

Decompositions of packed words and self duality of Word Quasisymmetric Functions

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Abstract. By Foissy's work, the bidendriform structure of the Word Quasisymmetric Functions Hopf algebra (WQSym) implies that it is isomorphic to its dual. However, the only known explicit isomorphism due to Vargas does not respect the bidendriform structure. This structure is entirely determined by so-called totally primitive elements (elements such that the two half-coproducts vanish). In this paper, we construct two bases indexed by two new combinatorial families called red (dual side) and blue (primal side) biplane forests in bijection with packed words. In those bases, primitive elements are indexed by biplane trees and totally primitive elements by a certain subset of trees. We carefully combine red and blue forests to get bicolored forests. A simple recoloring of the edges allows us to obtain the first explicit bidendriform automorphism of WQSym.

Résumé. Grâce aux travaux de Foissy, on sait que l'algèbre de Hopf WQSym est isomorphe à sa duale car bidendriforme. Cependant, le seul isomorphisme explicite connu (dû à Vargas) ne respecte pas la structure bidendriforme. Cette structure est entièrement déterminée par les éléments totalement primitifs (annulés par les demi-co-produits). Dans ce papier, nous construisons deux bases indexées par deux nouvelles familles combinatoire appelées forêts biplanes rouges (coté duale) et bleues (coté primale), en bijection avec les mots tassés. Dans ces bases, les éléments primitifs sont indexés par les arbres et les totalement primitifs par un certain sous-ensemble d'arbres. Nous combinons soigneusement les forêts rouges et bleues pour obtenir des forêts bicolores. Une simple recoloration des arêtes nous permet d'obtenir le premier automorphisme bidendriforme explicite de WQSym.

Keywords: bidendriform Hopf algebras, Word Quasisymmetric Functions, packed words, permutation, primitive elements, duality, tree, forest, global descents

Introduction

Combinatorial Hopf algebras are a common meeting point of different communities. On one hand, the operad theory allows to construct them frequently since the free algebra on one generator admits a Hopf structure. For many operads, one can make the

structure explicit using combinatorics, one of the most basic example being the free dendriform algebra on one generator realized as the Loday-Ronco hopf algebra of binary trees [LR98].

On the other hand, the theory of symmetric functions often proceeds through non-commutative lifting to better understand the identities. Hence, the community introduced a series of larger and larger Hopf algebras over a large variety of combinatorial structures. One of the first step was the introduction of the dual pair of quasi-symmetric functions and non-commutative symmetric functions [Ges83, GKL⁺95] to understand the inner product of characters through the descent algebra [MR95]. It leads to the discovery of the Malvenuto-Reutenauer algebra **FQSym** of permutations. Another example was the introduction by Poirier [Poi96] of an algebra of Young tableaux. In [DHT01], it was realized that it can lead to a very simple proof of the Littlewood-Richardson rule.

An early meeting between the symmetric function and the operad communities was the discovery [HNT04] that the same procedure allows to construct both the algebra of tableaux and the algebra of binary trees from the algebra of permutations. One just has to enforce some simple relation (respectively plactic and sylvester relation) in the variable of the polynomial realization.

Aside from the algebra of permutations, there is another non-commutative lifting of quasi-symmetric functions. Indeed Hivert's action on polynomials whose invariant are quasi-symmetric functions [DHT01, Hiv99] can be lifted to words. Here, its non-commutative invariants spans the Hopf algebra **WQSym** of packed words or, equivalently, surjections or even ordered set partitions. This algebra has various applications in the theory of free Lie algebras is closely related to the Solomon-Tits algebra and twisted descents, the development of which was motivated by the geometry of Coxeter groups, the study of Markov chains on hyperplane arrangements (see [NPT13] and the reference therein).

To better understand these algebra, one has to investigate their structure. For the binary tree algebra it was shown in [HNT04] that it is free as an algebra and isomorphic to its dual. Though those properties are quite obvious for the algebra **FQSym** of permutation, the situation of **WQSym** is much more difficult. Its first study is due to Bergeron-Zabrocki [BZ05]. They showed that it is free and co-free. However, it was only conjectured in [DHT01] that its primitive Lie algebra is free and that it is self dual. It is only by a deep theorem of Foissy [Foi07] that one can show that the second one is too. In particular, until Vargas's work [Var19], no concrete isomorphism was known.

Independently, Novelli-Thibon worked on parking functions which is a super-set of packed words. They endowed the hopf algebra of parking functions **PQSym** with a bidendriform bialgebra structure [NT07]. Then they describe **WQSym** as a subbidendriform bialgebra of **PQSym** [NT06]. Recall that a dendriform algebra is an abstraction of a shuffle algebra where the product is split in two half-products. If the

coproduct is also split, and certain compatibilities hold, one gets the notion of bidendriform bialgebra [Foi07].

Building on the work of Chapoton and Ronco [Ron00, Cha02], Foissy [Foi07] showed that the structure of a bidendriform bialgebra is very rigid. In particular, he defined a specific subspace called the space of totally primitive elements, and showed that it characterizes the whole structure. This does not only re-prove the freeness and co-freeness, as well as the freeness of the primitive lie algebra, but also shows that the structure of a bidendriform bialgebra depends only on its Hilbert series (the series of dimensions of its homogeneous components). In particular, any such algebra is isomorphic to its dual. However, Foissy's isomorphism is not fully explicit and depends on a choice of a basis of the totally primitive elements. To this end, one needs an explicit basis of the totally primitive elements. Foissy described such a construction for \mathbf{FQSym} [Foi11]. In this paper, we construct a far reaching generalization for packed words and \mathbf{WQSym} so that the basis described in [Foi11] is simply a restriction to permutations and \mathbf{FQSym} is a sub-bidendriform bialgebra of \mathbf{WQSym} . We provide two explicit bases of totally primitive elements, for \mathbf{WQSym} and its dual, using a bijection with certain families of trees called biplane.

We begin with a background section presenting Foissy's two rigidity structure theorems that prove, among other things, the self-duality of any bidendriform bialgebra ([Theorem 2](#) and [Corollary 3](#)). We then define the notion of packed words as well as the two specific bases (\mathbb{Q} and \mathbb{R}) of \mathbf{WQSym} and its dual, which will be the starting point of our combinatorial analysis.

[Section 2](#) is devoted to the combinatorial construction of biplane forests ([Definitions 67](#) and [102](#)) which are our first key ingredient. They record a recursive decomposition of packed words according to their global descents ([Lemma 16](#)) and positions of the maximum letter ([Lemma 42](#)) or the value of the last letter ([Lemma 81](#)). We show that the cardinalities of some specific sets of biplane trees match the dimensions of primitive and totally primitive elements ([Theorems 75](#) and [111](#)).

In [Section 3](#) we construct two new bases (\mathbb{O} and \mathbb{P} [Definitions 116](#) and [124](#)) of \mathbf{WQSym} and its dual which each contain as a subset a basis for the primitive and totally primitive elements (see [Theorems 118](#) and [126](#)). To do so we decompose the space of totally primitive elements as a certain direct sum which matches the combinatorial decomposition of packed words ([Lemmas 115](#) and [123](#)).

Finally in [Section 4](#) we make explicit how bases \mathbb{O} and \mathbb{P} are sufficient to have an infinite number of bidendriform automorphism of \mathbf{WQSym} . Then we give an explicit isomorphism based on an involution on packed words. The definition of the bijection require a new kind of forest mixing red and blue, namely bicolored-packed forests.

1 Background

1.1 Cartier-Milnor-Moore theorems for Bidendriform bialgebras

The goal of this section is to recall the elements of the definition of bidendriform bialgebras which are useful for the comprehension of this paper. We refer to [Foi07] for the full list of axioms.

A bialgebra is a vector space over a field K , endowed with an unitary associative product \cdot and a counitary coassociative coproduct Δ satisfying a compatibility relation called the Hopf relation $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$. In this paper all bialgebras are assumed to be graded and connected (*i.e.* the homogeneous component of degree 0 is K). They are therefore Hopf algebras, as the existence of the antipode is implied.

A **dendriform algebra** (see [Lod01, LR98, Ron00, Ron02]) A is a K -vector space, endowed with two binary bilinear operations \prec, \succ satisfying the following axioms, for all $a, b, c \in A$:

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c), \quad (1.1)$$

$$(a \succ b) \prec c = a \succ (b \prec c), \quad (1.2)$$

$$(a \prec b + a \succ b) \succ c = a \succ (b \succ c). \quad (1.3)$$

Adding together **Equations (1.1) to (1.3)** show that the product $a \cdot b := a \prec b + a \succ b$ is associative. Adding a subspace of scalars, this defines a unitary algebra structure on $K \oplus A$. In this paper, all the dendriform algebras are graded and have null 0-degree component so that the associated algebra is connected.

Dualizing, one gets a notion of **co-dendriform co-algebra** (see [Foi07]) which is a K -vector space with two binary co-operations (*i.e.*, linear maps $A \rightarrow A \otimes A$) denoted by $\Delta_{\prec}, \Delta_{\succ}$ satisfying the dual axioms of **Equations (1.1) to (1.3)**:

$$(\Delta_{\prec} \otimes \text{Id}) \circ \Delta_{\prec}(a) = (\text{Id} \otimes \Delta_{\prec} + \text{Id} \otimes \Delta_{\succ}) \circ \Delta_{\prec}(a), \quad (1.4)$$

$$(\Delta_{\succ} \otimes \text{Id}) \circ \Delta_{\prec}(a) = (\text{Id} \otimes \Delta_{\prec}) \circ \Delta_{\succ}(a), \quad (1.5)$$

$$(\Delta_{\prec} \otimes \text{Id} + \Delta_{\succ} \otimes \text{Id}) \circ \Delta_{\succ}(a) = (\text{Id} \otimes \Delta_{\succ}) \circ \Delta_{\succ}(a). \quad (1.6)$$

Adding together **Equations (1.4) to (1.6)** show that the reduced coproduct $\tilde{\Delta}(a) := \Delta_{\prec}(a) + \Delta_{\succ}(a)$ is co-associative. On $K \oplus A$, setting $\Delta(a) := 1 \otimes a + a \otimes 1 + \tilde{\Delta}(a)$ defines a co-associative and co-unitary coproduct.

