Automating the Verification of Floating-point Algorithms

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I will not talk much about SMT. I will not talk about floating-point arithmetic in general, but only about its use in a peculiar context: mathematical libraries (libm). I will focus on a specific procedure for verifying floating-point algorithms: the Gappa tool.
This Talk’s Content

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Why Floating-point Arithmetic?

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Speed of FP operations is high and deterministic, but all bets are off with respect to the quality of FP results: precision is known, but accuracy is not.
FP Arithmetic Quick Reference Card

**IEEE-754 FP numbers**

- **Exceptional values:** $\pm 0$, $\pm \infty$, Not-a-Number.
- **Finite values:**
  \[
  \mathbb{F} = \left\{ m \cdot \beta^e \in \mathbb{R} \mid m, e \in \mathbb{Z} \land |m| < \beta^p \land e_{\text{min}} \leq e \leq e_{\text{max}} \right\}
  \]
  with $\beta$, $p$, $e_{\text{min}}$, $e_{\text{max}}$ parameters of the format.
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FP arithmetic operations

Every operation shall be performed as if it first produced an intermediate result correct to infinite precision and with unbounded range, and then rounded that result.

- $a \oplus b = \circ(a + b)$, $a \otimes b = \circ(a \times b)$, and so on.
- Rounding to nearest: $\forall y \in \mathbb{F}$, $|\circ(x) - x| \leq |y - x|$.
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1. Prove correctness assuming that all operators are infinitely precise.
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There exist numerous automated tools for this job. But what if your algorithm is intricate or you need a formal proof?
Outline

1. **Introduction**
   - Context
   - Gallery
   - Gappa
   - Decidability

2. **Interval arithmetic and forward error analysis**

3. **Dealing with more intricate algorithms**

4. **The Gappa tool**

5. **Conclusion**
Example 1: Toy Elementary Function

```c
float toy_sin(float x) {
    assert(fabsf(x) <= 1.0f);
    if (fabsf(x) < 0x1p-5f) return x;
    return x * (1.0f - x * x * 0x28e9p-16f);
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Verification condition for accuracy

\[ \forall x \in \mathbb{F}_{32}, \ |x| \leq 1 \ \Rightarrow \ \left| \frac{\text{toy\_sin} \ x}{\sin x} - 1 \right| \leq 103 \cdot 2^{-16}. \]
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Optional hypothesis:

\[
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- Relative errors.
Example 2: Cody-Waite Argument Reduction for Exp

double exp(double x) {
    if (fabs(x) >= 800) return ...;
    double k = nearbyint(x * 0x1.71547652b82fep0);
    double t1 = x - k * 0xb.17217f7d1cp-4;
    double t2 = t1 - k * 0xf.79abc9e3b398p-48;
    ...
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Verification conditions

- $|t_2| \leq 0.35$,
- $|t_2 - (x - k \cdot 0xb.17217f7d1cf79abc9e3b398p-4)| \leq 2^{-55}$. 
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- Vanishing round-off errors.
Example 3: Itanium Division of 16-bit Unsigned Integers

