Automated Methods for Verifying Floating-point Algorithms

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Why Floating-point Arithmetic?

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Speed of FP operations is high and deterministic, but all bets are off with respect to the quality of FP results: precision is known, but accuracy is not.
Verifying Floating-point Algorithms

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There exist numerous automated tools for this job. But what if your algorithm is intricate or you need a formal proof?
### Scope and Constraints

**Scope**

- **real numbers** and basic operators: $+$, $\times$, $\div$, $\sqrt{}$;
- **radix-2** fixed- and FP arithmetic (no multi-precision);
- **logical formulas** (no control flow).
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#### Features

- compute **range** and **format** of expressions;
- bound **forward errors**.
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**Features**
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- bound **forward errors**.

**Constraints**
- handle complicated formulas (possibly with user help),
- generate Coq proofs that fit into **Flocq**’s formalism.
Outline

1 Introduction
   - Verification
   - The Flocq library
   - The Gappa tool

2 Interval arithmetic and forward error analysis

3 Dealing with more intricate algorithms

4 The Gappa tool
Why is FP Arithmetic Amenable to Formal Proof?

IEEE-754 standard for FP arithmetic

*Every operation shall be performed as if it first produced an intermediate result correct to infinite precision and with unbounded range, and then rounded that result.*
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Every operation shall be performed as if it first produced an intermediate result correct to infinite precision and with unbounded range, and then rounded that result.

- Concise specification, suitable for program verification.
- It is all about real numbers.
Exceptional Values

Floating-point computations can lead to **exceptional** behaviors:

- invalid operations: $\sqrt{-1}$,
- overflow: $2 \times 2 \times \cdots \times 2$. 
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When proving a FP algorithm, the very first step is to prove that
- exceptional behaviors cannot arise, or
- they are properly handled.
Floating-point computations can lead to exceptional behaviors:
  - invalid operations: $\sqrt{-1}$,
  - overflow: $2 \times 2 \times \cdots \times 2$.

When proving a FP algorithm, the very first step is to prove that
  - exceptional behaviors cannot arise, or
  - they are properly handled.

Today’s talk is not about floating-point exceptions. Let us assume that they are proved not to occur.

(This can be achieved by computing the range of expressions.)
Since there are no exceptional behaviors, floating-point numbers can be embedded into real numbers.
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Representable numbers

\[ \mathbb{F} = \{ m \cdot \beta^e \in \mathbb{R} \mid m, e \in \mathbb{Z} \land |m| < \beta^p \land e \geq e_{\text{min}} \} \]

with \( \beta \), \( p \), and \( e_{\text{min}} \) depending on the format.
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with \( \beta, p, \) and \( e_{\text{min}} \) depending on the format.

Rounding operators

The result of an addition \( a \oplus b \) is \( \circ(a + b) \)

with \( \circ : \mathbb{R} \rightarrow F \) a monotonic function that is the identity on \( F \).

\( \circ(\cdot) \) depends on the destination format and the rounding direction.
The Gappa Tool

Gappa 1.1: 11k lines of C++, 8k lines of Coq, GPL’d.
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Example (Cody-Waite argument reduction for exp)

```plaintext
x = float<ieee_64,ne>(dummyx);  # x is a double

Log2h = 0xb.17217f7d1cp-4;  # 42 bits out of 53
InvLog2 = 0x1.71547652b82fep0;
k = int<ne>(float<ieee_64,ne>(x*InvLog2));
t1 float<ieee_64,ne>= x - k*Log2h;

# prove that t1 is computed exactly
{ x in [0.7, 800] -> t1 = x - k*Log2h }

Log2h ~ 1/InvLog2;  # user hint
```

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Outline

1. Introduction

2. Interval arithmetic and forward error analysis
   - Preliminaries
   - Interval arithmetic
   - Forward error analysis
   - Example: fast sine

3. Dealing with more intricate algorithms

4. The Gappa tool
What We Want to Prove

- **Bounds** on program expressions:
  \[
  \forall x_1, \ldots, x_m \in \mathbb{R}, \; e_1 \in I_1 \land \ldots \land e_n \in I_n \Rightarrow e \in J
  \]
  with \( I_1, \ldots, I_n, J \) intervals with *nonsymbolic* bounds.
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  with \( I_1, \ldots, I_n, J \) intervals with nonsymbolic bounds.

