Special Session on Interval Arithmetic
IEEE Interval Standard WG – P1788

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Interval Arithmetic

Extension of real arithmetic operators to intervals.
(Intervals $\equiv$ closed connected subsets of real numbers.)

Property (Inclusion)

For $\mathbf{x}$ *and* $\mathbf{y}$ *intervals, and for* $\diamond \in \{+, -, \times, \div, \ldots\}$,

$$\forall x \in \mathbf{x},\ \forall y \in \mathbf{y}, \quad x \diamond y \in \mathbf{x} \diamond \mathbf{y}.$$

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Interval arithmetic is part of the toolbox for reliable computing.

Example: Newton Interval Method

Lemma (Newton interval Method)

Given a function f, its first derivative f', and some of their interval extensions $\mathbf{f}$ and $\mathbf{f}'$, the sequence $(\mathbf{x}_n)$ of intervals

$$\mathbf{x}_{n+1} = \mathbf{x}_n \cap \left(\mathbf{m}_n - \frac{\mathbf{f}(\mathbf{m}_n)}{\mathbf{f}'(\mathbf{x}_n)} \right) \quad \textit{with } \mathbf{m}_n = [mid(\mathbf{x}_n)]$$

converges, and each $\mathbf{x}_n$ contains all the roots of f in $\mathbf{x}_0$.

(Easy improvement: isolate roots into separate intervals.)

Reliability: all the roots are accounted for, even if the interval computations are over-approximated.

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- Constraint propagation; solving satisfiability problems.
- Global optimization.
- Solutions of systems of differential equations.
- Mathematical proofs, *e.g.*, Hales' proof of Kepler's conjecture.

Floating-Point Bounds

Representing enclosures with floating-point bounds:
$[a, b] = \{x \mid a \le x \le b\}$, $[a, +\infty)$, $(-\infty, b]$, $(-\infty, +\infty)$.

Property (Basic operators)

Given $u \in [\underline{u}, \overline{u}]$ *and* $v \in [\underline{v}, \overline{v}]$,

- $-u \in [-\overline{u}, -\underline{u}]$,
- $\sqrt{u} \in [down(\sqrt{\underline{u}}), up(\sqrt{\overline{u}})] \quad$ *for* $\underline{u} \ge 0$,
- $u + v \in [down(\underline{u} + \underline{v}), up(\overline{u} + \overline{v})]$,
- $\ldots$

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IEEE 754 provides portability for infinities, directed rounding, etc.

Some Other Interval Arithmetics

- Kaucher arithmetic: algebraization of bounded intervals

  $[a, b] + \text{opp}([a, b]) = [0, 0]$ with $\text{opp}([a, b]) = [-a, -b]$,
  $[a, b] \times \text{inv}([a, b]) = [1, 1]$ with $\text{inv}([a, b]) = [a^{-1}, b^{-1}]$.

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- C-set models:

  Infinities as first-class citizens.

History

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January 2008, Dagstuhl meeting:

Should interval arithmetic be pushed into IEEE 754R, or should it have its own separate standard?

Working Group

Approved by IEEE on 2008-06-12.

Officers accepted on 2008-11-08,
after being elected by the registered participants:

Chair	Nathalie Revol
Vice chairs	Baker Kearfott, John Pryce
Secretary	William Edmonson
Vote tabulator	George Corliss
Tech. editors	David Lester, John Pryce
Archivist	Guillaume Melquiond
Webmaster	Jürgen Wolff von Gudenberg

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Currently processing our 5th motion.

Participating to IEEE 1788

- Join the mailing-list.

  Send an email to `nathalie.revol@ens-lyon.fr` mentioning names, affiliation, and address.

- Register on the IEEE-SA website.

  (IEEE membership is not mandatory.)