Formal Verification for Numerical Computations, and the Other Way Around

Guillaume Melquiond

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A Bit of Vocabulary

**Computer arithmetic**

Art of representing *numbers* and performing *computations* on them in a mechanized fashion:

- integer arithmetic,
- fixed- and floating-point arithmetic.
A Bit of Vocabulary

**Computer arithmetic**

Art of representing *numbers* and performing *computations* on them in a mechanized fashion:

- integer arithmetic,
- fixed- and floating-point arithmetic.

**Formal verification**

Art of proving the *correctness* of a system with respect to a formal *specification*, using *mathematical* methods.

**Formal proof**: well-formed formulas related by inference rules.
Position of \( p_1 \) with respect to \( (p_2p_3) \)

\[
\text{sign} \left( (x_2 - x_1) \cdot (y_3 - y_1) - (y_2 - y_1) \cdot (x_3 - x_1) \right)
\]
Position of $p_1$ with respect to $(p_2p_3)$

$$\text{sign} \left( \frac{(x_2 - x_1) \cdot (y_3 - y_1)}{-(y_2 - y_1) \cdot (x_3 - x_1)} \right)$$
Orientation Detection

Position of $p_1$ with respect to $(p_2p_3)$

$$\text{sign} \left( \circ(\circ(x_2 - x_1) \cdot \circ(y_3 - y_1)) \right)$$

with $\circ(r) \in \mathbb{F}$ the closest floating-point number to $r \in \mathbb{R}$. 
Orientation Detection

Position of $p_1$ with respect to $(p_2 p_3)$

$$\text{sign} \left( \circ (\circ (x_2 - x_1) \cdot \circ (y_3 - y_1)) - \circ (\circ (y_2 - y_1) \cdot \circ (x_3 - x_1)) \right)$$

with $\circ (r) \in \mathbb{F}$ the closest floating-point number to $r \in \mathbb{R}$. 
Formal Verification for Numerical Computations

**Situation**

- Number representations cause subtle behaviors, if not counter-intuitive ones.
- Algorithms are more and more intricate, due to speed and space concerns.

How much can formal verification help?
Numerical Integrals in Modern Math Proofs

Double bubbles minimize (Hass, Schlafly, 2000)

“The proof parameterizes the space of possible solutions by a two-dimensional rectangle [...]. This rectangle is subdivided into 15,016 smaller rectangles which are investigated by calculations involving a total of 51,256 numerical integrals.”

∫\(\int_{-\infty}^{\infty} \left(0.5 \cdot \log(\tau^2 + 2.25) + \frac{4.1396}{\tau^2} + \log(\pi)\right) d\tau\)

“We compute the last integral numerically (from \(-10^5\) to \(10^5\)).”
Numerical Integrals in Modern Math Proofs

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Major arcs for Goldbach’s problem (Helfgott, 2013)

(Every odd number \(\geq 7\) is the sum of three prime numbers.)

\[
\int_{-\infty}^{\infty} \frac{(0.5 \cdot \log(\tau^2 + 2.25) + 4.1396 + \log \pi)^2}{0.25 + \tau^2} d\tau
\]

“We compute the last integral \textit{numerically} (from \(-10^5\) to \(10^5\)).”
Rigorous numerical integration

I need to evaluate some (one-variable) integrals that neither SAGE nor Mathematica can do symbolically. As far as I can tell, I have two options:

(a) Use GSL (via SAGE), Maxima or Mathematica to do numerical integration. This is really a non-option, since, if I understand correctly, the "error bound" they give is not really a guarantee.

(b) Cobble together my own programs using the trapezoidal rule, Simpson's rule, etc., and get rigorous error bounds using bounds I have for the second (or fourth, or what have you) derivative of the function I am integrating. This is what I have been doing.

Is there a third option? Is there standard software that does (b) for me?

na.numerical-analysis
Modern mathematical proofs rely more and more on computers, by sheer necessity.

There is a lingering doubt:
what if one of the computations went wrong?

