Numerical Computations and Formal Methods

Guillaume Melquiond

Proval, Laboratoire de Recherche en Informatique
INRIA Saclay–IdF, Université Paris Sud, CNRS

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Deductive Program Verification

1. Deductive program verification
   - Floyd-Hoare logic and weakest preconditions
   - A framework for program verification: Why
   - Gappa
2. Computing in a formal system
3. Decision procedures for arithmetic theories
4. Conclusion

Hoare Triple

Definition (Hoare triple)

$$\{\mathit{precondition}\} \quad \mathit{code} \quad \{\mathit{postcondition}\}.$$

Meaning of correctness:
If the precondition holds just before the code is executed,
the postcondition holds just after it has been executed.

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Note: the definition assumes the code terminates.
If it does not, any postcondition holds, including False.

Hoare Triple

```
{ x >= 0 }
y = floor(sqrt(x))
{ y >= 0 and y*y <= x < (y+1)*(y+1) }
```

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How to verify it?

- Compute the weakest precondition (Dijkstra, 1975) from the function and its specified postcondition.
- Prove that the specified precondition implies the weakest one.

A Framework for Program Verification: Why

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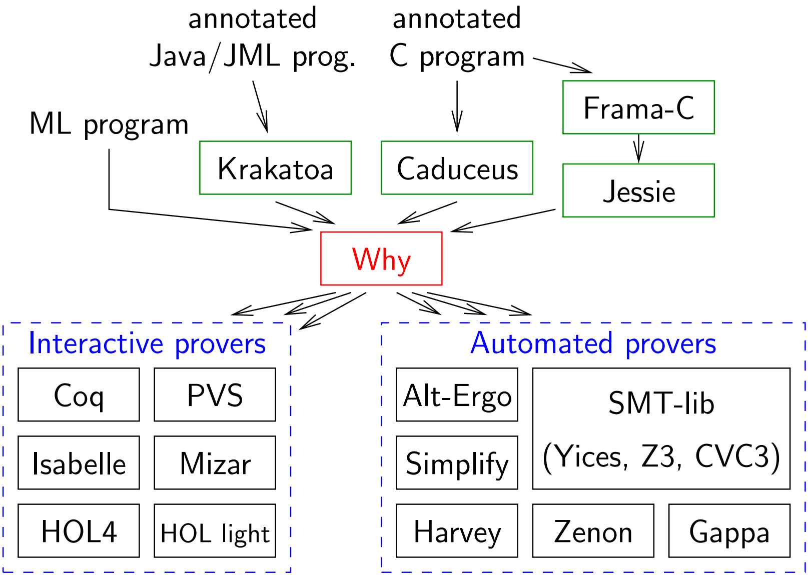
Various tools translate programing languages (C, Java) to the ML language.

Environment



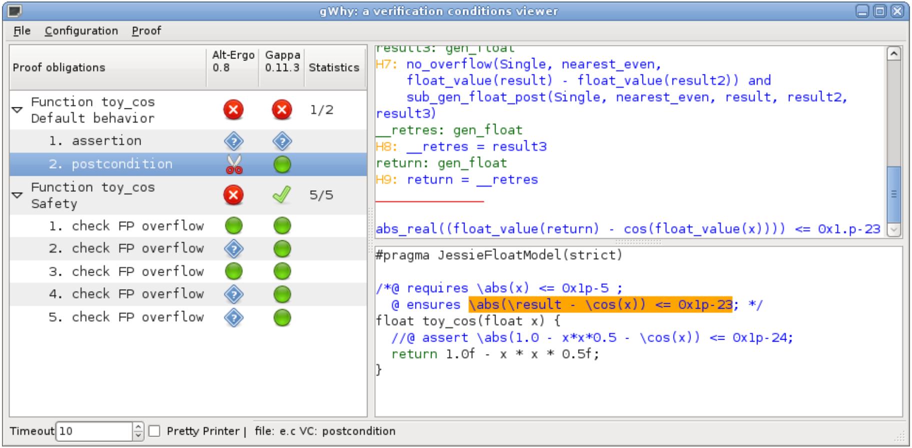
Toy Example: Cosine Around Zero

```
/*@ requires \abs(x) <= 0x1p-5 ;
  @ ensures \abs(\result - \cos(x)) <= 0x1p-23; */
float toy_cos(float x) {
  //@assert \abs(1.0-x*x*0.5 - \cos(x)) <= 0x1p-24;
  return 1.0f - x * x * 0.5f;
}
```

"\result" is the value returned by the function, that is:
$1 - 0.5 \cdot x^2$ with all the operations rounded to nearest binary32.