```haskell
// Inputs: dividend a in f6, divisor b in f7, 1 + 2^{-17} in f9
  frcpa.s1  f8,p6=f6,f7 ;;
(p6) fma.s1  f6=f6,f8,f0
(p6) fnma.s1  f7=f7,f8,f9 ;;
(p6) fma.s1  f8=f7,f6,f6 ;;
  fcvt.fx.trunc.s1  f8=f8
// Output: ⌊a/b⌋ in f8
```
Example 3: Itanium Division of 16-bit Unsigned Integers

\[
\begin{align*}
    y_0 & \approx 1/b \quad \text{[frcpa]} \\
    q_0 & = \circ(a \times y_0) \\
    e_0 & = \circ(1 + 2^{-17} - b \times y_0) \\
    q_1 & = \circ(e_0 \times q_0 + q_0) \\
    q & = \lfloor q_1 \rfloor
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with \(\circ(\cdot)\) rounding to nearest on Itanium’s 82-bit format.
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\forall a, b \in [1; 65535], \quad q = \lfloor a/b \rfloor
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- Vanishing round-off errors.
- Polynomial manipulations.
Example 4: Knuth’ TwoSum Algorithm

\[
\begin{align*}
    s &= a + b \\
    t &= s - a \\
    e &= (a - (s - t)) + (b - t)
\end{align*}
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Example 4: Knuth’ TwoSum Algorithm

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\begin{align*}
  s &= a + b \\
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Assuming no overflow occurs, \( s + e = a + b \).
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Verification condition

Assuming no overflow occurs, \( s + e = a + b \).

- Pointless infinitely-precise values.
- Pointless round-off errors.
Scope and Constraints of Gappa

**Scope**

- Only **real numbers**: no exceptional values.
- Basic arithmetic operations: $+$, $\times$, $\div$, $\sqrt{\cdot}$.
- **Radix-2** fixed- and FP arithmetic (no multi-precision).
- **Logical formulas** (no control flow).
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Features

- Compute **range** and **format** of expressions.
- Bound **forward errors**.
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**Features**
- Compute **range** and **format** of expressions.
- Bound **forward errors**.

**Constraints**
- Handle complicated formulas (possibly with some user help).
- Generate Coq proofs that fit into **Flocq**’s formalism.
- Answer instantly.
Why No Exceptional Values?

- **Safety** of most programs relies on their absence. In that case, range computation is sufficient.
Why No Exceptional Values?

- **Safety** of most programs relies on their absence. In that case, range computation is sufficient.

- Their propagation is purely *combinatorial* anyway. Just use your preferred SAT method.
The Gappa Tool

Gappa 1.1: 11k lines of C++, 8k lines of Coq, GPL'd.
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Example (Cody-Waite argument reduction for exp)

\[
\text{x} = \text{float<ieee}_64,\text{ne}> (\text{dummyx}); \quad \# \text{x is a double}
\]

\[
\text{Log2h} = 0xb.17217f7d1cp-4; \quad \# 42 \text{ bits out of 53}
\]

\[
\text{InvLog2} = 0x1.71547652b82fep0;
\]

\[
\text{k} = \text{int<ne>}(\text{float<ieee}_64,\text{ne}> (\text{x*InvLog2}));
\]

\[
\text{t1 float<ieee}_64,\text{ne}> = \text{x} - \text{k*Log2h};
\]

\[
\# \text{prove that t1 is computed exactly}
\]

\[
\{ \text{x in [0.7, 800]} \rightarrow \text{t1} = \text{x} - \text{k*Log2h} \}
\]

\[
\text{Log2h} \sim 1/\text{InvLog2}; \quad \# \text{user hint}
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k = \text{int}<\text{ne}>(\text{float}<\text{ieee}_64,\text{ne}>(x*\text{InvLog2})); \\
t1 = \text{float}<\text{ieee}_64,\text{ne}>= x - k*\text{Log2h};
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Generated Coq proof: 664 lines, 55 reasoning steps.
Effective rounding

\[ \circ(x) = \text{ulp}(x) \cdot \left\lfloor x / \text{ulp}(x) \right\rfloor, \]

with ulp(x) the distance between the 2 FP numbers surrounding x. Note: ulp is piecewise constant, e.g. 4091 ranges for binary64.
Rounding Operators and Decidability

Effective rounding

\[ \circ(x) = \text{ulp}(x) \cdot \lceil x / \text{ulp}(x) \rceil, \]

with \( \text{ulp}(x) \) the distance between the 2 FP numbers surrounding \( x \).

Note: \( \text{ulp} \) is piecewise constant, e.g. 4091 ranges for binary64.

Decidability of FP arithmetic