- **Bounds** on forward errors:
  \[ \forall x_1, \ldots, x_m \in \mathbb{R}, \ e_1 \in I_1 \land \ldots \land e_n \in I_n \Rightarrow \tilde{e} - e \in K \]
  with \( \tilde{e} \) and \( e \) two expressions with close values.
A Variety of Forward Errors

Example (Addition)
Let $u$ and $v$ be approximated by $\tilde{u}$ and $\tilde{v}$.
What is the error between $\circ(\tilde{u} + \tilde{v})$ and $u + v$?
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Let \( u \) and \( v \) be approximated by \( \tilde{u} \) and \( \tilde{v} \).
What is the error between \( \circ(\tilde{u} + \tilde{v}) \) and \( u + v \)?

Three errors are involved:

- between \( \tilde{u} \) and \( u \),
- between \( \tilde{v} \) and \( v \),
- round-off error between \( \circ(\tilde{u} + \tilde{v}) \) and \( \tilde{u} + \tilde{v} \).
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- *round-off* error between $\circ(\tilde{u} + \tilde{v})$ and $\tilde{u} + \tilde{v}$.

Each error bound might be either
- *absolute*: $\tilde{u} - u \in I$, or
- *relative*: $(\tilde{u} - u)/u \in I$. 
A Variety of Round-off Errors

The round-off error between $\circ(\tilde{u} + \tilde{v})$ and $\tilde{u} + \tilde{v}$ is

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- relatively bounded if $\tilde{u} + \tilde{v}$ is far enough from 0,
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- absolutely bounded if $\tilde{u}$ and $\tilde{v}$ are bounded,
- relatively bounded for FP formats with gradual underflow,
- relatively bounded if $\tilde{u} + \tilde{v}$ is far enough from 0,
- zero if $\tilde{u} + \tilde{v}$ is in a suitable fixed-point format,
- zero if $\tilde{u}/\tilde{v} \in [-2, -1/2]$ for FP formats with gradual underflow.
Interval arithmetic extends operations on real numbers to operations on closed connected subsets of real numbers.

Application

Instead of proving $\forall x \in [a, b], \ f(x) \in [c, d]$, you can prove $F([a, b]) \subseteq [c, d]$, assuming that $F$ is an interval extension of $f$. 
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**Application**

Instead of proving $\forall x \in [a, b], \ f(x) \in [c, d]$, you can prove $F([a, b]) \subseteq [c, d]$, assuming that $F$ is an interval extension of $f$.

Evaluating $F$ is easy; it involves operations on bounds only:

$$x \in [a, b] \land y \in [c, d] \Rightarrow x + y \in [a + c, b + d].$$

This makes interval arithmetic suitable for automatically proving bounds on real-valued expressions.
Interval Arithmetic and Dependencies

**Independent expressions**

If \( a \in [3, 5] \) and \( b \in [1, 2] \) are independent, then

\[
a - b \in [3 - 2, 5 - 1] = [1, 4]
\]

is the optimal enclosure.
Interval Arithmetic and Dependencies

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Correlated expressions

If we have \( a \in [1, 100] \), interval arithmetic gives

\[
(a + \varepsilon) - a \in [1 + \varepsilon, 100 + \varepsilon] - [1, 100] = [-99 + \varepsilon, 99 + \varepsilon]
\]

while the optimal enclosure is \([\varepsilon, \varepsilon]\).
Various methods solve the dependency issue:

- octogons,
- ellipsoids,
- zonotopes,
- Taylor/Chebyshev models,
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- decision procedures, e.g. simplex or CAD.

Unfortunately they are much costlier than interval arithmetic at execution time, and even worse at formalization time.
Leveraging Forward Error Analysis

Forward error analysis offers a simpler way to deal with dependencies.

- “the absolute error of the sum is the sum of the absolute errors”

\[(\tilde{u} + \tilde{v}) - (u + v) = (\tilde{u} - u) + (\tilde{v} - v)\]
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  \[(\tilde{u} + \tilde{v}) - (u + v) = (\tilde{u} - u) + (\tilde{v} - v)\]

- “the relative error of the product is the sum of the relative errors”
  
  \[\frac{\tilde{u}\tilde{v}}{uv} - 1 = \varepsilon_u + \varepsilon_v + \varepsilon_u\varepsilon_v\]

  with \(\varepsilon_u = \tilde{u}/u - 1\) and \(\varepsilon_v = \tilde{v}/v - 1\)
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- “the relative error of rounding operators is bounded”

\[\left|\frac{o(u)}{u} - 1\right| \leq 2^{-p}\text{ if } |u| \geq \ldots\]
Leveraging Forward Error Analysis

Forward error analysis:

1. \((\tilde{u} + \tilde{v}) - (u + v) = (\tilde{u} - u) + (\tilde{v} - v)\)
2. \((\tilde{u}\tilde{v})/(uv) - 1 = \varepsilon_u + \varepsilon_v + \varepsilon_u\varepsilon_v\)

This inductive rewriting works fine as long as

- errors are not correlated,
- expressions have the same inductive structure with correlated sub-expressions in the same places.