- Safety: none of the operations overflow nor are invalid.
- Correctness: the result is almost the mathematical cosine.

Frama-C/Jessie/Why + Gappa



Verifying Arithmetic Properties

Kind of properties:

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- Accuracy of results:
  - absolute error: $\forall \vec{x},\ f(\vec{x}) - g(\vec{x}) \in E$;

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  - absolute error: $\forall \vec{x},\ f(\vec{x}) - g(\vec{x}) \in E$;
  - relative error: $\forall \vec{x},\ \exists \varepsilon,\ f(\vec{x}) = g(\vec{x}) \times (1 + \varepsilon)$.

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Language of formulas:

- intervals with nonsymbolic bounds,
- expressions with mathematical operators (e.g., $\times$, tan) and rounding operators (e.g., $\lfloor \cdot \rceil$).

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Method: saturation over a set of theorems.

- Naive interval arithmetic:
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- Floating-/fixed-point arithmetic properties:
  $u \in 2^{-1074} \cdot \mathbb{Z} \Rightarrow \exists \varepsilon \in [-2^{-53}, 2^{-53}],\ \circ(u) = u \times (1 + \varepsilon)$.

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- Forward error analysis:
  $\tilde{u} \times \tilde{v} - u \times v = (\tilde{u} - u) \times v + u \times (\tilde{v} - v) + (\tilde{u} - u) \times (\tilde{v} - v)$.
- ...

Computing in a Formal System

Example: Peano's Arithmetic

Inductive definition of natural numbers:

```
type nat = O | S of nat    (* 5 = SSSSSO *)
```

Axioms for addition:

add0: $\forall b,\ \mathrm{O} + b = b$
addS: $\forall a\ b,\ (\mathrm{S}\ a) + b = a + (\mathrm{S}\ b)$

Example: Peano's Arithmetic

Deductive proof of $4 + (2 + 3) = 9$: (9 steps)

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{}{9 = 9}\ \text{reflexivity}}{0 + 9 = 9}\ \mathtt{add0} \\ \vdots\ \mathtt{addS} \times 4 \\ 4 + 5 = 9\end{array}}{4 + (0 + 5) = 9}\ \mathtt{add0}}{4 + (1 + 4) = 9}\ \mathtt{addS}}{4 + (2 + 3) = 9}\ \mathtt{addS}
$$

Introducing Computations into Proofs

Recursive definition of addition:

```
let rec plus x y =
  match x with
  | O -> y
  | S x' -> plus x' (S y)
```

Lemma `plus_xlate`: $\forall a\ b,\ a + b = \texttt{plus}\ a\ b$

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Proof of $4 + (2 + 3) = 9$: (4 steps)

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Type Theory and Conversion

Curry-Howard correspondence and type theory:

1. Proposition A holds if the type A is inhabited.
2. Convertible types have the same inhabitants.

$$\frac{p : A}{p : B}\ A \equiv_\beta B$$

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Encoding Expressions

Inductive definition of expressions on natural numbers:

```
type expr = Nat of nat | Add of expr * expr
let rec interp_expr e =
  match e with
  | Nat n -> n
  | Add (x, y) ->
    (interp_expr x) "+" (interp_expr y)
```

Proof of $4 + (2 + 3) = 9$:

$$\dfrac{\dfrac{???}{\texttt{interp\_expr}\ (\texttt{Add}\ (\texttt{Nat}\ 4, \texttt{Add}\ (\texttt{Nat}\ 2, \texttt{Nat}\ 3))) = 9}}{4 + (2 + 3) = 9}\ \beta\text{-reduction}$$

Evaluating Expressions

Evaluating expressions on natural numbers:

```
let rec eval_expr e =
  match e with
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  | Add (x, y) ->
    plus (eval_expr x) (eval_expr y)
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Proof of $4 + (2 + 3) = 9$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{9 = 9}\ \text{reflexivity}}{\texttt{eval\_expr (Add (Nat 4,} \ldots\texttt{))} = 9}\ \beta\text{-reduction}}{\texttt{interp\_expr (Add (Nat 4,} \ldots\texttt{))} = 9}\ \texttt{expr\_xlate}}{4 + (2 + 3) = 9}\ \beta\text{-reduction}$$

Relational Operators

Equality is usually a native concept, while comparisons are not.

Comparing natural numbers:

```
let rec le x y =
  match x, y with
  | O   , _    -> true
  | S _ , O    -> false
  | S x', S y' -> le x' y'
```

Lemma: $\forall a \forall b \quad \texttt{le}\ a\ b = \mathit{true} \Leftrightarrow a \leq b$

Encoding Comparisons

Inductive definition of relations on natural expressions:

```
type prop = Le of expr * expr
let interp_prop p =
  match p with
  | Le (x, y) ->
    (interp_expr x) "<=" (interp_expr y)
let eval_prop p =
  match p with
  | Le (x, y) -> le (eval_expr x) (eval_expr y)
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```

Proof of $4 + (2 + 3) \leq 5 + 6$:

$$\dfrac{\dfrac{\dfrac{}{\mathit{true} = \mathit{true}}\ \text{reflexivity}}{\texttt{eval\_prop (Le (Add ..., Add ...))} = \mathit{true}}\ \beta\text{-reduction}}{\dfrac{\texttt{interp\_prop (Le (Add ..., Add ...))}}{4 + (2 + 3) \leq 5 + 6}\ \beta\text{-reduction}}\ \texttt{prop\_xlate}$$

Some Formalizations of Arithmetic in Coq

- Integers as lists of bits:
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  enclosures for expressions of elementary functions.
- Real numbers as streams of integer words.

Enclosures for Expressions of Elementary Functions

Example:

$$\forall x \in [2^{-20}, 1],\ \left|\frac{x \times (1 - 10473 \cdot 2^{-16} \cdot x^2)}{\sin x} - 1\right| \leq 102 \cdot 2^{-16}.$$

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Example:

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Method: order-1 Taylor interval computations and bisection.

Interval Approaches: Relative Error of a Rounded Sine

Relative error between $\sin x$ and the binary32 Horner evaluation of a degree-3 polynomial for $x \in [2^{-20}, 1]$:

```
Theorem rounded_sine :
  forall x y,
  y = rnd(x * rnd(1 - rnd(rnd(x*x) * (10473/65536)))) ->
  1/1048576 <= x <= 1 ->
  Rabs(y - sin x) <= 103 / 65536 * Rabs(sin x).
Proof.
intros.
set (My := x * (1 - (x*x) * (10473/65536))).
assert (Rabs(My - sin x) <= 102 / 65536 * Rabs(sin x)).
  (* method error *)
  apply helper. admit.
  unfold My.
  abstract interval with
    (i_bisect_diff x, i_depth 40, i_nocheck).
unfold My in H1.
gappa. (* global error *)
Qed.
```

Interval Approaches: Square Root

- Fully computational approach:

$$f([\underline{u},\overline{u}]) = \begin{cases} \left[\triangledown\sqrt{\underline{u}}, \vartriangle\sqrt{\overline{u}}\right] & \text{if } 0 \le u, \\ \perp & \text{otherwise.} \end{cases}$$

Correctness lemma: $\forall x \in [\underline{u},\overline{u}],\ \sqrt{x} \in f([\underline{u},\overline{u}])$.

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- Oracle-based approach:

$$f([\underline{u},\overline{u}],[\underline{v},\overline{v}]) = 0 \le \overline{v} \land \overline{u} \le \overline{v}^2 \land \begin{cases} 0 \le \underline{u} & \text{if } \underline{v} \le 0 \\ \underline{v}^2 \le \underline{u} & \text{otherwise.} \end{cases}$$

  Correctness lemma:
  $\forall x \in [\underline{u},\overline{u}],\ f([\underline{u},\overline{u}],[\underline{v},\overline{v}]) = \mathit{true} \Rightarrow \sqrt{x} \in [\underline{v},\overline{v}]$.

Computing with (Approximate) Reals: Issues

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- Decidability?
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Decision Procedures for Arithmetic Theories

1. Deductive program verification
2. Computing in a formal system
3. Decision procedures for arithmetic theories
   - Quantifier elimination
   - Theory $(\mathbb{C}, +, \times, =)$
   - Theory $(\mathbb{Q}, +, =, <)$
   - $\forall$-formulas, ideals, and cones
4. Conclusion

Quantifier Elimination

Definition (Quantifier elimination)

A theory T in a first-order language L admits QE if,
for any formula $p \in L$, there is a quantifier-free formula $q \in L$ such that $T \models p \Leftrightarrow q$ and q has no other free variables than p.

Sufficient condition: any formula "$\exists x,\ \alpha_1 \wedge \cdots \wedge \alpha_n$" admits QE.