For bounded quantifications, the following theories are decidable:

- \((\mathbb{R}, +, \circ(\cdot))\),
- \((\mathbb{F}, \oplus, \otimes, \ldots)\).
Outline

1. Introduction

2. Interval arithmetic and forward error analysis
   - Preliminaries
   - Interval arithmetic
   - Forward error analysis
   - Example 1: toy elementary function

3. Dealing with more intricate algorithms

4. The Gappa tool

5. Conclusion
What We Want to Prove

- **Bounds** on program expressions:
  \[
  \forall x_1, \ldots, x_m \in \mathbb{R}, \ e_1 \in l_1 \land \ldots \land e_n \in l_n \Rightarrow e \in J
  \]
  with \(l_1, \ldots, l_n, J\) intervals with **nonsymbolic** bounds.
What We Want to Prove

- **Bounds** on program expressions:
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  with \( I_1, \ldots, I_n, J \) intervals with **nonsymbolic** bounds.

- **Bounds** on **forward errors**:
  \[ \forall x_1, \ldots, x_m \in \mathbb{R}, \ e_1 \in I_1 \land \ldots \land e_n \in I_n \Rightarrow \tilde{e} - e \in K \]
  with \( \tilde{e} \) and \( e \) two expressions with close values.
A Variety of Forward Errors

Example (Addition)

Let $u$ and $v$ be approximated by $\tilde{u}$ and $\tilde{v}$. What is the error between $\circ(\tilde{u} + \tilde{v})$ and $u + v$?
A Variety of Forward Errors

Example (Addition)

Let \( u \) and \( v \) be approximated by \( \tilde{u} \) and \( \tilde{v} \).
What is the error between \( \circ(\tilde{u} + \tilde{v}) \) and \( u + v \)?

Three errors are involved:

- between \( \tilde{u} \) and \( u \),
- between \( \tilde{v} \) and \( v \),
- round-off error between \( \circ(\tilde{u} + \tilde{v}) \) and \( \tilde{u} + \tilde{v} \).
A Variety of Forward Errors

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- between $\tilde{v}$ and $v$,
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Each error bound might be either

- **absolute**: $\tilde{u} - u \in I$, or
- **relative**: $(\tilde{u} - u)/u \in I$. 
A Variety of Round-off Errors

The round-off error between \( \circ(\tilde{u} + \tilde{v}) \) and \( \tilde{u} + \tilde{v} \) is

- absolutely bounded if \( \tilde{u} \) and \( \tilde{v} \) are bounded,
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The round-off error between $\circ(\tilde{u} + \tilde{v})$ and $\tilde{u} + \tilde{v}$ is

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- relatively bounded for FP formats with gradual underflow,
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- absolutely bounded if \( \tilde{u} \) and \( \tilde{v} \) are bounded,
- relatively bounded for FP formats with \textit{gradual underflow},
- relatively bounded if \( \tilde{u} + \tilde{v} \) is far enough from 0,
- zero if \( \tilde{u} + \tilde{v} \) is in a suitable fixed-point format,
- zero if \( \tilde{u}/\tilde{v} \in [-2, -1/2] \) for FP formats with gradual underflow.
Interval Arithmetic

**Interval arithmetic** extends operations on real numbers to operations on closed **connected subsets** of real numbers.

**Application**

Instead of proving \( \forall x \in [a, b],\ f(x) \in [c, d] \),
you can prove \( F([a, b]) \subseteq [c, d] \),
assuming that \( F \) is an **interval extension** of \( f \).
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**Application**

Instead of proving $\forall x \in [a, b], \ f(x) \in [c, d]$, you can prove $F([a, b]) \subseteq [c, d]$, assuming that $F$ is an interval extension of $f$.

Evaluating $F$ is easy; it involves operations on bounds only:

$$x \in [a, b] \land y \in [c, d] \Rightarrow x + y \in [a + c, b + d].$$

This makes interval arithmetic suitable for automatically proving bounds on real-valued expressions.
Interval Arithmetic and Dependencies

**Independent expressions**

If \( a \in [3, 5] \) and \( b \in [1, 2] \) are independent, then

\[
a - b \in [3 - 2, 5 - 1] = [1, 4]
\]

is the optimal enclosure.
Interval Arithmetic and Dependencies

**Independent expressions**

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is the optimal enclosure.

**Correlated expressions**

If we have \( a \in [1, 100] \), interval arithmetic gives
\[