A **bidendriform bialgebra** is a K -vector space which is both a dendriform algebra and a co-dendriform co-algebra satisfying a set of four relations relating respectively \prec and \succ with $\Delta_{\prec}, \Delta_{\succ}$ (see [Foi07] for more details). In these equations, we use a kind of

Einstein notation where $\tilde{\Delta}(a) = a' \otimes a''$ and $\Delta_\alpha(b) = b'_\alpha \otimes b''_\alpha$ with $\alpha \in \{\prec, \succ\}$.

$$\Delta_\succ(a \succ b) = a' b'_\succ \otimes a'' \succ b''_\succ + b'_\succ \otimes a \succ b''_\succ + a b'_\succ \otimes b''_\succ + a' \otimes a'' \succ b + a \otimes b, \quad (1.7)$$

$$\Delta_\succ(a \prec b) = a' b'_\succ \otimes a'' \prec b''_\succ + b'_\succ \otimes a \prec b''_\succ + a' \otimes a'' \prec b, \quad (1.8)$$

$$\Delta_\prec(a \succ b) = a' b'_\prec \otimes a'' \succ b''_\prec + b'_\prec \otimes a \succ b''_\prec + a b'_\prec \otimes b''_\prec, \quad (1.9)$$

$$\Delta_\prec(a \prec b) = a' b'_\prec \otimes a'' \prec b''_\prec + b'_\prec \otimes a \prec b''_\prec + a' b \otimes a'' + b \otimes a. \quad (1.10)$$

Adding those four relations shows that \cdot and Δ as defined above defines a proper bi-algebra.

We recall here the relevant results of Foissy [Foi07] on the rigidity of bidendriform bialgebras based on the works of Chapoton and Ronco [Ron00, Cha02].

Let A be a bidendriform bialgebra. We denote $\text{Prim}(A) := \text{Ker}(\tilde{\Delta})$ the set of **primitive** elements of A . We also denote by $\mathcal{A}(z)$ and $\mathcal{P}(z)$ the Hilbert series of A and $\text{Prim}(A)$ defined as $\mathcal{A}(z) := \sum_{n=1}^{+\infty} \dim(A_n) z^n$ and $\mathcal{P}(z) := \sum_{n=1}^{+\infty} \dim(\text{Prim}(A_n)) z^n$. The present work is based on two analogues of the Cartier-Milnor-Moore theorems [Foi07] which we present now. The first one is extracted from the proof of [Foi11, Proposition 6]:

Proposition 1. *Let A be a bidendriform bialgebra and let $p_1 \dots p_n \in \text{Prim}(A)$. Then the map*

$$p_1 \otimes p_2 \otimes \dots \otimes p_n \mapsto p_1 \prec (p_2 \prec (\dots \prec p_n) \dots). \quad (1.11)$$

is an isomorphism of co-algebras from $T^+(\text{Prim}(A))$ (the non trivial part of the tensor algebra with deconcatenation as coproduct) to A . As a consequence, taking a basis $(p_i)_{i \in I}$ of $\text{Prim}(A)$, the family $(p_{w_1} \prec (p_{w_2} \prec (\dots \prec p_{w_n}) \dots))_w$ where $w = w_1 \dots w_n$ is a non empty word on I defines a basis of A . This implies the equality of Hilbert series $\mathcal{A} = \mathcal{P} / (1 - \mathcal{P})$.

One can further analyze $\text{Prim}(A)$ using the so-called **totally primitive** elements of A defined as $\text{TPrim}(A) := \text{Ker}(\Delta_\prec) \cap \text{Ker}(\Delta_\succ)$. The associated Hilbert serie is defined as $\mathcal{T}(z) := \sum_{n=1}^{+\infty} \dim(\text{TPrim}(A_n)) z^n$. Recall that a brace algebra is a K -vector space A together with an n -multilinear operation denoted as $\langle \dots \rangle$ for all $n \geq 2$ which satisfies certain relations (see [Ron00] for details).

Theorem 2 ([Foi11, Theorem 4 and 5]). *Let A be a bidendriform bialgebra. Then $\text{Prim}(A)$ is freely generated as a brace algebra by $\text{TPrim}(A)$ with brackets given by*

$$\langle p_1, \dots, p_{n-1}; p_n \rangle := \sum_{i=0}^{n-1} (-1)^{n-1-i} (p_1 \prec (p_2 \prec (\dots \prec p_i) \dots)) \succ p_n \prec ((\dots (p_{i+1} \succ p_{i+2}) \succ \dots) \succ p_{n-1}).$$

A basis of $\text{Prim}(A)$ is described by ordered trees that are decorated with elements of $\text{TPrim}(A)$ where p_n is the root and p_1, \dots, p_{n-1} are the children (see [Ron00, Cha02,

[Foi11]). This is reflected on their Hilbert series as [Foi07, Corollary 37]: $\mathcal{T} = \mathcal{A}/(1 + \mathcal{A})^2$ or equivalently $\mathcal{P} = \mathcal{T}(1 + \mathcal{A})$.

Using [Proposition 1](#) and [Theorem 2](#) together with a dimension argument, one can show the two following corollaries:

Corollary 3 ([Foi11, Theorem 2]). *Let A be a bidendriform bialgebra. Then A is freely generated as a dendriform algebra by $\text{TPrim}(A)$.*

Corollary 4 ([Foi02a, Foi02b]). *A basis of A is described by ordered forests of ordered trees that are decorated with a basis of $\text{TPrim}(A)$.*

On this basis, the product can be described using grafting (see Proposition 28 in [Foi02b]) and the coproduct as the deconcatenation of forests that are word of trees (see Theorem 35 equation 7.(c) in [Foi02a]).

1.2 Packed words

The algebra **WQSym** is a Hopf algebra whose bases are indexed by ordered set partitions or equivalently surjections or even packed words. In this paper, we use the latter which we define now.

In this paper we will deal with words over the alphabet of positive integers $\mathbb{N}_{>0}$. We start with basic notations: First, $\max(w)$ is the maximum letter of the word w with the convention that $\max(\epsilon) = 0$. Then $|w|$ is the length (or size) of the word w . The concatenation of the two words u and v is denoted as $u \cdot v$. The shift of a word w of a value i is denoted by $w^{[i]}$. Once that said, $u/v := u^{[\max(v)]} \cdot v$ (resp. $u \setminus v := u \cdot v^{[\max(u)]}$) is the left-shifted (resp. right-shifted) concatenation of the two words where all the letters of the left (resp. right) word are shifted by the maximum of the right (resp. left) word: $1121/3112 = 44543112$ and $1121 \setminus 3112 = 11215334$. We also use the notation $u|_{\leq i}$ (resp. $u|_{> i}$) for the subword containing all letters smaller (resp. strictly greater) than a value i .

Definition 5. A word over the alphabet $\mathbb{N}_{>0}$ is **packed** if all the letters from 1 to its maximum m appears at least once. By convention, the empty word ϵ is packed. For $n \in \mathbb{N}$, we denote by \mathbf{PW}_n the set of all packed words of length (also called size) n and $\mathbf{PW} = \bigsqcup_{n \in \mathbb{N}} \mathbf{PW}_n$ the set of all packed words.

n	1	2	3	4	5	6	7	8	9	OEIS
\mathbf{PW}_n	1	3	13	75	541	4 683	47 293	545 835	7 087 261	A000670

Table 1: Number of packed words of size smaller than 9.

Definition 6. The packed word $u := \text{pack}(w)$ associated with a word over the alphabet $\mathbb{N}_{>0}$ is obtained by the following process: if $b_1 < b_2 < \dots < b_r$ are the distinct letters occurring in w , then u is the image of w by the homomorphism $b_i \mapsto i$.

A word u is packed if and only if $\text{pack}(u) = u$.

Example 7. The word 4152142 is not packed because the letter 3 does not appear while the maximum letter is $5 > 3$. Meanwhile $\text{pack}(4152142) = 3142132$ is a packed word. Here are all packed words of size 1, 2 and 3 in lexicographic order:

1, 11 12 21, 111 112 121 122 123 132 211 212 213 221 231 312 321

The fonction $\text{pack}(w)$ is the analogue of the **standardization** $\text{std}(w)$ that returns a permutation.

Definition 8. The standardized word $\text{std}(w)$ associated with a word over the alphabet $\mathbb{N}_{>0}$ is obtained by iteratively scanning w from left to right, and labelling the occurrences of its smallest letter, then labelling the occurrences of the next one, and so on.

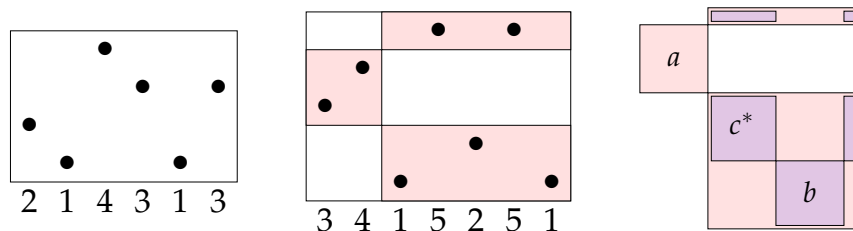
Example 9. For example, $\text{std}(4152142) = \text{std}(3142132) = 5173264$.