Because of the two-step verification process, the above often holds.
Example: Sine Around Zero

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Example (Toy sine)

```c
float toy_sin(float x) {
    if (fabsf(x) < 0x1p-5f) return x;
    return x * (1.0f - x * x * 0x28e9p-16f);
}
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An actual implementation of sin would
- use more than just 2 polynomials, and/or
- perform an argument reduction.

But the proof process is the same!
Approximating a Mathematical Function

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1. Find an approximation $\hat{g}$ of $g$ that uses only real operations that can be approximated by your floating-point unit.

Bound the method error $\frac{\hat{g}(x)}{g(x)} - 1$. 

Proving correctness is just a matter of computing tight bounds for these expressions.
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Proving correctness is just a matter of computing tight bounds for these expressions.
Method Error (Relative)

Method error: \( \frac{x \cdot (1 - x^2 \cdot 10473 \cdot 2^{-16})}{\sin x} - 1 \).

Interval analysis knows how to bound such an expression.
Binary32 Round-off Error (Relative)

Round-off error: \[
\frac{\circ(x \cdot (1-\circ(x^2 \cdot 10473 \cdot 2^{-16})))}{x \cdot (1-x^2 \cdot 10473 \cdot 2^{-16})} - 1.
\]

Gappa knows how to bound such an expression.
(And how to compose method and round-off errors.)
Correctness Statement in Coq

Notation fsub x y :=
  (round radix2 binary32Fmt rndNE (x - y)).
Notation fmul x y :=
  (round radix2 binary32Fmt rndNE (x * y)).

Definition fsin x :=
  if Rle_lt_dec (pow2 (-5)) (Rabs x) then
    fmul x (fsub 1 (fmul (fmul x x)
      (10473 * pow2 (-16))))
  else x.

Lemma sine_spec : forall x, Rabs x <= 1 ->
  Rabs (fsin x - sin x) <= 103* pow2 (-16) *
  Rabs (sin x).
Proof Sketch in Coq

**Lemma** sine_spec : \( \forall x, \text{Rabs } x \leq 1 \rightarrow \)

\( \text{Rabs } (\text{fsin } x - \sin x) \leq 103 \times \text{pow2 } (-16) \times \)

\( \text{Rabs } (\sin x). \)

**Proof.**

intros \( x \) \( \text{Bx}. \) unfold \( \text{fsin}. \)

case \( \text{Rle_lt_dec} \); intros \( \text{Bx'}. \)

\(- (* |x| \geq 1/32, degree-3 approx *) \)

assert (\( \text{Rabs } (x \times (1 - x \times x \times (10473 \times \text{pow2 } (-16))) - \))

\( \sin x) \leq 102 \times \text{pow2 } (-16) \times \text{Rabs } (\sin x)). \)

(* bound the method error *)

interval with (\( \text{i_bisect_diff } x). \)

(* bound the round-off and total errors *)

gappa.

\(- (* |x| < 1/32, degree-1 approx *) \)

destruct (\( \text{MVT_cor2 } \sin \cos). \)

interval.

Qed.
@rnd = float<ieee_32,ne>;
x = rnd(dummyx);  # x is a float

# floating-point implementation
y rnd= x * (1 - x*x * 0x28E9p-16);
# infinitely-precise computation
My = x * (1 - x*x * 0x28E9p-16);

{|x| in [1b-5,1] /\  
  # relative method error
  |My -/ sin_x| <= 1.55e-3 –>

  # relative total error
  |y -/ sin_x| <= 1.551e-3 }
Outline

1. Introduction

2. Interval arithmetic and forward error analysis

3. Dealing with more intricate algorithms
   - Example: Cody-Waite argument reduction
   - Example: Integer division on Itanium

4. The Gappa tool
Intricate Algorithms

For some algorithms, bounding errors is not sufficient, as they might rely on various tricks:

- exact computations,
- error compensations,
- convergent iterations,
- and so on.
Cody-Waite Argument Reduction