Can proofs that rely on numerical computations be redeemed?
1. Introduction

2. Numerical computations for formal verification: Guaranteed quadrature using Coq

3. Formal verification for numerical computations: Getting a verified GMP using Why3

4. Conclusion
Some Relevant Tools

Coq

formal system, rich and expressive, but proofs are tedious
Some Relevant Tools

- **Coq**: automatic prover for floating-point properties
- **Gappa**: formal prover for floating-point properties
Some Relevant Tools

Coq can ask Gappa for a formal proof
Some Relevant Tools

- **Why3**: Platform for deductive program verification, depends on external provers.
- **Coq**: Formal verification tool.
- **Gappa**: Library for numerical computations.
Some Relevant Tools

Gappa is one of the many automated solvers supported by Why3
Some Relevant Tools

Why3’s standard library is verified with Coq; users can use Coq as a last resort.
Outline

1 Introduction

2 Numerical computations for formal verification:
   Guaranteed quadrature using Coq
   - Some relevant Coq libraries
   - Example: a proper definite integral
   - Interval arithmetic
   - Helfgott’s integrals using CoqInterval

3 Formal verification for numerical computations:
   Getting a verified GMP using Why3

4 Conclusion
Numerical Computations for Formal Verification

Objective

Can we produce a convincing proof of

\[
\int_{-\infty}^{\infty} \frac{(0.5 \cdot \log(\tau^2 + 2.25) + 4.1396 + \log \pi)^2}{0.25 + \tau^2} d\tau \leq 226.844
\]
Some Relevant Coq Libraries

Coq’s standard library (Mayero, Desmettre)

Reals
Some Relevant Coq Libraries

- Flocq: formalization of floating-point arithmetic (Boldo, M.)
- Reals
Some Relevant Coq Libraries

helper libraries and realizations to interface the tools with Coq (M.)

- Reals
- Flocq
- Gappa
- Why3
Some Relevant Coq Libraries

IEEE-754 semantics preservation for a C compiler (Boldo, Jourdan, Leroy, M.)
Some Relevant Coq Libraries

- **Reals**
- **Flocq**
  - Gappa
  - Why3
  - CompCert
- **Coquelicot**

user-friendly formalization of real analysis (Boldo, Lelay, M.)
Some Relevant Coq Libraries

- Reals
- Flocq
- Coquelicot
- Gappa
- Why3
- CompCert
- CoqInterval

interval-based tactics for formally verifying bounds (M., Martin-Dorel, Mayero, Pasca, Rideau, Théry, Mahboubi, Sibut-Pinote)
Some Relevant Coq Libraries

- Reals
  - Flocq
    - Gappa
    - Why3
    - CompCert
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    - CoqInterval
Example: a Proper Definite Integral

Example (Mathematics and Coq script)

\[
\int_0^1 \tan t^2 dt \leq \frac{2}{5}.
\]
Example: a Proper Definite Integral

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\[ \int_{0}^{1} \tan t^2 dt \leq \frac{2}{5}. \]

Goal RInt (fun t => tan (t*t)) 0 1 <= 2/5.
Proof. interval. Qed.
Example: a Proper Definite Integral

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\int_0^1 \tan^2 t \, dt \leq \frac{2}{5}.
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Goal \( \text{RInt (fun t => tan (t*t)) 0 1 <= 2/5.} \)

Proof. interval. Qed.

integral from Coquelicot
tactic from CoqInterval

Formal proof is instant!
Interval Arithmetic

**Definition (Intervals)**

Connected closed subsets of $\mathbb{R}$, e.g., $x = [\underline{x}; \bar{x}]$ or $[\underline{x}; +\infty)$. 
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**Definition (Interval extension)**

$f : \mathbb{I}^n \rightarrow \mathbb{I}$ is an interval extension of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ if

\[
\forall x_1, \ldots, x_n \in \mathbb{I}, \ \forall x_1, \ldots, x_n \in \mathbb{R}, \quad x_1 \in x_1 \land \ldots \land x_n \in x_n \Rightarrow f(x_1, \ldots, x_n) \in f(x_1, \ldots, x_n).
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Interval Arithmetic

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$$\forall x_1, \ldots, x_n \in \mathbb{I}, \forall x_1, \ldots, x_n \in \mathbb{R},$$

$$x_1 \in x_1 \land \ldots \land x_n \in x_n \Rightarrow f(x_1, \ldots, x_n) \in f(x_1, \ldots, x_n).$$