For the reader familiar with ordered set partitions, there is a classical bijection between packed words and ordered set partitions. The one corresponding to a packed word $w_1 \cdot w_2 \cdot \dots \cdot w_n$ is obtained by placing the index i into the w_i -th block.

Example 10. The word 121 is associated with $\{\{13\}, \{2\}\}$ and the word 113223 with $\{\{12\}, \{45\}, \{36\}\}$.

To depict some definitions or lemmas, we will use box diagrams with Cartesian coordinates for packed words. On these diagrams, positions are from left to right (as reading direction) and values are from bottom to top. These diagrams will also be used to represent different decompositions with different colors. Transparency will order the decompositions.

Example 11. Here we have three examples: the representation of the packed word 214313. Then the word 3415251 decomposed with red-factorization (see [Lemma 42](#)). Finally the general case of the red-blue-factorization (see [Definition 135](#)) where it can be seen clearly that the blue-factorization is done after the red-factorization thanks to transparency.



Global descent are defined in [AS05] on permutations, here we generalise the definition on packed words.

Definition 12. A **global descent** of a packed word w is a position c such that all the letters before or at position c are strictly greater than all letters after position c .

Example 13. The global descents of $w = 54664312$ are the positions 5 and 6. Indeed, all letters of 54664 are greater than the letters of 312 and this is also true for 546643 and 12.

Definition 14. A packed word w is **irreducible** if it is non empty and it has no global descent.

n	1	2	3	4	5	6	7	8	9	OEIS
p_n	1	2	8	48	368	3 376	35 824	430 512	5 773 936	A095989

Table 2: Number of irreducible packed words of size smaller than 9.

Example 15. The word $w' = 21331$ is irreducible.

Lemma 16. Each word w admits a unique factorization as $w = w_1/w_2/\dots/w_k$ such that w_i is irreducible for all i .

Example 17. The global descent decomposition of 54664312 is 21331/1/12. The word $n \cdot n - 1 \cdot \dots \cdot 1$ has 1/1/.../1 as global descent decomposition.

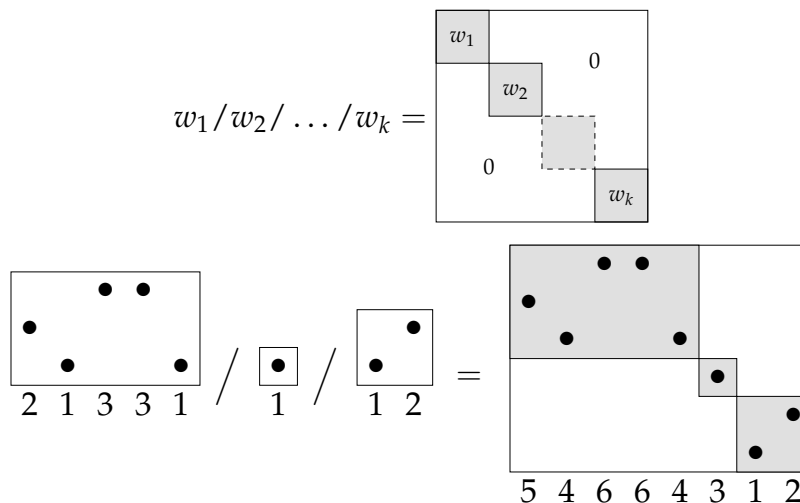


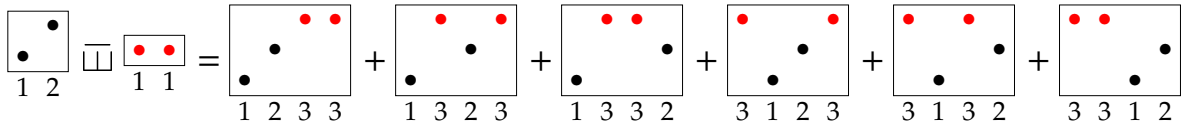
Figure 1: Box diagrams: global descent decomposition.

Definition 18. $u \sqcup v$ denotes the shuffle product of the two words. It is recursively defined by $u \sqcup \epsilon := \epsilon \sqcup u := u$ and

$$ua \sqcup vb := (u \sqcup vb) \cdot a + (ua \sqcup v) \cdot b \quad (1.12)$$

where u and v are words and a and b are letters. Analogously to the shifted concatenation, one can define the right shifted-shuffle $u \sqcup v := u \sqcup v^{[\max(u)]}$ where all the letters of the right word v are shifted by the maximum of the left word u .

Example 19. $12 \sqcup 11 = 12 \sqcup 33 = 1233 + 1323 + 1332 + 3123 + 3132 + 3312$.



Definition 20 ([Lod01, Example 5.4.(a)]). The recursive definition of the shuffle product Equation (1.12) contains two summands. The two half shuffle products on words \prec and \succ are defined respectively by:

$$ua \prec vb := (u \sqcup vb) \cdot a \quad \text{and} \quad ua \succ vb := (ua \sqcup v) \cdot b. \quad (1.13)$$

Example 21. $12 \prec 33 = 1332 + 3132 + 3312$ and $12 \succ 33 = 1233 + 1323 + 3123$.

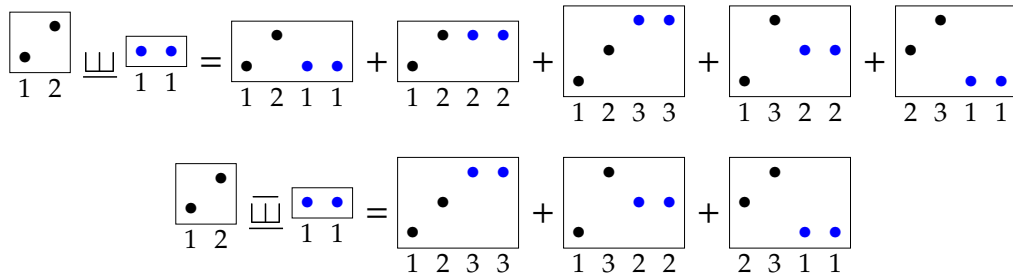
Definition 22. $u \sqcup\sqcup v$ denote the dualisation of the deconcatenation using the function $\text{pack}(w)$ of Definition 6.

$$u \sqcup\sqcup v := \sum_{\substack{u=\text{pack}(u') \\ v=\text{pack}(v')}} u' \cdot v' \quad (1.14)$$

where u, v and $u' \cdot v'$ are packed words. We also use the non-overlapping shuffle product on values by adding the constraint that letters of the two parts are distinct:

$$u \sqcup\sqcup\sqcup v := \sum_{\substack{u=\text{pack}(u') \\ v=\text{pack}(v') \\ \forall i, j, u'_i \neq v'_j}} u' \cdot v'. \quad (1.15)$$

Example 23. $12 \sqcup\sqcup 11 = 1211 + 1222 + 1233 + 1322 + 2311$, $12 \sqcup\sqcup\sqcup 11 = 1233 + 1322 + 2311$.



Remark 24. Using the classical bijection between ordered set partitions and packed words (see [Example 10](#)), the product $\underline{\sqcup}$ is equivalent to the shifted shuffle on ordered set partitions defined in [\[BZ05\]](#).

Definition 25. Analogously to the two half shuffle product of [Definition 20](#), we split the two products $\underline{\sqcup}$ and $\underline{\sqcup}$ in two parts.

$$u \ll v := \sum_{\substack{u=\text{pack}(u') \\ v=\text{pack}(v') \\ \max(u') > \max(v')}} u' \cdot v' \quad \text{and} \quad u \gg v := \sum_{\substack{u=\text{pack}(u') \\ v=\text{pack}(v') \\ \max(u') \leq \max(v')}} u' \cdot v'. \quad (1.16)$$

$$u \preceq v := \sum_{\substack{u=\text{pack}(u') \\ v=\text{pack}(v') \\ \forall i,j, u'_i \neq v'_j \\ \max(u') > \max(v')}} u' \cdot v' \quad \text{and} \quad u \succeq v := \sum_{\substack{u=\text{pack}(u') \\ v=\text{pack}(v') \\ \forall i,j, u'_i \neq v'_j \\ \max(u') < \max(v')}} u' \cdot v'. \quad (1.17)$$

Example 26.

$$\begin{aligned} 12 \ll 11 &= 1211 + 1322 + 2311 & \text{and} & \quad 12 \gg 11 = 1222 + 1233. \\ 12 \preceq 11 &= 1322 + 2311 & \text{and} & \quad 12 \succeq 11 = 1233. \end{aligned}$$

To sum up in a few words, in $u \prec v$ the last letter is coming from u and the rest is shuffled, in $u \preceq v$ the maximum value is coming from u and the rest is shuffled.

1.3 The Hopf algebra of word-quasisymmetric functions \mathbf{WQSym}

We are now in position to define the Hopf algebra of word-quasisymmetric functions \mathbf{WQSym} . It was first defined as a Hopf algebra in [\[Hiv99\]](#). Novelli-Thibon proved later that \mathbf{WQSym} and its dual are bidendriform bialgebras [\[NT06, Theorems 2.5 and 2.6\]](#). Their products and coproducts in the monomial basis $(\mathbb{M}_w)_{w \in \mathbf{PW}}$ involve overlapping-shuffle. However, to deal with the bidendriform structure, it will be easier for us to chose, among the various bases known in the literature [\[Hiv99, BZ05, NT06, Var19\]](#) a basis where the shuffles are non-overlapping. Therefore, for \mathbf{WQSym}^* , we take the basis denoted $(\mathbb{Q}_w)_{w \in \mathbf{PW}}$ of [\[BZ05, Equation 23\]](#) using the classical bijection between ordered set partitions and packed words (see [Example 10](#)). For the primal \mathbf{WQSym} , we define the dual basis denoted $(\mathbb{R}_w)_{w \in \mathbf{PW}}$. In this section we transfer the bidendriform structure on the bases $(\mathbb{Q}_w)_{w \in \mathbf{PW}}$ and $(\mathbb{R}_w)_{w \in \mathbf{PW}}$.

