   Without any loss of generality, let us assume $p = 1$ and $p \geq q$.
   Only two cases:
   - $p = 1$ and $q = 1^- = 1 - \frac{1}{2}\text{ulp}(1)$,
   - $p = 1$ and $q = 1^{--} = 1 - \text{ulp}(1)$.

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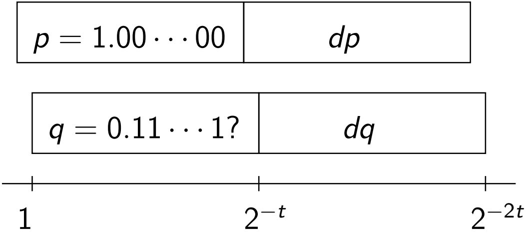

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   $d \geq p - q \leq 2^{-t}$ and $dp - dq \leq 2^{-t} + 2^{-1-t}$,
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   3.3 When dp and dq have opposite signs and $dp - dq \le 0$,
   $dp \ominus dq = dp - dq$.
   (Either $dp - dq = -2^{-t}$ or as in 3.1).

What is missing?

The proof is not complete since the hypotheses do not match the conditional execution.

Two steps may be required:

1. Modify the algorithm: change the boolean expression to $3 \otimes |d| \leq p \oplus q$ for the slow path.
2. Prove that the slow path still works correctly when both $3 \cdot |p - q| > p + q$ and $3 \otimes |p \ominus q| \leq p \oplus q$ hold.

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Two potential reasons why the slow path should still work:

- $p \ominus q = p - q$ when $\frac{p}{2+2^{-t}} \leq q \leq (2 + 2^{-t}) \cdot p$,
- if the subtraction is not exact, the additional error-term is bounded by $\text{ulp}(d)$.

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Future work:

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Future work:

- Complete the proof.
- Prove the modified algorithm ($3 \longrightarrow 1024$) when extra-precision is available for intermediate results.

Questions?

Formal development: http://lipforge.ens-lyon.fr/www/pff/

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