**Lemma (Naive quadrature)**

$$\int_u^v f(t) \, dt \in (v - u) \cdot f(\text{hull}(u, v)).$$
Effective Interval Extensions

Definition (Floating-point numbers and directed rounding)

\[ \mathbb{F} = \{ m \cdot \beta^e \in \mathbb{R} \mid m, e \in \mathbb{Z} \land |m| < \beta^0 \}. \]

\[ \forall u, v \in \mathbb{F}, \quad \nabla (u \diamond v) \leq u \diamond v \leq \Delta (u \diamond v). \]
Effective Interval Extensions

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\[ F = \{ m \cdot \beta^e \in \mathbb{R} \mid m, e \in \mathbb{Z} \land |m| < \beta^0 \}. \]

\[ \forall u, v \in F, \quad \nabla (u \odot v) \leq u \odot v \leq \triangle (u \odot v). \]

Interval extensions of +, −, ×

If \( u \in u = [\underline{u}; \overline{u}] \) and \( v \in v = [\underline{v}; \overline{v}] \), then (by monotony)

\[ u + v \in [\nabla (u + v); \triangle (u + v)] \overset{\text{def}}{=} u + v, \]

\[ u - v \in [\nabla (u - v); \triangle (u - v)] \overset{\text{def}}{=} u - v, \]

\[ u \cdot v \in [\min (\nabla (u \cdot v), \nabla (u \cdot v), \nabla (u \cdot v)), \max (\triangle (u \cdot v), \triangle (u \cdot v), \triangle (u \cdot v), \triangle (u \cdot v))]. \]
Quadrature

\[ \int_0^1 \tan t^2 \, dt \leq 1.557 \]
Quadrature

\[ \int_0^1 \tan t^2 \, dt \leq 0.482 \]
Formally Reliable Computations

What is needed?

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Formally Reliable Computations

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1. Formalize floating-point rounding and verify basic arithmetic operators.
2. Verify basic interval operators and automatic differentiation.
3. Formalize real analysis: power series, integrals, etc.
4. Verify interval extensions of elementary functions.
5. Support rigorous polynomial approximations based on Taylor models.
Formally Reliable Computations

What is needed?

1. Formulate floating-point rounding and verify basic arithmetic operators.

2. Verify basic interval operators and automatic differentiation.

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5. Support rigorous polynomial approximations based on Taylor models.

6. Verify quadrature algorithms for proper and improper integrals.
Formally Reliable Computations

What is needed?

1. Formalize floating-point rounding and verify basic arithmetic operators. [ARITH’11]
2. Verify basic interval operators and automatic differentiation. [IJCAR’08]
3. Formalize real analysis: power series, integrals, etc. [CPP’12, MCS’15]
4. Verify interval extensions of elementary functions. [RNC’08, I&C’12]
5. Support rigorous polynomial approximations based on Taylor models. [JAR’16]
6. Verify quadrature algorithms for proper and improper integrals. [ITP’16, JAR’18]
Details: Improving Enclosures

Definition (Polynomial enclosure)

\[(\vec{p}, \Delta) \in \mathbb{I}^{n+1} \times \mathbb{I} \text{ encloses } f \text{ over } x \ni x_0, \text{ if } \exists p \in \mathbb{R}[X], \]

\[\left(\forall i < n, \ p_i \in p_i \right) \land \forall x \in x, \ f(x) - p(x - x_0) \in \Delta.\]
Definition (Polynomial enclosure)

\((\vec{p}, \Delta) \in \mathbb{I}^{n+1} \times \mathbb{I} \) encloses \( f \) over \( x \ni x_0 \), if \( \exists p \in \mathbb{R}[X], \)
\[
(\forall i < n, \ p_i \in p_i) \land \forall x \in x, \ f(x) - p(x - x_0) \in \Delta.
\]

Lemma (Improved quadrature)

If \((p, \Delta)\) encloses \( f \) over \([u; v]\), and if \( P \) is a primitive of \( p \), then
\[
\int_{u}^{v} f \in P(v) - P(u) + (v - u) \cdot \Delta.
\]
Quadrature with a Degree-3 Polynomial

\[ p(t) + \Delta \]

\[ \tan(t^2) \]

\[ \int_0^1 \tan t^2 \, dt \leq 0.681 \]
Helfgott’s Integrals

Proper integral, in the MathOverflow post

\[
\int_0^1 \left| (x^4 + 10x^3 + 19x^2 - 6x - 6) e^x \right| \, dx \simeq 11.14731055.
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INTLAB got this integral wrong.