**Goal:** compute \( \exp x \) for \( |x| \leq 800 \).

**Argument reduction:** replace \( x \) by a value close to 0, so that \( \exp \) can be approximated by a *small polynomial*. 
Cody-Waite Argument Reduction

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- **Idea 1:** use \( \exp x = 2^k \exp(x - k \log 2) \) with \( k \) an integer.
Goal: compute $\exp x$ for $|x| \leq 800$.

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- **Issue:** how to compute $x - k \log 2$ accurately?
- **Idea 2:** use $\log 2 = \ell_h + \ell_l + \varepsilon$ with $\varepsilon$ close to negligible.

$$\exp x = 2^k \exp((x - k\ell_h) - k\ell_l) \exp(-k\varepsilon).$$
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- **Implementation:** evaluate $(x - k\ell_h) - k\ell_l$ with FP arithmetic.

$$\exp x = 2^k \exp(\circ(\ldots)) \exp(\delta - k\varepsilon).$$
Cody-Waite Argument Reduction

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  \exp x = 2^k \exp((x - k\ell_h) - k\ell_l) \exp(-k\varepsilon).
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- **Implementation:** evaluate \( (x - k\ell_h) - k\ell_l \) with FP arithmetic.
  \[
  \exp x = 2^k \exp(\circ(\ldots)) \exp(\delta - k\varepsilon).
  \]
- **Issue:** how much is \( \delta \)?
Cody-Waite Argument Reduction

Example (Cody-Waite argument reduction for exp, part 1)

\[
\begin{align*}
\Log2h & = 0xb.17217f7d1cp^{-4}; \quad \# \text{42 bits out of 53} \\
\Log2l & = 0xf.79abc9e3b398p^{-48}; \\
\InvLog2 & = 0x1.71547652b82fep0; \\
k & = \text{int}<\text{ne}>(\text{rnd}(x*\InvLog2)); \\
t1 \text{ rnd} & = x - k*\Log2h;
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k &= \text{int}<\text{ne}>(\text{rnd}(x*\text{InvLog2})); \\
t1 \text{ rnd} &= x - k*\text{Log2h}; \\
\end{align*}
\]

Proof.

1. \(|x| \leq 800\), so \(|k| < 2048\), so \(k\) fits on 11 bits.
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k &= \text{int}<\text{ne}> (\text{rnd}(x*\text{InvLog2})); \\
t1 \text{ rnd}= x - k*\text{Log2h};
\end{align*}
\]

Proof.

1. \(|x| \leq 800\), so \(|k| < 2048\), so \(k\) fits on 11 bits.

2. \(\ell_h\) fits on 42 bits, so \(\circ(\ell_h) = k\ell_h\).
Cody-Waite Argument Reduction

Example (Cody-Waite argument reduction for exp, part 1)

\[
\begin{align*}
\text{Log2}h &= 0xb.17217f7d1cp-4; \quad \# \text{42 bits out of 53} \\
\text{Log2}l &= 0xf.79abc9e3b398p-48; \\
\text{InvLog2} &= 0x1.71547652b82fep0; \\
k &= \text{int}<\text{ne}>(\text{rnd}(x*\text{InvLog2})); \\
t1 \text{ rnd} &= x - k*\text{Log2}h;
\end{align*}
\]

Proof.

1. \(|x| \leq 800\), so \(|k| < 2048\), so \(k\) fits on 11 bits.
2. \(l_h\) fits on 42 bits, so \(\circ(kl_h) = kl_h\).
3. \(l_h^{-1} \approx \text{InvLog2}\), so \(x \approx kl_h\).

Guillaume Melquiond
Automated Methods for Verifying Floating-point Algorithms
Cody-Waite Argument Reduction

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k &= \text{int <ne>}(\text{rnd}(x \ast \text{InvLog2})); \\
t1 \ \text{rnd} &= x - k \ast \text{Log2h};
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Proof.

1. \(|x| \leq 800\), so \(|k| < 2048\), so \(k\) fits on 11 bits.
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3. \(\ell_h^{-1} \approx \text{InvLog2}\), so \(x \approx k\ell_h\).
4. So \(\circ(x - \circ(k\ell_h)) = x - k\ell_h\) by Sterbenz.
Exact Computations

For intricate algorithms, ranges of expressions are not enough. You also need to know how many bits you need to represent them.
Cody-Waite Argument Reduction

Example (Cody-Waite argument reduction for exp)