Following Novelli-Thibon, we start from the basis $(\mathbb{M}_w)_{w \in \mathbf{PW}}$ and compute expressions of half product [Equation \(1.22\)](#) and half coproduct [Equations \(1.23\) and \(1.24\)](#) in the basis $(\mathbb{Q}_w)_{w \in \mathbf{PW}}$. Then we dualise these operations [Equations \(1.25\) to \(1.27\)](#) to define the basis $(\mathbb{R}_w)_{w \in \mathbf{PW}}$. The Hopf algebra product and reduced coproduct are respectively recovered as the sum of the half products and half coproducts.

The monomial word-quasisymmetric function of a totally ordered alphabet \mathcal{A} associated to the packed word u is the linear combination of words defined by

$$\mathbb{M}_u := \sum_{\substack{w \in \mathcal{A}^*, \\ \text{pack}(w)=u}} w.$$

It turns out that the concatenation of two such elements is a sum of $(\mathbb{M}_u)_{u \in \mathbf{PW}}$ so that $\mathbf{WQSym} := \text{Vect}(\mathbb{M}_u \mid u \in \mathbf{PW})$ is an algebra. This can be refined to a bidendriform bialgebra structure. The operations \ll, \gg, Δ_{\ll} and Δ_{\gg} on

$$(\mathbf{WQSym})_+ := \text{Vect}(\mathbb{M}_u \mid u \in \mathbf{PW}_n, n \geq 1)$$

are defined in the following way: for all $u = u_1 \cdots u_n \in \mathbf{PW}_{n \geq 1}$ and $v \in \mathbf{PW}_{m \geq 1}$,

$$\mathbb{M}_u \ll \mathbb{M}_v := \sum_{w \in u \ll v} \mathbb{M}_w, \quad \text{and} \quad \mathbb{M}_u \gg \mathbb{M}_v := \sum_{w \in u \gg v} \mathbb{M}_w. \quad (1.18)$$

$$\Delta_{\ll}(\mathbb{M}_u) := \sum_{i=u_n}^{\max(u)-1} \mathbb{M}_{u|_{\leq i}} \otimes \mathbb{M}_{\text{pack}(u|_{>i})}, \quad (1.19)$$

$$\Delta_{\gg}(\mathbb{M}_u) := \sum_{i=1}^{u_n-1} \mathbb{M}_{u|_{\leq i}} \otimes \mathbb{M}_{\text{pack}(u|_{>i})}. \quad (1.20)$$

Example 27.

$$\begin{aligned} \mathbb{M}_{112} \ll \mathbb{M}_{12} &= \mathbb{M}_{11312} + \mathbb{M}_{11423} + \mathbb{M}_{22312} + \mathbb{M}_{22413} + \mathbb{M}_{33412}, \\ \mathbb{M}_{112} \gg \mathbb{M}_{12} &= \mathbb{M}_{11212} + \mathbb{M}_{11213} + \mathbb{M}_{11223} + \mathbb{M}_{11234} + \mathbb{M}_{11323} + \mathbb{M}_{11324} + \mathbb{M}_{22313} + \mathbb{M}_{22314}, \\ \Delta_{\ll}(\mathbb{M}_{212536434}) &= \mathbb{M}_{2123434} \otimes \mathbb{M}_{\text{pack}(56)} + \mathbb{M}_{21253434} \otimes \mathbb{M}_{\text{pack}(6)}, \\ &= \mathbb{M}_{2123434} \otimes \mathbb{M}_{12} + \mathbb{M}_{21253434} \otimes \mathbb{M}_1, \\ \Delta_{\gg}(\mathbb{M}_{212536434}) &= \mathbb{M}_1 \otimes \mathbb{M}_{\text{pack}(22536434)} + \mathbb{M}_{212} \otimes \mathbb{M}_{\text{pack}(536434)} + \mathbb{M}_{21233} \otimes \mathbb{M}_{\text{pack}(5644)} \\ &= \mathbb{M}_1 \otimes \mathbb{M}_{11425323} + \mathbb{M}_{212} \otimes \mathbb{M}_{314212} + \mathbb{M}_{21233} \otimes \mathbb{M}_{2311}. \end{aligned}$$

Theorem 28. [NT06, Theorem 2.5] $((\mathbf{WQSym})_+, \ll, \gg, \Delta_{\ll}, \Delta_{\gg})$ is a bidendriform bialgebra.

As we said, we want a basis without overlapping-shuffle for the product. Following [BZ05], we first define a partial order on packed words then we define the new basis.

Definition 29. [BZ05] We say that the packed word u is smaller than v for the relation \leq_* if u and v have the same standardization and if $u_i = u_j$ implies $v_i = v_j$ for all i and j .

$$u \leq_* v \iff \text{std}(u) = \text{std}(v) \text{ and } (u_i = u_j \implies v_i = v_j).$$

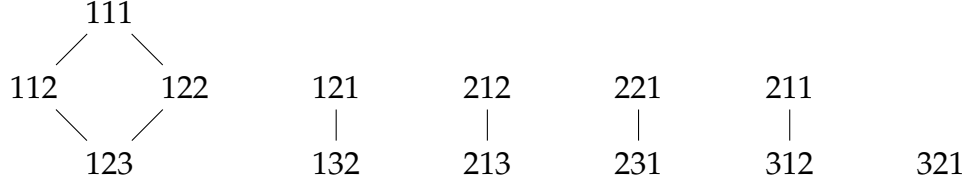


Figure 2: The Hasse diagram of (\mathbf{PW}_3, \leq_*) .

We give two immediate lemmas on this order that are useful.

Lemma 30. For $u \leq_* v$, let m_u (resp. m_v) be the set of positions of occurrences of the maximum value letters in u (resp. v). Then m_u is included in m_v and all positions in m_v that are not in m_u are smaller to the minimum of m_u .

Proof. It is immediate with the definition of \leq_* . \square

Lemma 31. For $u \leq_* v$, let i and i' such that $u|_{\leq i}$ and $v|_{\leq i'}$ are of the same size then $u|_{\leq i} \leq_* v|_{\leq i'}$ and $u|_{> i} \leq_* v|_{> i'}$.

Proof. It is immediate with the definition of \leq_* . \square

Now we can recall ([BZ05, Equation 23]) the definition of the basis $(Q_w)_{w \in \mathbf{PW}}$

$$Q_u := \sum_{u \leq_* v} M_v. \quad (1.21)$$

Example 32.

$$\begin{aligned} Q_{123} &= M_{123} + M_{122} + M_{112} + M_{111} & Q_{43132} &= M_{43132} + M_{32121} \\ Q_{412234} &= M_{412234} + M_{312223} + M_{311123} + M_{211112} & Q_{2131} &= M_{2131} + M_{2121} \end{aligned}$$

It is proved in [BZ05, Theorem 17] that the product in basis $(Q_w)_{w \in \mathbf{PW}}$ is

$$Q_u Q_v = \sum_{w \in u \sqcup v} Q_w.$$

Thanks to [Lemma 30](#), we have the two expressions for the two half products.

$$Q_u \preceq Q_v := \sum_{w \in u \preceq v} Q_w, \quad \text{and} \quad Q_u \succeq Q_v := \sum_{w \in u \succeq v} Q_w. \quad (1.22)$$

For the coproduct, we start with the definition of the coproduct in basis \mathbb{M} .

$$\begin{aligned}
\Delta(\mathbb{Q}_u) &= \sum_{v \geq_* u} \Delta(\mathbb{M}_v) \\
&= \sum_{v \geq_* u} \left(\sum_{i=0}^{\max(v)} \mathbb{M}_{v|_{\leq i}} \otimes \mathbb{M}_{v|_{> i}} \right) \\
&= \sum_{i=0}^{\max(u)} \left(\sum_{v \geq_* u|_{\leq i}} \mathbb{M}_v \otimes \sum_{v' \geq_* u|_{> i}} \mathbb{M}_{v'} \right) \quad (\text{by Lemma 31}) \\
&= \sum_{i=0}^{\max(u)} \mathbb{Q}_{u|_{\leq i}} \otimes \mathbb{Q}_{u|_{> i}}.
\end{aligned}$$

Then, with Lemma 30 we have the two expressions for the two half coproducts.

$$\Delta_{\preceq}(\mathbb{Q}_u) := \sum_{i=u_n}^{\max(u)-1} \mathbb{Q}_{u|_{\leq i}} \otimes \mathbb{Q}_{\text{pack}(u|_{> i})}, \quad (1.23)$$

$$\Delta_{\succeq}(\mathbb{Q}_u) := \sum_{i=1}^{u_n-1} \mathbb{Q}_{u|_{\leq i}} \otimes \mathbb{Q}_{\text{pack}(u|_{> i})}. \quad (1.24)$$

Example 33.

$$\begin{aligned}
\mathbb{Q}_{1312} \preceq \mathbb{Q}_{12} &= \mathbb{Q}_{151234} + \mathbb{Q}_{151324} + \mathbb{Q}_{151423} + \mathbb{Q}_{252314} + \mathbb{Q}_{252413} + \mathbb{Q}_{353412}, \\
\mathbb{Q}_{1312} \succeq \mathbb{Q}_{12} &= \mathbb{Q}_{131245} + \mathbb{Q}_{141235} + \mathbb{Q}_{141325} + \mathbb{Q}_{242315}, \\
\Delta_{\preceq}(\mathbb{Q}_{212536434}) &= \mathbb{Q}_{2123434} \otimes \mathbb{Q}_{\text{pack}(56)} + \mathbb{Q}_{21253434} \otimes \mathbb{Q}_{\text{pack}(6)}, \\
&= \mathbb{Q}_{2123434} \otimes \mathbb{Q}_{12} + \mathbb{Q}_{21253434} \otimes \mathbb{Q}_1, \\
\Delta_{\succeq}(\mathbb{Q}_{212536434}) &= \mathbb{Q}_1 \otimes \mathbb{Q}_{\text{pack}(22536434)} + \mathbb{Q}_{212} \otimes \mathbb{Q}_{\text{pack}(536434)} + \mathbb{Q}_{21233} \otimes \mathbb{Q}_{\text{pack}(5644)} \\
&= \mathbb{Q}_1 \otimes \mathbb{Q}_{11425323} + \mathbb{Q}_{212} \otimes \mathbb{Q}_{314212} + \mathbb{Q}_{21233} \otimes \mathbb{Q}_{2311}.
\end{aligned}$$