VNODE/LP cannot compute it.
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**Improper integral, in the proof**

\[
\int_{-\infty}^{+\infty} \frac{(0.5 \cdot \log(\tau^2 + 2.25) + 4.1396 + \log \pi)^2}{0.25 + \tau^2} \, d\tau \in [226.849; 226.850].
\]
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\in [226.849; 226.850].
\]

Helfgott was using \( \leq 226.844 \).
Outline

1. Introduction

2. Numerical computations for formal verification: Guaranteed quadrature using Coq

3. Formal verification for numerical computations: Getting a verified GMP using Why3
   - Using Why3
   - Example: decrementing a long integer
   - Square root and fixed-point arithmetic
   - Error analysis using Gappa

4. Conclusion
Arbitrary-Precision Integer Arithmetic

The GNU Multiple Precision arithmetic library (GMP)

- Free software, widely used.
- State-of-the-art algorithms, unmatched performances.
Arbitrary-Precision Integer Arithmetic

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- Highly intricate algorithms written in low-level C and ASM.
- Ill-suited for random testing.

GMP 5.0.4: “Two bugs in multiplication [...] with extremely low probability [...] Two bugs in the gcd code [...] For uniformly distributed random operands, the likelihood is infinitesimally small.”
Arbitrary-Precision Integer Arithmetic

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Objectives
- Produce a verified library compatible with GMP.
- Attain performances comparable to a no-assembly GMP.
- Focus on the low-level mpn layer.
The Why3 Workflow

GMP library
The Why3 Workflow

- Rieu-Helft
- GMP library
- WhyML library
- Specification

Why3 Workflow Diagram:

- Rieu-Helft
- GMP library
- WhyML library
- Specification

The Why3 Workflow involves the following steps:

1. Specification
2. WhyML library
3. GMP library
4. Rieu-Helft

The workflow starts with a specification, which is translated into WhyML. The WhyML library is then used to formalize the code, which is further refined using the GMP library. Finally, the Rieu-Helft framework is applied to verify the correctness of the implementation.
The Why3 Workflow

Marché, M., Rieu-Helft

GMP library

Specification

C memory model

WhyML library

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Formal Verification for Numerical Computations, and the Other Way Around
The Why3 Workflow

C memory model \rightarrow WhyML library \rightarrow GMP library \rightarrow Specification

Filliâtre, Marché, M., Paskevich, and many others

Why3 \rightarrow Verification conditions
The Why3 Workflow

- **GMP library**
- **Specification**
- **WhyML library**
- **Why3**
- **Verification conditions**
- **SMT solvers**

- **C memory model**
The Why3 Workflow

- **GMP library**
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- GMP library
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The Why3 Workflow

- **GMP library**
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  - WhyML library
  - C memory model
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  - Verification conditions
    - SMT solvers
      - Coq
      - Gappa
  - Verified C library
Example: Subtracting 1 to a Long Integer

Original macro (simplified from 18-line `mpn_decr_u`)

```c
#define mpn_decr_1(p)  \
    mp_ptr __p = (p);  \
    while (*((__p++)-- == 0) ;
```

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Formal Verification for Numerical Computations, and the Other Way Around

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#define mpn_decr_1(p) \
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```

Conversion to WhyML

```whyml
let wmpn_decr_1 (p: ptr uint64) : unit =
  let ref lp = 0 in
  let ref i = 0 in
  while lp = 0 do
    lp <- get_ofs p i;
    set_ofs p i (sub_mod lp 1);
    i <- i + 1;
  done
```
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Conversion to WhyML

```whyml
let wmpn_decr_1 (p: ptr uint64) : unit
  ensures { value p = value (old p) - 1 }
  =
    let ref lp = 0 in
    let ref i = 0 in
    while lp = 0 do
      lp <- get_ofs p i;
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```

Conversion to WhyML