```plaintext
@rnd = float<ieee_64,ne>;
x = rnd(dummyx);  # x is a double

# Cody-Waite argument reduction
Log2h = 0xb.17217f7d1cp-4;  # 42 bits out of 53
Log2l = 0xf.79abc9e3b398p-48;
InvLog2 = 0x1.71547652b82fep0;
k = int<ne>(rnd(x*InvLog2));
t1 rnd= x - k*Log2h;
t2 rnd= t1 - k*Log2l;

# exact values
T1 = x - k*Log2h;
T2 = T1 - k*Log2l;

{ x in [0.3, 800] ->
  t1 = T1 \/
  T1 in [-0.35,0.35] \/
  t2 - T2 in ? }

Log2h ~ 1/InvLog2;

# try harder!
T1 $ x;
```
Intel **Itanium** processors have no hardware divisor. How to efficiently perform a division with just add and mul?
Intel **Itanium** processors have no hardware divisor.
How to efficiently perform a **division** with just add and mul?

**Example (Division of 16-bit unsigned integers on Itanium)**

```plaintext
// Inputs: dividend a in f6, divisor b in f7, 1 + 2^{-17} in f9
frcpa.s1 f8, p6=f6, f7
(p6) fma.s1 f6=f6, f8, f0
(p6) fnma.s1 f7=f7, f8, f9
(p6) fma.s1 f8=f7, f6, f6
fcvt.fx.trunc.s1 f8=f8
// Output: \lfloor a/b \rfloor in f8
```
Example (Division of 16-bit unsigned integers on Itanium)

\[ y_0 \approx \frac{1}{b} \quad [\text{frcpa}] \]
\[ q_0 = \circ(a \times y_0) \]
\[ e_0 = \circ(1 + 2^{-17} - b \times y_0) \]
\[ q_1 = \circ(e_0 \times q_0 + q_0) \]
\[ q = \lfloor q_1 \rfloor \]

with \(\circ(\cdot)\) rounding to nearest on the extended 82-bit format.

Correctness of the division

\[ \forall a, b \in [1; 65535], \quad q = \lfloor a/b \rfloor. \]
Correctness Statement in Coq

Notation fma x y z :=
  (round radix2 register_fmt rndNE (x * y + z)).

Axiom fr CPA : R -> R.
Axiom fr CPA spec : forall x : R,
  1 <= Rabs x <= 65536 ->
  generic_format radix2 (FLT_exp _ 11) (fr CPA x) /\n  Rabs (fr CPA x - 1/x) <= 4433*pow2 (-21) * Rabs(1/x).

Definition div_u16 a b :=
  let y0 := fr CPA b in
  let q0 := fma a y0 0 in
  let e0 := fnma b y0 (1 + pow2 (-17)) in
  let q1 := fma e0 q0 q0 in
  Zfloor q1.

Lemma div_u16 spec : forall a b,
  (1 <= a <= 65535)%Z ->
  (1 <= b <= 65535)%Z ->
  div_u16 a b = (a / b)%Z.
Proof Sketch

**Theorem (Exclusion zones)**

Given $a$ and $b$ positive integers.

If $0 \leq a \times (q_1/(a/b) - 1) < 1$, then $\lfloor q_1 \rfloor = \lfloor a/b \rfloor$. 

Notice the relative error between the FP value $q_1$ and the real $a/b$. So proving the correctness is just a matter of bounding this error.
Proof Sketch

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Proof Sketch Continued

Bounding the method error $\hat{q}_1 - a/b$ and the round-off error $q_1 - \hat{q}_1$ and composing them does not work at all.
Bounding the method error $\hat{q}_1 - a/b$ and the round-off error $q_1 - \hat{q}_1$ and composing them does not work at all.

What the developers knew when designing the algorithm:

- If not for $2^{-17}$, the code would perform a Newton iteration:
  $$\hat{q}_1/(a/b) - 1 = -\varepsilon_0^2 \text{ with } \varepsilon_0 = y_0/(1/b) - 1.$$
- By taking into account $2^{-17}$,
  $$\hat{q}_1/(a/b) - 1 = -\varepsilon_0^2 + (1 + \varepsilon_0) \cdot 2^{-17}.$$
Proof Sketch, the Coq Version

Lemma div_u16_spec : forall a b,
  (1 <= a <= 65535)%Z -> (1 <= b <= 65535)%Z ->
  div_u16 a b = (a / b)%Z.
Proof.
intros a b Ba Bb.
apply Zfloor_imp.
cut (0 <= b * q1 - a < 1).
lra.
set (err := (q1 - a / b) / (a / b)).
replace (b * q1 - a) with (a * err) by field.