Property

A formula is decidable in a theory QE if it has no free variables.

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Property

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Example on $\mathbb{N}$: $\quad \forall x,\ 1 \leq x \Rightarrow \exists y,\ y < x.$

- $\neg(\exists x,\ 1 \leq x \wedge \neg(\exists y,\ y < x))$
- $\neg(\exists x,\ 1 \leq x \wedge \neg(0 < x))$
- $\neg(1 \leq 0)$

Arithmetic Theories and Quantifier Elimination

Decidable theories:

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- $(\mathbb{Q}, +, \lfloor\cdot\rfloor, =, <)$ — Weispfenning

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Undecidable theory:

- $(\mathbb{Z}, +, \times, =, <)$ Tarski, Gödel

Theory $(\mathbb{C}, +, \times, =)$

Given $\exists x,\ p_1(x) = 0 \land \cdots \land p_m(x) = 0 \land q_1(x) \neq 0 \land \cdots \land q_n(x) \neq 0.$

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1. Reducing to $\exists x,\ P(x) = 0 \land Q(x) \neq 0$:
   - $q_1(x) \neq 0 \land \cdots \land q_n(x) \neq 0 \Leftrightarrow q_1(x) \times \cdots \times q_n(x) \neq 0.$

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   - $q_1(x) \neq 0 \land \cdots \land q_n(x) \neq 0 \Leftrightarrow q_1(x) \times \cdots \times q_n(x) \neq 0.$
   - $c^k \times p_i(x) = p_j(x) \times q(x) + r(x)$, so

     $$p_i(x) = 0 \land p_j(x) = 0 \Leftrightarrow \begin{cases} r(x) = 0 \land p_j(x) = 0 & \text{if } c \neq 0 \\ p_i(x) = 0 \land p_j^*(x) = 0 & \text{if } c = 0 \end{cases}$$

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2. Cases:
   - $(\exists x,\ Q(x) \neq 0) \Leftrightarrow \neg$(coefs of Q are zero).
   - $(\exists x,\ P(x) = 0) \Leftrightarrow \neg(\ldots)$

Theory $(\mathbb{C}, +, \times, =)$

Given $\exists x,\ p_1(x) = 0 \wedge \cdots \wedge p_m(x) = 0 \wedge q_1(x) \neq 0 \wedge \cdots \wedge q_n(x) \neq 0$.

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   - $(\exists x,\ P(x) = 0) \Leftrightarrow \neg(\ldots)$
   - $(\exists x,\ P(x) \neq 0 \Rightarrow Q(x) \neq 0) \Leftrightarrow \neg(P|_x Q^n)$.

Theory $(\mathbb{Q}, +, =, <)$

Quantifier elimination of linear constraints:

- $(\exists x,\ x = \vec{a} \cdot \vec{y} \wedge P[x, \vec{y}]) \Leftrightarrow P[\vec{a} \cdot \vec{y}, \vec{y}]$.

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- $(\exists x,\ \bigwedge_i x < \vec{a}_i \cdot \vec{y} \wedge \bigwedge_j x > \vec{b}_j \cdot \vec{y}) \Leftrightarrow \bigwedge_{i,j} 0 < (\vec{a}_i - \vec{b}_j) \cdot \vec{y}$.

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Quantifier elimination of linear constraints:

- $(\exists x,\ x = \vec{a} \cdot \vec{y} \wedge P[x, \vec{y}]) \Leftrightarrow P[\vec{a} \cdot \vec{y}, \vec{y}]$.
- $(\exists x,\ \bigwedge_i x < \vec{a}_i \cdot \vec{y} \wedge \bigwedge_j x > \vec{b}_j \cdot \vec{y}) \Leftrightarrow \bigwedge_{i,j} 0 < (\vec{a}_i - \vec{b}_j) \cdot \vec{y}$.

Special case: closed $\exists$-formulas of conjunctions.
Methods: simplex, interior point.

$\forall$-Formulas, Ideals, and Cones

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Methods: Gröbner bases, semi-definite programming, ...

Suitable for oracles: verifying ideal membership ($\Leftarrow$) is just a single polynomial equality.

Conclusion

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Conclusion

- Deductive verification allows to certify arbitrary programs.

  But proof obligations lack structure,
  making it difficult for automated provers.

- Numerical computations are not incompatible
  with formal systems.

  They can be used to prove mathematical theorems.

- There are powerful but slow methods for proving
  some large sets of proof obligations.

  Oracle-based approaches can dramatically increase
  performances on specific subsets.

Questions?