(a + \varepsilon) - a \in [1 + \varepsilon, 100 + \varepsilon] - [1, 100] = [-99 + \varepsilon, 99 + \varepsilon]
\]
while the optimal enclosure is \([\varepsilon, \varepsilon]\).
Various methods solve the dependency issue:

- octogons,
- ellipsoids,
- zonotopes,
- Taylor/Chebyshev models,
- decision procedures, e.g. simplex or CAD.

Unfortunately they are much costlier than interval arithmetic at execution time, and even worse at formalization time.
Leveraging Forward Error Analysis

Some well-known results of the standard model of FP arithmetic:

- “The absolute error of the sum is the sum of the absolute errors.”
  \[(\tilde{u} + \tilde{v}) - (u + v) = (\tilde{u} - u) + (\tilde{v} - v).\]
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- “The absolute error of the sum is the sum of the absolute errors.”
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- “The relative error of the product is the sum of the relative errors.”
  \[\frac{\tilde{u}\tilde{v}}{uv} - 1 = \varepsilon_u + \varepsilon_v + \varepsilon_u\varepsilon_v\]
  with \(\varepsilon_u = \frac{\tilde{u}}{u} - 1\) and \(\varepsilon_v = \frac{\tilde{v}}{v} - 1\).
Leveraging Forward Error Analysis

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  \[
  \frac{\tilde{u}\tilde{v}}{uv} - 1 = \varepsilon_u + \varepsilon_v + \varepsilon_u\varepsilon_v
  \]
  with \(\varepsilon_u = \tilde{u}/u - 1\) and \(\varepsilon_v = \tilde{v}/v - 1\).

- “The relative round-off error is bounded.”
  \[
  \left| \frac{\circ(u)}{u} - 1 \right| \leq 2^{-p} \text{ if } |u| \geq \ldots
  \]
Leveraging Forward Error Analysis

Rewriting system:

\[
\begin{align*}
(\tilde{u} + \tilde{v}) - (u + v) &\rightarrow (\tilde{u} - u) + (\tilde{v} - v) \\
(\tilde{u}\tilde{v})/(uv) - 1 &\rightarrow \varepsilon_u + \varepsilon_v + \varepsilon_u\varepsilon_v
\end{align*}
\]

Sufficient as long as

- errors are not correlated,
- expressions have the same inductive structure with correlated sub-expressions in the same places.

Because of the 2-step design/verification process, these hypotheses often hold.
Example 1: Toy Elementary Function

How to efficiently compute \( \sin x \) for \( |x| \leq 1 \) with a relative accuracy bounded by \( 103 \cdot 2^{-16} \)?
Example 1: Toy Elementary Function

How to **efficiently** compute $\sin x$ for $|x| \leq 1$ with a **relative accuracy** bounded by $103 \cdot 2^{-16}$?

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An actual implementation of sin would

- use more than just 2 polynomials, and/or
- perform an argument reduction.

But the proof process is the same!
Approximating a Mathematical Function

How to compute an accurate FP approximation of $g(x)$ for any $x$?
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How to compute an accurate FP approximation of $g(x)$ for any $x$?

1. Find an approximation $\hat{g}$ of $g$ that uses only real operations that can be approximated by your floating-point unit.

   Bound the method error $\hat{g}(x)/g(x) - 1$. 
Approximating a Mathematical Function

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1. Find an approximation $\hat{g}$ of $g$ that uses only real operations that can be approximated by your floating-point unit.
   Bound the method error $\hat{g}(x)/g(x) - 1$.

2. Write $\tilde{g}$ that implements $\hat{g}$ with floating-point operations.
   Bound the round-off error $\tilde{g}(x)/\hat{g}(x) - 1$.
Approximating a Mathematical Function

How to compute an **accurate** FP approximation of $g(x)$ for any $x$?

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Proving correctness is just a matter of computing tight bounds for these expressions.
Method Error (Relative)

Method error: \( \frac{x \cdot (1 - x^2 \cdot 10473 \cdot 2^{-16})}{\sin x} - 1 \).

Interval analysis knows how to bound such an expression.
Binary32 Round-off Error (Relative)

Round-off error: \[
\frac{o(x \cdot o(1 - o(x^2 \cdot 10473 \cdot 2^{-16})))}{x \cdot (1 - x^2 \cdot 10473 \cdot 2^{-16})} - 1.
\]

Gappa knows how to bound such an expression.
(And how to compose method and round-off errors.)
Gappa Script, as Written by a Human

**Example (Relative error for a toy sin implementation)**