set (y0 := frcpa b).
set (Mq0 := a * y0 + 0).
set (Me0 := 1 + pow2 (-17) - b * y0).
set (Mq1 := Me0 * Mq0 + Mq0).
set (eps0 := (y0 - 1 / b) / (1 / b)).
assert (((Mq1 - a / b) / (a / b) =
  -(eps0 * eps0) + (1 + eps0) * pow2 (-17)) by field.
generalize (frcpa_spec b) (FIX_format_Z2R radix2 a)
  (FIX_format_Z2R radix2 b).
gappa.
Qed.
Convergent Algorithms

If you know some clever property about an algorithm, don’t expect automatic tools to infer it, just tell them about it.
The Gappa Script, as Written by a Human

Example (Division of 16-bit unsigned integers on Itanium)

@rnd = float<x86_80,ne>;

# algorithm with no rounding operators
q0 = a * y0;
e0 = 1 + 1b-17 - b * y0;
q1 = q0 + e0 * q0;

# notations for relative errors
eps0 = (y0 - 1 / b) / (1 / b);
err = (q1 - a / b) / (a / b);

{ # a and b are integers
  @FIX(a, 0) \ a in [1,65535] /
  @FIX(b, 0) \ b in [1,65535] /
  # specification of frcpa
  @FLT(y0, 11) \ |eps0| <= 0.00211373 /
  # Newton’s iteration, almost
  err = -(eps0 * eps0) + (1 + eps0) * 1b-17 ->

  # the separation hypothesis is satisfied
  err in [0,1] \ a * err in [0,0.99999] /
  # all the computations are exact
  rnd(q0) = q0 \ rnd(e0) = e0 \ rnd(q1) = q1 }

# try harder!
 rnd(q1) = q1 $ 1 / b;
Outline

1. Introduction
2. Interval arithmetic and forward error analysis
3. Dealing with more intricate algorithms
4. The Gappa tool
   - Supported properties
   - Proof process
   - Theorem database
   - Conclusion
A Few Words About Gappa

Starting from a formula, Gappa saturates a set of theorems to infer new properties until it encounters a contradiction.
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Supported properties

- **BND**\((x, I)\) \(\equiv x \in I\)
- **ABS**\((x, I)\) \(\equiv |x| \in I\)
- **REL**\((x, y, I)\) \(\equiv \exists \varepsilon \in I, \ x = y \cdot (1 + \varepsilon)\)
- **FIX**\((x, e)\) \(\equiv \exists m \in \mathbb{Z}, \ x = m \cdot 2^e\)
- **FLT**\((x, p)\) \(\equiv \exists m, e \in \mathbb{Z}, \ x = m \cdot 2^e \land |m| < 2^p\)
- **NZR**\((x)\) \(\equiv x \neq 0\)
- **EQL**\((x, y)\) \(\equiv x = y\)

To prove `div u16`, Gappa tried to apply 17k theorems. The final proof infers \(\sim 80\) properties.
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4. Generate a formal proof from the trace.
Theorem Database

- Naive interval arithmetic:
  \[ u \in [u, \bar{u}] \wedge v \in [v, \bar{v}] \Rightarrow u + v \in [u + v, \bar{u} + \bar{v}]. \]
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  \[ \tilde{u} \times \tilde{v} - u \times v = (\tilde{u} - u) \times v + u \times (\tilde{v} - v) + (\tilde{u} - u) \times (\tilde{v} - v). \]
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  \[ \text{FLT}(x, p) \land \text{FLT}(y, q) \Rightarrow \text{FLT}(x \times y, p + q). \]
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  \[ \text{FLT}(x, p) \land \text{FLT}(y, q) \Rightarrow \text{FLT}(x \times y, p + q). \]

- And so on.
# Theorem Database

<table>
<thead>
<tr>
<th>Category</th>
<th>Thm</th>
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Conclusion

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But with a bit of help from the user, it can make short work of intricate algorithms.

And it generates formal proofs!
Questions?

Gappa:  http://gappa.gforge.inria.fr/