```whyml
let wmpn_decr_1 (p: ptr uint64) (ghost sz: int32) : unit 
  ensures { value p sz = value (old p) sz - 1 }
  = 
  let ref lp = 0 in 
  let ref i = 0 in 
  while lp = 0 do 
    lp <- get_ofs p i; 
    set_ofs p i (sub_mod lp 1); 
    i <- i + 1; 
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Conversion to WhyML

```whyml
let wmpn_decr_1 (p: ptr uint64) (ghost sz: int32) : unit
    requires { valid p sz }
    requires { 1 <= value p sz }
    ensures { value p sz = value (old p) sz - 1 }
= 
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```

Extraction to C

```c
void wmpn_decr_1(uint64_t * p) {
    uint64_t lp; int32_t i;
    lp = 0;
    i = 0;
    while (lp == 0) {
        lp = p[i];
        p[i] = lp - 1;
        i = i + 1;
    }
}
```
Example: 64-bit Square Root

Original GMP code

```c
mp_limb_t mpn_sqrtrem1(mp_ptr rp, mp_limb_t a0) {
    mp_limb_t a1, x0, t2, t, x2;
    unsigned abits = a0 >> (GMP_LIMB_BITS - 1 - 8);
    x0 = 0x100 | invsqrttab[abits - 0x80];
    /* x0 is now an 8 bits approximation of 1/sqrt(a0) */
    a1 = a0 >> (GMP_LIMB_BITS - 1 - 32);
    t = (mp_limb_signed_t) (CNST_LIMB(0x2000000000000000) - 0x30000 - a1 * x0 * x0) >> 16;
    x0 = (x0<<16) + ((mp_limb_signed_t)(x0*t) >> (16+2));
    /* x0 is now a 16 bits approximation of 1/sqrt(a0) */
    ...
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- Table lookup, **Newton iteration** toward $1/\sqrt{a}$,
- modified Newton iteration toward $a/\sqrt{a}$, correcting step.
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    ...
```

- Table lookup, **Newton iteration** toward \(1/\sqrt{a}\), modified Newton iteration toward \(a/\sqrt{a}\), correcting step.
- Hand-coded **fixed-point** arithmetic.
A Small Fixed-Point Arithmetic Theory for Why3