```gappa
@rnd = float<ieee_32,ne>;
x = rnd(dummyx);  # x is a float

# floating-point implementation
y rnd= x * (1 - x*x * 0x28E9p-16);
# infinitely-precise computation
My = x * (1 - x*x * 0x28E9p-16);

{|x| in [1b-5,1] \/
 # relative method error
 |My -/ sin_x| <= 1.55e-3 ->

 # relative total error
 |y -/ sin_x| <= 1.551e-3 }
```
Outline

1. Introduction

2. Interval arithmetic and forward error analysis

3. Dealing with more intricate algorithms
   - Example 2: Cody-Waite argument reduction for exp
   - Example 3: Itanium division of 16-bit unsigned integers

4. The Gappa tool

5. Conclusion
Intricate Algorithms

For some algorithms, bounding errors is not sufficient, as they might rely on various tricks:

- exact computations,
- error compensations,
- convergent iterations,
- and so on.
Example 2: Cody-Waite Argument Reduction for Exp

**Goal:** compute $\exp x$ for $|x| \leq 800$.

**Argument reduction:** replace $x$ by a value close to 0, so that $\exp$ can be approximated by a small polynomial.
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- **Implementation:** evaluate $(x - k\ell_h) - k\ell_l$ with FP arithmetic.

$$
\exp x = 2^k \exp(o(\ldots)) \exp(\delta - k\varepsilon).
$$
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- **Implementation:** evaluate $(x - k\ell_h) - k\ell_l$ with FP arithmetic.

$$\exp x = 2^k \exp(\circ(\ldots)) \exp(\delta - k\varepsilon).$$

- **Issue:** how much is $\delta$?
Example 2: Cody-Waite Argument Reduction for Exp

Example (Cody-Waite argument reduction for exp, part 1)

Log2h = 0xb.17217f7d1cp-4; # 42 bits out of 53
Log2l = 0xf.79abc9e3b398p-48;
InvLog2 = 0x1.71547652b82fep0;
k = int<ne>(rnd(x*InvLog2));
t1 rnd = x - k*Log2h;
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Proof.

1 |x| ≤ 800, so |k| < 2048, so k fits on 11 bits.
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k = int<ne>(rnd(x*InvLog2));
t1 rnd= x - k*Log2h;

Proof.

1. \(|x| \leq 800\), so \(|k| < 2048\), so \(k\) fits on 11 bits.
2. \(\ell_h\) fits on 42 bits, so \(\circ(k\ell_h) = k\ell_h\).
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t1 rnd= x - k*Log2h;

Proof.

1. $|x| \leq 800$, so $|k| < 2048$, so $k$ fits on 11 bits.
2. $l_h$ fits on 42 bits, so $\circ(kl_h) = kl_h$.
3. $l_h^{-1} \approx \text{InvLog2}$, so $x \approx kl_h$. 
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k = int <ne> (rnd (x * InvLog2));
t1 rnd = x - k*Log2h;

Proof.

1. $|x| \leq 800$, so $|k| < 2048$, so $k$ fits on 11 bits.
2. $\ell_h$ fits on 42 bits, so $\circ (k\ell_h) = k\ell_h$.
3. $\ell_h^{-1} \approx \text{InvLog2}$, so $x \approx k\ell_h$.
4. So $t_1 = \circ (x - \circ (k\ell_h)) = x - k\ell_h$ by Sterbenz’ lemma.
Example 2: Cody-Waite Argument Reduction for Exp

@rnd = float<ieee_64,ne>;
x = rnd(dummyx);  # x is a double

# Cody-Waite argument reduction
Log2h = 0xb.17217f7d1cp-4;  # 42 bits out of 53
Log2l = 0xf.79abc9e3b398p-48;
InvLog2 = 0x1.71547652b82fep0;
k = int<ne>(rnd(x*InvLog2));
t1 rnd = x - k*Log2h;
t2 rnd = t1 - k*Log2l;

# exact values
T1 = x - k*Log2h;
T2 = T1 - k*Log2l;

{ |x| in [0.3, 800] ->
  t1 = T1 \/
  T1 in [-0.35,0.35] \/
  t2 - T2 in ? }
Example 3: Itanium Division of 16-bit Unsigned Integers

Intel Itanium processors have no hardware divisor. How to efficiently perform a division with just add and mul?