Finally we define $\prec, \succ, \Delta_{\prec}$ and Δ_{\succ} on $(\mathbf{WQSym}^*)_+ := \text{Vect}(\mathbb{R}_u \mid u \in \mathbf{PW}_n, n \geq 1)$ by dualizing half products and half coproducts of the basis $(\mathbb{Q}_w)_{w \in \mathbf{PW}}$ in the following way: for all $u = u_1 \cdots u_n \in \mathbf{PW}_{n \geq 1}$ and $v \in \mathbf{PW}_{m \geq 1}$,

$$\mathbb{R}_u \prec \mathbb{R}_v := \sum_{w \in u \prec v^{[\max(u)]}} \mathbb{R}_w, \quad \text{and} \quad \mathbb{R}_u \succ \mathbb{R}_v := \sum_{w \in u \succ v^{[\max(u)]}} \mathbb{R}_w. \quad (1.25)$$

$$\Delta_{\prec}(\mathbb{R}_u) := \sum_{\substack{i=k \\ \{u_1, \dots, u_i\} \cap \{u_{i+1}, \dots, u_n\} = \emptyset \\ u_k = \max(u)}}^{n-1} \mathbb{R}_{\text{pack}(u_1 \dots u_i)} \otimes \mathbb{R}_{\text{pack}(u_{i+1} \dots u_n)}, \quad (1.26)$$

$$\Delta_{\succ}(\mathbb{R}_u) := \sum_{\substack{i=1 \\ \{u_1, \dots, u_i\} \cap \{u_{i+1}, \dots, u_n\} = \emptyset \\ u_k = \max(u)}}^{k-1} \mathbb{R}_{\text{pack}(u_1 \dots u_i)} \otimes \mathbb{R}_{\text{pack}(u_{i+1} \dots u_n)}. \quad (1.27)$$

Example 34.

$$\begin{aligned} \mathbb{R}_{211} \prec \mathbb{R}_{12} &= \mathbb{R}_{21341} + \mathbb{R}_{23141} + \mathbb{R}_{23411} + \mathbb{R}_{32141} + \mathbb{R}_{32411} + \mathbb{R}_{34211}, \\ \mathbb{R}_{221} \succ \mathbb{R}_{12} &= \mathbb{R}_{21134} + \mathbb{R}_{21314} + \mathbb{R}_{23114} + \mathbb{R}_{32114}, \\ \Delta_{\prec}(\mathbb{R}_{2125334}) &= \mathbb{R}_{2123} \otimes \mathbb{R}_{112} + \mathbb{R}_{212433} \otimes \mathbb{R}_1, \\ \Delta_{\succ}(\mathbb{R}_{2125334}) &= \mathbb{R}_{212} \otimes \mathbb{R}_{3112}. \end{aligned}$$

Theorem 35. $((\mathbf{WQSym})_+, \preceq, \succeq, \Delta_{\prec}, \Delta_{\succ})$ and $((\mathbf{WQSym}^*)_+, \prec, \succ, \Delta_{\prec}, \Delta_{\succ})$ are two dual bidendriform bialgebras.

From now on $\text{Prim}(\mathbf{WQSym})$ and $\text{TPrim}(\mathbf{WQSym})$ are respectively abbreviated to **Prim** and **TPrim**. Moreover, we denote homogeneous components using indices and dualization using a $*$ in exponent as in \mathbf{Prim}_n^* . We give the first values of the dimensions $a_n := \dim(\mathbf{WQSym}_n)$, $p_n := \dim(\mathbf{Prim}_n)$ and $t_n := \dim(\mathbf{TPrim}_n)$:

n	1	2	3	4	5	6	7	8	9	OEIS
a_n	1	3	13	75	541	4 683	47 293	545 835	7 087 261	A000670
p_n	1	2	8	48	368	3 376	35 824	430 512	5 773 936	A095989
t_n	1	1	4	28	240	2 384	26 832	337 168	4 680 272	

Table 3: Dimensions of homogeneous components for **WQSym**, **Prim** and **TPrim**.

Though the numbers $(t_n)_n$ are easy to obtain thanks to the relation of **Theorem 2:** $\mathcal{T} = \mathcal{A}/(1 + \mathcal{A})^2$, no combinatorial interpretation existed. The first results of this paper are two different subsets of packed words that are counted by these dimensions (red-irreducible and blue-irreducible).

2 Decorated forests

In this paper we will generalize twice the construction of [Foi11], one for **WQSym** and one for its dual. This section is devoted to the combinatorial ingredient, that is a notion of biplane forests suitable for indexing the various bases of primitive elements. Each

time, we start by decomposing packed words through global descents and removal of specific letters. We then perform those decompositions recursively, encoding the result in a forests. We hence obtain so called biplane forests, which are in bijection with packed words. Later, the recursive structure of forests will be understood as a chaining of brace and dendriform operations generating some elements of **WQSym** or its dual. This will allow us to construct two bases of respectively **TPrim** and its dual by characterizing a subfamily of biplane trees.

From now on, we associate the color **blue** to the primal (**WQSym**) and the color **red** to the dual (**WQSym***). We start by explaining the construction on **WQSym*** (**red**) then we dualize the construction to the primal **WQSym** (**blue**).

2.1 Dual (**Red**)

For the red side **WQSym***, the decomposition of packed words is made through global descents and removal of maximum values. One step of this decomposition is called the red-factorization.

2.1.1 Decomposition of packed words through maximums

In this section, we define two combinatorial operations on packed words (ϕ_I and \blacktriangleright) and the red-factorization that uses them. The unary operation ϕ_I inserts new maximums in a word in positions I . A word that cannot be factorized $u \blacktriangleright v$ in a non trivial way is called red-irreducible. Red-irreducible words will index our basis of **TPrim***.

Definition 36. Fix $n \in \mathbb{N}$ and $w \in \mathbf{PW}_n$. We write $m' := \max(w) + 1$. For any $p > 0$ and any subset $I \subseteq [1, \dots, n + p]$ of cardinality p , we define $\phi_I(w) := u_1 \dots u_{n+p}$ as the packed word of length $n + p$ obtained by inserting p occurrences of the letter m' in w so that they end up in positions $i \in I$. In other words $u_i = m'$ if $i \in I$ and w is obtained from $\phi_I(w)$ by removing all occurrences of m' . Notice that $\phi_I(w)$ is only defined if $n + p \geq i_p$.

Example 37. $\phi_{2,4,7}(1232) = 1424324$ and $\phi_{1,2,3}(\epsilon) = 111$.

Notations 38. For the rest of this paper, $I = [i_1, \dots, i_p]$ will always denote a non-empty ($p > 0$) list of increasing non-zero integers. For any integer k , $I' = I + k$ denote the list $I' = [i_1 + k, \dots, i_p + k]$. Let \mathbf{PW}_n^I denote the set of packed words of size n whose maximums are in positions $i \in I$. This way $\phi_I(w) \in \mathbf{PW}_{n+p}^I$ for any $w \in \mathbf{PW}_n$.

Lemma 39. Let $n \in \mathbb{N}$ and $p > 0$, for any $I = [i_1, \dots, i_p] \subseteq [1, \dots, n + p]$ of size p , ϕ_I is a bijection from \mathbf{PW}_n to the \mathbf{PW}_{n+p}^I .

Moreover, for any $W \in \mathbf{PW}_\ell$ where $\ell > 0$ there exists a unique pair (I, w) where $I \subseteq [1 \dots \ell]$ and w is packed, such that $W = \phi_I(w)$.

The box diagram that pictures this lemma is $W = \begin{array}{|c|} \hline \\ \hline w \\ \hline \end{array} \leftarrow I$.

Proof. Let $W \in \mathbf{PW}_\ell$ with $\ell > 0$ and m the value of the maximum letter of W . Let $I = [i_1, \dots, i_p] \subseteq [1, \dots, \ell]$ be the list of the positions of m in W and let w be the word obtain by removing all occurrences of m in W , then $W = \phi_I(w)$. If $\phi_I(u) = \phi_J(v)$ then positions of maximum values are the same so $I = J$ and words obtain by removing these maximum values are also the same so $u = v$. \square

Definition 40. Let $u, v \in \mathbf{PW}$ with $v \neq \epsilon$. By Lemma 39, there is a unique pair (I, v') such that $v = \phi_I(v')$. Let $I' = I + |u|$, we define $u \blacktriangleright v := \phi_{I'}(u/v')$. In other words, we remove the maximum letter of the right word, perform a left shifted concatenation and reinsert the removed letters as new maximums.

Example 41. $2123 \blacktriangleright 322312 = 2123 \blacktriangleright \phi_{1,4}(2212) = \phi_{1+4,4+4}(43452212) = 4345622612$.