```plaintext
type fxp = { ival: uint64;
  ghost rval: real; ghost iexp: int }

invariant { rval = floor_at rval iexp }

invariant { ival = mod (floor (rval *. pow2 (-iexp)))
  (uint64'maxInt + 1) }
```
A Small Fixed-Point Arithmetic Theory for Why3

type fxp = { ival: uint64;
    ghost rval: real; ghost iexp: int }

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    (uint64't-maxInt + 1) }

Converted WhyML code

let sqrt1 (rp: ptr uint64) (a0: uint64): uint64 =
    let a = fxp_init a0 (-64) in
    let x0 = rsa_estimate a in
    let a1 = fxp_lsr a 31 in
    let m1 = fxp_sub (fxp_init 0x2000000000000000 (-49))
        (fxp_init 0x30000 (-49)) in
    let t1' = fxp_sub m1 (fxp_mul (fxp_mul x0 x0) a1) in
    let t1 = fxp_asr t1' 16 in
    let x1 = fxp_add (fxp_lsl x0 16)
        (fxp_asr' (fxp_mul x0 t1) 18 1) in
    ...

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Formal Verification for Numerical Computations, and the Other Way Around
Newton iteration toward $1/\sqrt{a}$

- Recurrence: $x_{i+1} = x_i + x_i \cdot (1 - a \cdot x_i^2)/2$.
- Relative error: $x_i = a^{-1/2} \cdot (1 + \varepsilon_i)$.
- Quadratic convergence: $|\varepsilon_{i+1}| \leq \frac{3}{2} |\varepsilon_i|^2$.

But what the code actually computes is

$$\tilde{x}_1 = \tilde{x}_0 + \nabla - \frac{33}{2} \left( 1 - 3 \cdot \nabla - \frac{33}{2} \cdot (a \cdot \tilde{x}_0) \cdot \tilde{x}_2 \right)/2,$$

with $\nabla_k(r) = \lfloor r \cdot 2^{-k} \rfloor \cdot 2^k$.

What to do about...
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$$\tilde{x}_1 = \tilde{x}_0 + \nabla_{-24} (\tilde{x}_0 \cdot \nabla_{-33} (1 - 3 \cdot 2^{-33} - \nabla_{-33} (a) \cdot \tilde{x}_0^2) / 2),$$

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Error Analysis

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- Rounding errors? That is what Gappa is designed to handle.
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What to do about...

- Rounding errors? That is what Gappa is designed to handle.
- Magical constants? Critical for soundness; hinted to Gappa.
## Gappa, in a Nutshell

### Proof search by saturating over ~ 200 theorems

- **Interval arithmetic:** \( \forall u, v \in \mathbb{R}, \forall u, v, w \in \mathbb{I}, \)

  \[ u \in u \land v \in v \land u + v \subseteq w \Rightarrow u + v \in w; \]
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  \[ u \in u \land v \in v \land u + v \subseteq w \Rightarrow u + v \in w; \]

- **Rounding properties:**  \( \forall x \in \mathbb{R}, \forall k \in \mathbb{Z}, \forall e \in \mathbb{I}, \)
  \[ [−2^k; 0] \subseteq e \Rightarrow \nabla_k(x) − x \in e; \]
Gappa, in a Nutshell

Proof search by saturating over \( \sim 200 \) theorems

- Interval arithmetic: \( \forall u, v \in \mathbb{R}, \forall u, v, w \in \mathbb{I}, u \in u \land v \in v \land u + v \subseteq w \Rightarrow u + v \in w; \)

- Rounding properties: \( \forall x \in \mathbb{R}, \forall k \in \mathbb{Z}, \forall e \in \mathbb{I}, [-2^k; 0] \subseteq e \Rightarrow \nabla_k(x) - x \in e; \)

- Error analysis: \( \forall u, v, \tilde{u}, \tilde{v} \in \mathbb{R}, \tilde{u} \cdot \tilde{v} - u \cdot v = (\tilde{u} - u) \cdot v + u \cdot (\tilde{v} - v) + (\tilde{u} - u) \cdot (\tilde{v} - v). \)
What if Gappa Fails?

Help it by providing equalities

```coq
let sqrt1 (rp: ptr uint64) (a0: uint64): uint64
  ensures {result*result <= a0 < (result+1)*(result+1)}
  =
...
let ghost rsa = pure { 1. /. sqrt a } in
let ghost e0 = pure { (x0 -. rsa) /. rsa } in
let ghost ea1 = pure { (a1 -. a) /. a } in
let ghost mx1 = pure { x0 +. x0 *. t1' *. 0.5 } in
assert { (mx1 -. rsa) /. rsa =
  -0.5 *. (e0*.e0 *. (3.+e0) +. (1.+e0) *
    (1. -. m1 +. (1.+e0)*(1.+e0) *. ea1)) };
...
```
What if Gappa Fails?

Help it by providing equalities