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Intel Itanium processors have no hardware divisor. How to efficiently perform a division with just add and mul?

\[
y_0 \approx 1/b \quad [\text{frcpa}]
q_0 = \circ(a \times y_0)
\]
\[
e_0 = \circ(1 + 2^{-17} - b \times y_0)
\]
\[
q_1 = \circ(e_0 \times q_0 + q_0)
\]
\[
q = \lfloor q_1 \rfloor
\]

with \(\circ(\cdot)\) rounding to nearest on the extended 82-bit format.

Correctness of the division

\[\forall a, b \in [1; 65535], \quad q = \lfloor a/b \rfloor.\]
Proof Sketch

**Theorem (Exclusion zones)**

Given a and b positive integers.
If $0 \leq a \times (q_1/(a/b) - 1) < 1$, then $\lfloor q_1 \rfloor = \lfloor a/b \rfloor$. 

Notice the relative error between the FP value $q_1$ and the real $a/b$. So proving the correctness is just a matter of bounding this error.
Proof Sketch

**Theorem (Exclusion zones)**

*Given* \(a\) and \(b\) positive integers. If \(0 \leq a \times \left(q_1/(a/b) - 1\right) < 1\), then \(\lfloor q_1 \rfloor = \lfloor a/b \rfloor\).

Notice the *relative error* between the FP value \(q_1\) and the real \(a/b\). So proving the correctness is just a matter of *bounding* this error.
Bounding the method error $\hat{q}_1 - a/b$ and the round-off error $q_1 - \hat{q}_1$ and composing them does not work at all. There is some error compensation.
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What the developers knew when designing the algorithm:

- If not for $2^{-17}$, the code would perform a Newton iteration:
  $\hat{q}_1/(a/b) - 1 = -\varepsilon_0^2$ with $\varepsilon_0 = y_0/(1/b) - 1$.
- By taking $2^{-17}$ into account,
  $\hat{q}_1/(a/b) - 1 = -\varepsilon_0^2 + (1 + \varepsilon_0) \cdot 2^{-17}$. 
Example (Division of 16-bit unsigned integers on Itanium)

@\texttt{rnd} = float<\texttt{x86}_80,\texttt{ne}>;

\begin{verbatim}
# algorithm with no rounding operators
q0 = a * y0;
e0 = 1 + \texttt{b}-17 - b * y0;
q1 = q0 + e0 * q0;

# notations for relative errors
eps0 = (y0 - 1 / b) / (1 / b);
err = (q1 - a / b) / (a / b);

{ # a and b are integers
  @\texttt{FIX}(a, 0) /\ a \texttt{ in } [1,65535] /\n  @\texttt{FIX}(b, 0) /\ b \texttt{ in } [1,65535] /\n  # specification of frcpa
  @\texttt{FLT}(y0, 11) /\ |\texttt{eps0}| \leq 0.00211373 /\n  # Newton's iteration, almost
  err = -(\texttt{eps0} * \texttt{eps0}) + (1 + \texttt{eps0}) * \texttt{b}-17 ->

  # the separation hypothesis is satisfied
  err \texttt{ in } [0,1] /\ a \texttt{ * } err \texttt{ in } [0,0.99999] /\n  # all the computations are exact
  \texttt{rnd}(q0) = q0 /\ \texttt{rnd}(e0) = e0 /\ \texttt{rnd}(q1) = q1

# try harder!
\texttt{rnd}(q1) = q1 $ 1 / b;
\end{verbatim}
Lemma div_u16_spec : \forall a b, 
  (1 <= a <= 65535) \% \mathbb{Z} \Rightarrow (1 <= b <= 65535) \% \mathbb{Z} \Rightarrow 
  \text{div}_u16\ a\ b = (a / b)\%\mathbb{Z}.

Proof.
intros a b Ba Bb.
apply Zfloor_imp.
cut (0 <= b * q1 - a < 1).
lra.
set (err := (q1 - a / b) / (a / b)).
replace (b * q1 - a) with (a * err) by field.
set (y0 := frcpa b).
set (Mq0 := a * y0 + 0).
set (Me0 := 1 + pow2 (-17) - b * y0).
set (Mq1 := Me0 * Mq0 + Mq0).
set (eps0 := (y0 - 1 / b) / (1 / b)).
assert (((Mq1 - a / b) / (a / b) = 
  -(eps0 * eps0) + (1 + eps0) * pow2 (-17)) by field.
  generalize (frcpa_spec b) (\text{FIX}_\text{format}_\text{Z2R} \text{radix}2\ a) 
  (\text{FIX}_\text{format}_\text{Z2R} \text{radix}2\ b).