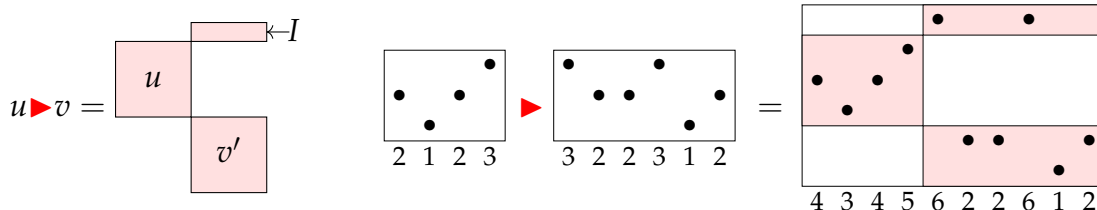


Figure 3: Box digrams: the operation \blacktriangleright

Lemma 42. Let w be an irreducible packed word. There exists a unique factorization of the form $w = u \blacktriangleright v$ which maximizes the size of u . In this factorization, let v' and I be such that $v = \phi_I(v')$, then

- either $v' = \epsilon$ and $I = [1, \dots, p]$ for some p ,
- or the global descent decomposition $v' = v_1 / \dots / v_r$ of v' satisfies the inequalities $1 \leq i_1 \leq |v_1|$, and $1 \leq (|I| + |v'|) + 1 - i_p \leq |v_r|$ with $I = [i_1, \dots, i_p]$.

We call it the **red-factorization** of a word.

Example 43. Here is a first detailed example of a red-factorization of an irreducible packed word:

Consider the irreducible packed word $w = 543462161$.

- The first step is to remove all the occurrences of the maximal value but keep in memory the positions in the initial word. We get $w' = 5434_21_1$ which is a packed word, but is not irreducible.

- The second step is to decompose the new word w' in irreducible factors $w' = 1/212/_1/1_1$. We still keep in memory the positions of the removed value. (when we have the choice, we cut to the left of the removed value.)
- We can distinguish two groups of factors, those strictly before the first maximum withdrawn and the others $w' = 1/212 \ / \ _1/1_1$.
- Finally, by numbering the positions of the maximum removed value in the right factor (positions 1 and 4), we get the following decomposition of w (see [Definition 36](#) for ϕ and [Definition 40](#) for \blacktriangleright):
 $w = 543462161 = (1/212)\blacktriangleright\phi_{1,4}(1/11) = (3212)\blacktriangleright\phi_{1,4}(211) = 3212\blacktriangleright 32131$.

Example 44. Here are some other red-factorizations:

$$\begin{aligned} 21331 &= 1\blacktriangleright\phi_{2,3}(11) = 1\blacktriangleright 1221 & 1231 &= \epsilon\blacktriangleright\phi_3(121) = \epsilon\blacktriangleright 1231 \\ 1233 &= 12\blacktriangleright\phi_{1,2}(\epsilon) = 12\blacktriangleright 11 & 111 &= \epsilon\blacktriangleright\phi_{1,2,3}(\epsilon) = \epsilon\blacktriangleright 111 \\ 56434126 &= 1\blacktriangleright\phi_{1,7}(212/12) = 1\blacktriangleright\phi_{1,7}(43412) = 1\blacktriangleright 5434125 \end{aligned}$$

Proof. Let w be irreducible and let (I, w') be the unique pair such that $w = \phi_I(w')$ according to [Lemma 39](#). By [Lemma 16](#), we write $w' = w'_1/w'_2/\dots/w'_k$, the unique decomposition into irreducibles. Let ℓ be such that w'_ℓ is the last factor which is entirely before the first removed maximum, it is the only choice to maximize the size of u . Then with $r = k - \ell$ we can rewrite w' as $(u_1/\dots/u_\ell)/(v_1/\dots/v_r)$. Now we get I' by subtracting $|u_1/\dots/u_\ell|$ to all parts of I ($I' = I - |u_1/\dots/u_\ell|$) and we obtain

$$w = u_1/\dots/u_\ell\blacktriangleright\phi_{I'}(v_1/\dots/v_r)$$

with $i'_1 \leq |v_1|$ or $r = 0$.

In the case of $v' \neq \epsilon$, the inequality $(|v'| + |I|) + 1 - i_p \leq |v_r|$ is always true otherwise w would not be irreducible. \square

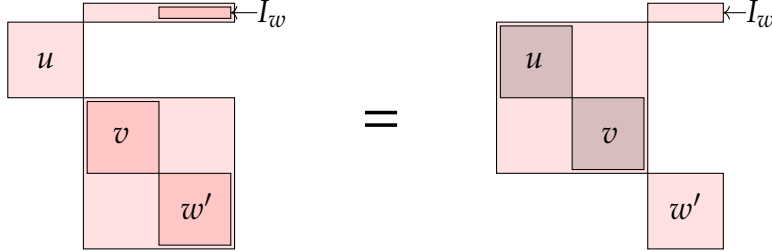
Definition 45. A packed word w is said to be **red-irreducible** if w is irreducible and the equality $w = u\blacktriangleright v$ implies that $u = \epsilon$ (and $w = v$).

Here are all red-irreducible packed words of size 1, 2, 3 and 4 in lexicographic order:

$$\begin{aligned} &1, \quad 11, \quad 111 \ 121 \ 132 \ 212, \\ &1111 \ 1121 \ 1132 \ 1211 \ 1212 \ 1221 \ 1231 \ 1232 \ 1243 \ 1312 \ 1321 \ 1322 \ 1323 \ 1332 \\ &1342 \ 1423 \ 1432 \ 2112 \ 2121 \ 2122 \ 2132 \ 2143 \ 2212 \ 2312 \ 2413 \ 3123 \ 3132 \ 3213. \end{aligned}$$

Here are some useful lemmas on the operation \blacktriangleright .

Lemma 46. For any $u, v, w \in \mathbf{PW}$ with $w \neq \epsilon$, we have $u \blacktriangleright (v \blacktriangleright w) = (u/v) \blacktriangleright w$.

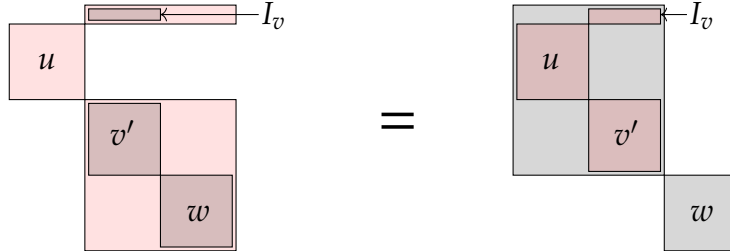


Proof. Let $u, v, w \in \mathbf{PW}$ with $w \neq \epsilon$ and let w' and I_w such that $w = \phi_{I_w}(w')$.

$$\begin{aligned}
 u \blacktriangleright (v \blacktriangleright w) &= u \blacktriangleright (\phi_{I_w+|v|}(v/w')) \\
 &= \phi_{I_w+|v|+|u|}(u/(v/w')) \\
 &= \phi_{I_w+|v|+|u|}((u/v)/w') \\
 &= (u/v) \blacktriangleright \phi_{I_w}(w') \\
 &= (u/v) \blacktriangleright w.
 \end{aligned}$$

□

Lemma 47. For any $u, v, w \in \mathbf{PW}$ with $v \neq \epsilon$, we have $u \blacktriangleright (v/w) = (u \blacktriangleright v)/w$.



Proof. Let $u, v, w \in \mathbf{PW}$ with $v \neq \epsilon$ and let v' and I_v such that $v = \phi_{I_v}(v')$.

$$\begin{aligned}
 u \blacktriangleright (v/w) &= u \blacktriangleright \phi_{I_v}(v'/w) \\
 &= \phi_{I_v+|u|}(u/(v'/w)) \\
 &= \phi_{I_v+|u|}((u/v')/w) \\
 &= \phi_{I_v+|u|}(u/v')/w \\
 &= (u \blacktriangleright v)/w.
 \end{aligned}$$

□

Remark 48. Adding the associativity of shifted concatenation $u/(v/w) = (u/v)/w$, the two operations \blacktriangleright and $/$ verify relations of the *skew-duplicial operad* [BDO20].

Corollary 49. For any $u, v \in \mathbf{PW}$, we have that $u \blacktriangleright v$ is irreducible if and only if v is irreducible.

Proof. By contradiction, if $v = v_1/v_2$ then by Lemma 47 $u \blacktriangleright v = (u \blacktriangleright v_1)/v_2$. Now if $u \blacktriangleright v = w_1/w_2$, as the position of the first maximum of $u \blacktriangleright v$ is greater than $|u|$ we have that $w_1 = w'_1 \cdot w''_1$ such that $\text{pack}(w'_1) = u$. We also have that $\text{pack}(w''_1)/w_2 = v$. □

Proposition 50. For any word w , $w = u \blacktriangleright v$ is the red-factorization of w if and only if v is red-irreducible.

Proof. Let $w \in \mathbf{PW}$ and let $u \blacktriangleright v$ be the red-factorization of w . Let v_1 and v_2 such that $v = v_1 \blacktriangleright v_2$, then $(u/v_1) \blacktriangleright v_2 = w$ by Lemma 46, but in the red-factorization the size of u is maximized so $|(u/v_1)| \leq |u|$ and then we have that $v_1 = \epsilon$ so v is red-irreducible.

Let $w \in \mathbf{PW}$ and let u and v such that $w = u \blacktriangleright v$ and v is red-irreducible. By contradiction, suppose that there exists u', v' such that $w = u' \blacktriangleright v'$ with $|u| < |u'|$ and $v' \neq \epsilon$. Then necessarily u is a prefix of u' . Let u'' such that $u' = u \cdot u''$, then $\text{pack}(u'') \blacktriangleright v' = v$. But v is red-irreducible. So the size of u is maximal if v is red-irreducible. \square

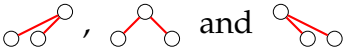

For the reader who is familiar with ordered set partitions, all the definitions in Section 2.1 can be easily written with these. However in Section 2.2 it is easier to do all the definitions on packed words and in Section 4 we must have the same object on both sides to explicit the isomorphism. So we decided to stick to packed words.