```plaintext
let sqrt1 (rp: ptr uint64) (a0: uint64): uint64
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  assert \{(mx1 -. rsa) /. rsa =
    -0.5 *. (e0*.e0 *. (3.+e0) +. (1.+e0) *. (1. -. m1 +. (1.+e0)*.(1.+e0) *. ea1)) \};
  ...
```

Four equalities are needed by **Gappa**:

- they hardly mention rounding errors;
- all of them are proved with straightforward **Coq** scripts.
A Verified Library

Supported operations

- Addition, subtraction, comparison.
- Multiplication: quadratic, and Toom-Cook 2 and 2.5.
- Division: “schoolbook”.
- Square root: divide-and-conquer.
Outline

1. Introduction

2. Numerical computations for formal verification: Guaranteed quadrature using Coq

3. Formal verification for numerical computations: Getting a verified GMP using Why3

4. Conclusion
Contributions

Floating-point arithmetic

- Formalization of high-level properties: rounding to odd, FMA, successor, etc.
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**Miscellaneous**
- Formalization of real analysis, with a focus on usability.
- Verification of a 3-point scheme for the wave equation.
- Safety invariant computation for hybrid systems.
- Procedure for linear integer arithmetic.
- Disproval of Masser-Gramain’s conjecture: $\sum_k 1/(\pi r_k^2) = \ldots$
Handbook of Floating-Point Arithmetic, Muller et al., 2010, 2018.

Proof assistants and mathematics

Make the Coq environment as friendly to use as computer algebra systems.
What is Next?

Proof assistants and mathematics
Make the Coq environment as friendly to use as computer algebra systems.

Deductive program verification for the masses
Make Why3 and related tools a natural choice to verify algorithms and libraries, especially when it comes to floating-point code.
Question Time
Details: Computing tan

Approximation of \( \tan x \) for \( x \in [-\frac{1}{2}; \frac{1}{2}] \)

1. Formally prove \( \tan x = \frac{\sin x}{\sqrt{1 - \sin^2 x}} \),
2. and \( 0 \leq (-1)^n \left( \frac{\sin x}{x} - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k}}{(2k + 1)!} \right) \leq \frac{x^{2n}}{(2n + 1)!} \).
3. Implement \( \tan : \mathbb{F} \to \mathbb{I} \) using extensions of +, −, ×, ÷, \( \sqrt{\cdot} \).
4. For \( x \not\in [-\frac{1}{2}; \frac{1}{2}] \), prove and implement an argument reduction.

Interval extension of \( \tan \)

\[ \forall x \in [\underline{x}; \overline{x}] \subset (-\frac{\pi}{2}; \frac{\pi}{2}), \quad \tan x \in \text{hull}(\tan \underline{x}, \tan \overline{x}). \]
Polynomial approximation of tan (also known as Taylor model)

1. Formally prove $\forall x \in x, \exists \xi \in x$, 
   $$
   \tan x = \sum_{k=0}^{n} \frac{\tan^{(k)} x_0}{k!} (x - x_0)^k + \frac{\tan^{(n+1)} \xi}{(n+1)!} (x - x_0)^{n+1},
   $$

2. and $\frac{\tan^{(k)} u}{k!} = q_k(\tan u)$ with $q_{i+1} = \frac{1}{i+1}(q'_i + X^2 q'_i) \in \mathbb{R}[X]$.

3. Implement the enclosure $(\vec{p}, \Delta)$ of tan over $x$ as $\vec{p} = (q_0(\tan x_0), \ldots, q_n(\tan x_0))$ and $\Delta = q_{n+1}(\tan x) \ldots$ using the interval extensions of $q_i$ and tan.

Remark

Computing a Taylor model is a lot more efficient when the function has a linear differential equation, e.g., exp, sin, etc.
Automating the Error Analysis

**Objective**

Formally prove that,
if \( X_0 = \tilde{x}_0 \cdot 2^{-8} \) approximates \( rsa = 1/\sqrt{a} \) with 8 bits of accuracy,
then \( X_1 = \tilde{x}_1 \cdot 2^{-64} \) approximates \( rsa \) with 16 bits of accuracy.

**Gappa script**

\[
\begin{align*}
X_1 &= X_0 + \text{fixed}<-24,\text{dn}>(\text{fixed}<-33,\text{dn}>(1 - 3\cdot 33 - X_0 - rsa) / X_0)) * 0.5); \\
rsa &= 1 / \text{sqrt}(a); \\
\{ &a \text{ in } [0.25,1] \land |(X_0 - rsa) / X_0| <= 1b-8 -> \} \\
(X_1 - rsa) / X_1 \text{ in } ? \}
\end{align*}
\]

- Implicit quantification on free variables: \( \forall a, X_0 \in \mathbb{R} \).
- Arithmetic operators on real numbers.
- **Rounding operators**: \( \text{fixed}<k,\text{dn}>(x) = [x \cdot 2^{-k}] \cdot 2^k \).
Reflection for Why3, in a Nutshell

When everything else has failed...

If the current verification condition is $Q(y)$, and if one has already verified the following WhyML function

```whyml
let prove (x: t): bool
  requires { P x }
  ensures { result = True -> Q x }
= ...
```

then one just needs to prove that $P(y)$ holds and to check that $(prove y)$ evaluates to True.

Remarks

- `prove` can make use of effects: mutability, exceptions, etc.
- If the current VC is not of the form $Q(y)$, Why3 uses reification to guess a suitable value for $x$. 