gappa.
Qed.
Outline

1. Introduction
2. Interval arithmetic and forward error analysis
3. Dealing with more intricate algorithms
4. The Gappa tool
   - Supported properties
   - Proof process
   - Theorem database
5. Conclusion
A Few Words About Gappa

Starting from a formula, Gappa saturates a set of theorems to infer new properties until it encounters a contradiction.
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Supported properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Formulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>BND(x, I)</td>
<td>x ∈ I</td>
</tr>
<tr>
<td>ABS(x, I)</td>
<td></td>
</tr>
<tr>
<td>REL(x, y, I)</td>
<td>∃ε ∈ I, x = y · (1 + ε)</td>
</tr>
<tr>
<td>FIX(x, e)</td>
<td>∃m ∈ ℤ, x = m · 2^e</td>
</tr>
<tr>
<td>FLT(x, p)</td>
<td>∃m, e ∈ ℤ, x = m · 2^e ∧</td>
</tr>
<tr>
<td>NZR(x)</td>
<td>x ≠ 0</td>
</tr>
<tr>
<td>EQL(x, y)</td>
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### Supported properties

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<td>(</td>
</tr>
<tr>
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</tr>
<tr>
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To prove \texttt{div\_u16}, Gappa tried to apply 17k theorems. The final proof infers \~80 properties.
Proof Process

Given a logical formula about some expressions $e_1, \ldots, e_n$, Gappa performs the following steps:
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4. Generate a **formal proof** from the trace.
Naive interval arithmetic:

\[ u \in [u, \bar{u}] \land v \in [v, \bar{v}] \Rightarrow u + v \in [u + v, \bar{u} + \bar{v}]. \]
Theorem Database

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Floating- and fixed-point arithmetic properties:

\[ u \in 2^{-1074} \cdot \mathbb{Z} \Rightarrow \exists \varepsilon \in [-2^{-53}, 2^{-53}], \circ(u) = u \times (1 + \varepsilon). \]
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- Precision handling:
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# Theorem Database

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<tbody>
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<td>Interval arithmetic</td>
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<td>Miscellaneous</td>
<td>27</td>
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<td><strong>Total</strong></td>
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   - Example 4: Knuth’ TwoSum Algorithm
   - Conclusion
AltErgo + Gappa
Joint work with Sylvain Conchon and Cody Roux

Motivations

- Make it possible to verify programs that handle more than just FP arithmetic, e.g. arrays.
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**Implementation features**

- Incremental backward reasoning.
- Pattern-matching modulo equality.
Example 4: Knuth’ TwoSum Algorithm

\[
\begin{align*}
  s &= a + b \\
  t &= s - a \\
  e &= (a - (s - t)) + (b - t)
\end{align*}
\]

Assuming no overflow occurs, \( s + e = a + b \).
Example 4: Knuth’ TwoSum Algorithm

\[ s = a + b \]
\[ t = s - a \]
\[ e = (a - (s - t)) + (b - t) \]

Assuming no overflow occurs, \( s + e = a + b \).

This example is out of the reach of Gappa. Yet its bounded instance is in the decidable fragment of FP arithmetic.
Conclusion

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And it generates formal proofs!
Questions?

Gappa:  http://gappa.gforge.inria.fr/