2.1.2 Red-forests from decomposed packed words using ϕ

We now apply recursively the red-factorization of the previous section to construct a bijection between packed words and a certain kind of trees that we now define.

Definition 51. An unlabeled **biplane tree** is an ordered tree (sometimes also called planar) whose children are organized in a pair of two (possibly empty) ordered forests, which we call the left and right forests, a forest being an ordered list of trees.

In the picture, we naturally draw the children of the left (resp. right) forest on the left (resp. right) of their father.

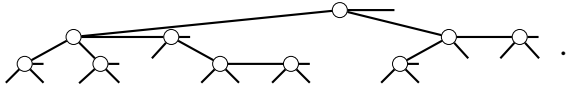
Example 52. The biplane trees  are different. Indeed in the first case, the left forest contains two trees and the right forest is empty, in the second case both forests contain exactly one tree while in the third case the left forest is empty and the right contains two trees. Here is an example of a bigger biplane tree where the root has two trees in both left and right forests .

Definition 53. A **skeleton biplane tree** is a biplane tree where no node has a right forest.

These skeleton biplane trees can also be seen as planar trees. In [Foi11] we have planar trees recursively labeled by planar trees. Skeleton biplane trees are similar to these planar trees, we prefer to see them as biplane tree with no right forest in order to keep some consistency.

Definition 54. The **size** of a biplane tree is the number of node in the tree.

Remark 55. Biplane forests \mathfrak{F} (i.e. ordered list of biplane trees \mathfrak{T}) are counted by the sequence A001764 in OEIS [SI20] whose explicit formula is $a(n) = \binom{3n}{n}/(2n+1)$. Biplane forests are in bijection with ternary trees. The bijection is the following, in a biplane forest a node has a first left child and a first right child and a right brother. A consequence is that unlabeled biplane trees are counted by the sequence A006013 in OEIS [SI20] whose explicit formula is $a(n) = \binom{3n+1}{n+1}$. Indeed, biplane trees are in bijection with pair of ternary trees. Here is an example of the ternary tree in bijection with the biplane forest constituted of one tree, the big biplane tree in [Example 52](#)



n	0	1	2	3	4	5	6	7	8	OEIS
\mathfrak{F}_n	1	1	3	12	55	273	1 428	7 752	43 263	A001764
\mathfrak{T}_n	0	1	2	7	30	143	728	3 876	21 318	A006013

Table 4: Number of biplane forests and biplane trees.

Remark 56. As we can see on OEIS [SI20], sequence A006013 which counts unlabeled biplane trees is the dimensions of the free L-algebra on one generator (see [Phi07]). It would be interesting to investigate the link between L-algebras and bidendriform bialgebras using biplane trees.

In our construction we will deal with labeled biplane trees with colored edges. For a labeled biplane tree, we denote by $\text{Node}_{\mathbf{R}}(x, f_{\ell}, f_r)$ the tree whose edges are colored in red, root is labeled by x and whose left (resp. right) forest is given by f_{ℓ} (resp. f_r). We also denote by $[t_1, \dots, t_k]$ a forest of k trees. The edge color (for now, only red) will play a role later in the paper.

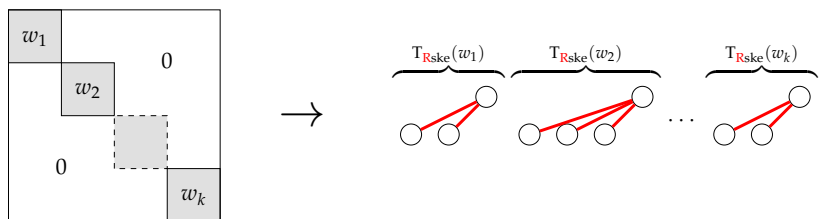
Example 57. $\text{Node}_{\mathbf{R}}((1), [], []) = \textcircled{1}$, and $\text{Node}_{\mathbf{R}}((1, 3), [], [\text{Node}_{\mathbf{R}}((1), [], [])]) = \textcircled{1,3} \text{---} \textcircled{1}$.

We now apply recursively the global descent decomposition and the red-factorization of [Lemmas 16](#) and [42](#). We obtain an algorithm which takes a packed word and returns a biplane forest where nodes are decorated by red-irreducible packed words:

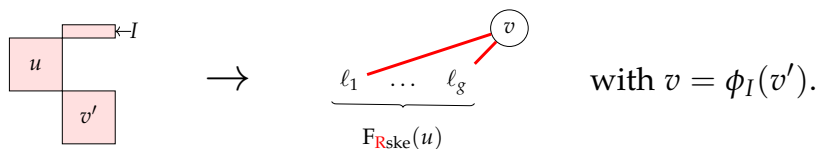
Definition 58. We now define two functions $F_{\mathbf{R}\text{ske}}$ and $T_{\mathbf{R}\text{ske}}$. These functions transform respectively a packed word and an irreducible packed word into respectively a skeleton biplane forest and a skeleton biplane tree labeled by red-irreducible words. These functions are defined in a mutual recursive way as follows:

- $F_{\mathbf{R}\text{ske}}(\epsilon) = []$ (empty forest),

- for any packed word w , let $w_1/w_2/\dots/w_k$ be the global descent decomposition of w , then $F_{\text{Rske}}(w) := [T_{\text{Rske}}(w_1), T_{\text{Rske}}(w_2), \dots, T_{\text{Rske}}(w_k)]$.

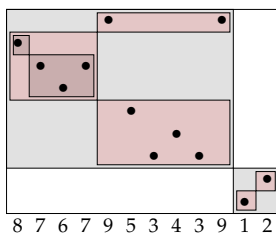


- for any irreducible packed word w , let $w = u \blacktriangleright v$ be the red-factorization of w . We define $T_{\text{Rske}}(w) := \text{Node}_{\mathbf{R}}(v, F_{\text{Rske}}(u), [\])$.



Example 59. Let $w = 876795343912$, the global descent decomposition of Lemma 16 gives $w = w_1/w_2$ with $w_1 = 6545731217$ and $w_2 = 12$. Now, we have the red-factorization of w_1 and w_2 using Lemma 42 as

$$w_1 = 3212 \blacktriangleright \phi_{1,6}(3121) = (1/212) \blacktriangleright 431214 \quad \text{and} \quad w_2 = 1 \blacktriangleright \phi_1(\epsilon) = 1 \blacktriangleright 1.$$



It gives the following forest:

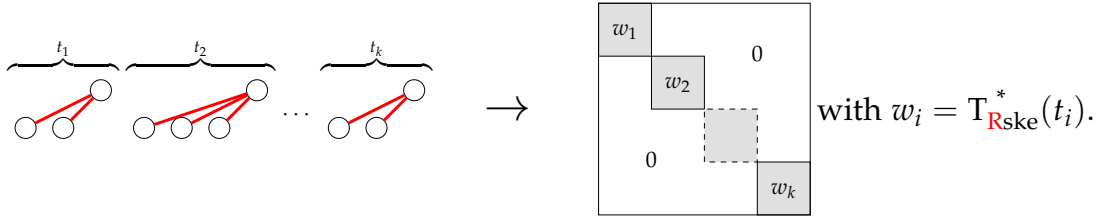
$$\begin{aligned} F_{\text{Rske}}(876795343912) &= [T_{\text{Rske}}(6545731217), T_{\text{Rske}}(12)] \\ &= F_{\text{Rske}}(3212) \quad F_{\text{Rske}}(1) \\ &= \text{Forest with nodes } (1), (212), (431214) \quad \text{Forest with nodes } (1), (1). \end{aligned}$$

Definition 60. A labeled biplane forest (resp. tree) is a **red-skeleton forest** (resp. **tree**) if it is labeled by red-irreducible words and no node has a right child.

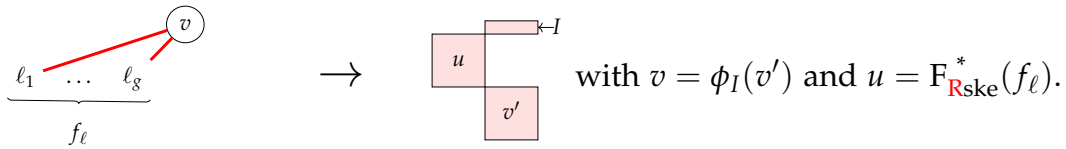
We want to prove that the functions F_{Rske} and T_{Rske} are bijections. To do that we first define two functions that are the inverses.

Definition 61. We now define two functions F_{Rske}^* and T_{Rske}^* that transform respectively a red-skeleton forest and tree into packed words. These functions are defined in a mutual recursive way as follows:

- $F_{\text{Rske}}^*([]) = \epsilon$,
- for any red-skeleton forest $f = [t_1, \dots, t_k]$, we define $F_{\text{Rske}}^*(f) := T_{\text{Rske}}^*(t_1) / \dots / T_{\text{Rske}}^*(t_k)$.



- for any red-skeleton tree $t = \text{Node}_{\text{R}}(v, f_{\ell}, [])$, we define $T_{\text{Rske}}^*(t) := F_{\text{Rske}}^*(f_{\ell}) \blacktriangleright v$.



Lemma 62. The functions F_{Rske} and F_{Rske}^* (resp. T_{Rske} and T_{Rske}^*) are two converse bijections between packed words and red-skeleton forests (resp. irreducible packed words and red-skeleton trees). That is to say $F_{\text{Rske}}^{-1} = F_{\text{Rske}}^*$ and $T_{\text{Rske}}^{-1} = T_{\text{Rske}}^*$.

Proof. We start to prove that domain and codomain are as announced (see Items (a) and (b) bellow), then we prove that the functions F_{Rske} and F_{Rske}^* (resp. T_{Rske} and T_{Rske}^*) are inverse to each other (see Items (c) and (d) bellow).

(a) By [Definition 58](#), a forest (resp. tree) obtain by F_{Rske} (resp. T_{Rske}) is a red-skeleton forest (resp. tree). Indeed, thanks to [Proposition 50](#) nodes are labeled by red-irreducible words because a red-factorization is done and nodes have no right children.

(b) We prove by a mutual induction that F_{Rske}^* returns a packed word and that T_{Rske}^* returns an irreducible packed word. Indeed, we do an induction on the size of the forest or the tree. Here is our induction hypothesis for $n \in \mathbb{N}$:

$$\begin{aligned} \forall t, \text{red-skeleton tree of size } \leq n, T_{\text{Rske}}^*(t) \text{ is an irreducible packed word,} \\ \forall f, \text{red-skeleton forest of size } \leq n, F_{\text{Rske}}^*(f) \text{ is a packed word.} \end{aligned} \quad (2.1)$$

The base case ($n = 0$) is given by the first item of [Definition 61](#) (i.e. $F_{\text{Rske}}^*([]) = \epsilon$ the empty packed word).

Now, let us fix $n \geq 1$ and suppose that the hypothesis (2.1) holds. Let $f = [t_1, \dots, t_k]$ be a red-skeleton forest of size $n + 1$.

- If $k = 1$, then f is reduced to a single tree t . We need to prove that $T_{\text{Rske}}^*(t)$ is an irreducible packed word (which also gives that $F_{\text{Rske}}^*(f)$ is a packed word as in this case $F_{\text{Rske}}^*(f) = T_{\text{Rske}}^*(t)$). Let $t = \text{Node}_{\mathbf{R}}(v, f_\ell, [])$ be a red-skeleton tree of size $n + 1$ (notice that the word v can be of any size). The size of f_ℓ is n , so by induction $F_{\text{Rske}}^*(f_\ell)$ is a packed word, as v is red-irreducible it is by definition irreducible so by [Corollary 49](#) $T_{\text{Rske}}^*(t) = F_{\text{Rske}}^*(f_\ell) \blacktriangleright v$ is irreducible.

- If $k \geq 2$, i.e., the forest contains at least two trees, since all trees are of size at least one, t_1, \dots, t_k are at most of size n , so we have by induction that $T_{\text{Rske}}^*(t_1), \dots, T_{\text{Rske}}^*(t_k)$ are irreducible packed words. $F_{\text{Rske}}^*(f)$ is the shifted concatenation of $T_{\text{Rske}}^*(t_1), \dots, T_{\text{Rske}}^*(t_k)$ and thus it is a packed word.

(c) We now prove by a mutual induction on the size of the forest or the tree that, for $n \in \mathbb{N}$:

$$\begin{aligned} \forall t, \text{red-skeleton tree of size } \leq n, T_{\text{Rske}}(T_{\text{Rske}}^*(t)) &= t, \\ \forall f, \text{red-skeleton forest of size } \leq n, F_{\text{Rske}}(F_{\text{Rske}}^*(f)) &= f. \end{aligned} \quad (2.2)$$

The base case ($n = 0$) is given by the first item of [Definitions 58](#) and [61](#) as $F_{\text{Rske}}(F_{\text{Rske}}^*([])) = F_{\text{Rske}}(\epsilon) = []$.

Now let us fix $n \geq 1$ and suppose that the hypothesis (2.2) holds. Let $f = [t_1, \dots, t_k]$ be a red-skeleton forest of size $n + 1$.

- If $k = 1$, then the forest f is reduced to a single tree t , then it is sufficient to prove $T_{\text{Rske}}(T_{\text{Rske}}^*(t)) = t$ (as in this case $F_{\text{Rske}}^*(f) = T_{\text{Rske}}^*(t)$). Let $t = \text{Node}_{\mathbf{R}}(v, f_\ell, [])$ a red-skeleton tree of size $n + 1$. As the label v is a red-irreducible packed word, with the induction hypothesis on $F_{\text{Rske}}^*(f_\ell)$ and with [Proposition 50](#), $F_{\text{Rske}}^*(f_\ell) \blacktriangleright v$ is the red-factorization so:

$$\begin{aligned} T_{\text{Rske}}(T_{\text{Rske}}^*(t)) &= T_{\text{Rske}}(F_{\text{Rske}}^*(f_\ell) \blacktriangleright v) \\ &= \text{Node}_{\mathbf{R}}(v, F_{\text{Rske}}(F_{\text{Rske}}^*(f_\ell), [])) \\ &= \text{Node}_{\mathbf{R}}(v, f_\ell, []) = t. \end{aligned}$$

- If $k \geq 2$, since all trees are of size at least one, they are at most of size n , so we have by induction that:

$$\begin{aligned} F_{\text{Rske}}(F_{\text{Rske}}^*(f)) &= F_{\text{Rske}}(T_{\text{Rske}}^*(t_1) / \dots / T_{\text{Rske}}^*(t_k)) \\ &\quad \text{as } T_{\text{Rske}}^*(t_i) \text{ are irreducible packed words} \\ &= [T_{\text{Rske}}(T_{\text{Rske}}^*(t_1)), \dots, T_{\text{Rske}}(T_{\text{Rske}}^*(t_k))] \\ &= [t_1, \dots, t_k] = f. \end{aligned}$$

(d) Finally we prove by a mutual induction on the size of the word w that, for $n \in \mathbb{N}$:

$$\begin{aligned} \forall v \in \mathbf{PW}, \text{irreducible packed word of size } \leq n, T_{\mathbf{R}\text{ske}}^*(T_{\mathbf{R}\text{ske}}(v)) &= v, \\ \forall w \in \mathbf{PW}, \text{packed word of size } \leq n, F_{\mathbf{R}\text{ske}}^*(F_{\mathbf{R}\text{ske}}(w)) &= w. \end{aligned} \quad (2.3)$$

The base case ($n = 0$) is given by the first item of [Definitions 58](#) and [61](#) as $F_{\mathbf{R}\text{ske}}^*(F_{\mathbf{R}\text{ske}}(\epsilon)) = F_{\mathbf{R}\text{ske}}^*([]) = \epsilon$.

Now let us fix $n \geq 1$ and suppose that the hypothesis (2.3) holds. Let $w \in \mathbf{PW}_{n+1}$ a packed word of size $n + 1$. Let $w = w_1/w_2/\dots/w_k$ be the global descent decomposition of w .

• If $k = 1$, the packed word w is irreducible then $F_{\mathbf{R}\text{ske}}(w) = [T_{\mathbf{R}\text{ske}}(w)]$ so we need to prove that $T_{\mathbf{R}\text{ske}}^*(T_{\mathbf{R}\text{ske}}(w)) = w$. Let $w = u \blacktriangleright v$ be the red-factorization of w , then we can use the induction hypothesis on u , indeed as v is not empty the size of u is smaller than n :

$$\begin{aligned} T_{\mathbf{R}\text{ske}}^*(T_{\mathbf{R}\text{ske}}(w)) &= T_{\mathbf{R}\text{ske}}^*(\text{Node}_{\mathbf{R}}(v, F_{\mathbf{R}\text{ske}}(u), [])) \\ &= F_{\mathbf{R}\text{ske}}^*(F_{\mathbf{R}\text{ske}}(u)) \blacktriangleright v \\ &= u \blacktriangleright v = w. \end{aligned}$$

• If $k \geq 2$, then we use the induction hypothesis on each factors, so we have:

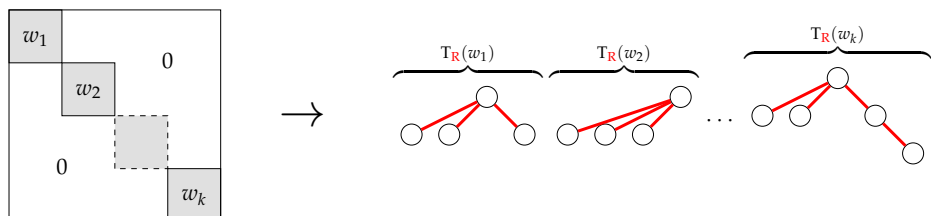
$$\begin{aligned} F_{\mathbf{R}\text{ske}}^*(F_{\mathbf{R}\text{ske}}(w)) &= F_{\mathbf{R}\text{ske}}^*([T_{\mathbf{R}\text{ske}}(w_1), \dots, T_{\mathbf{R}\text{ske}}(w_k)]) \\ &= T_{\mathbf{R}\text{ske}}^*(T_{\mathbf{R}\text{ske}}(w_1)) / \dots / T_{\mathbf{R}\text{ske}}^*(T_{\mathbf{R}\text{ske}}(w_k)) \\ &= w_1 / \dots / w_k = w. \end{aligned} \quad \square$$

Now that we have the red-skeleton, we will add right forests to every nodes to obtain biplane trees. For every node, if v is the red-irreducible word in label, with $\phi_I(v') = v$, then I is the new label and $F_{\mathbf{R}}(v')$ is the new right forest.

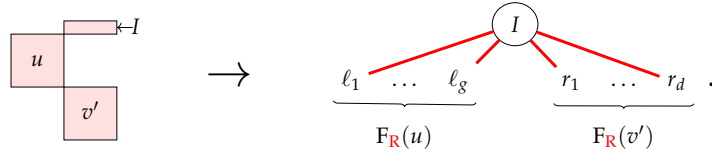
Here is the formal definition of $F_{\mathbf{R}}(w)$ and $T_{\mathbf{R}}(w)$ which are very similar to [Definition 58](#), only the third item is different. The labels are now lists of integers.

Definition 63. The forest $F_{\mathbf{R}}(w)$ (resp. tree $T_{\mathbf{R}}(w)$) associated to a packed word (resp. irreducible packed word) w are defined in a mutual recursive way as follows:

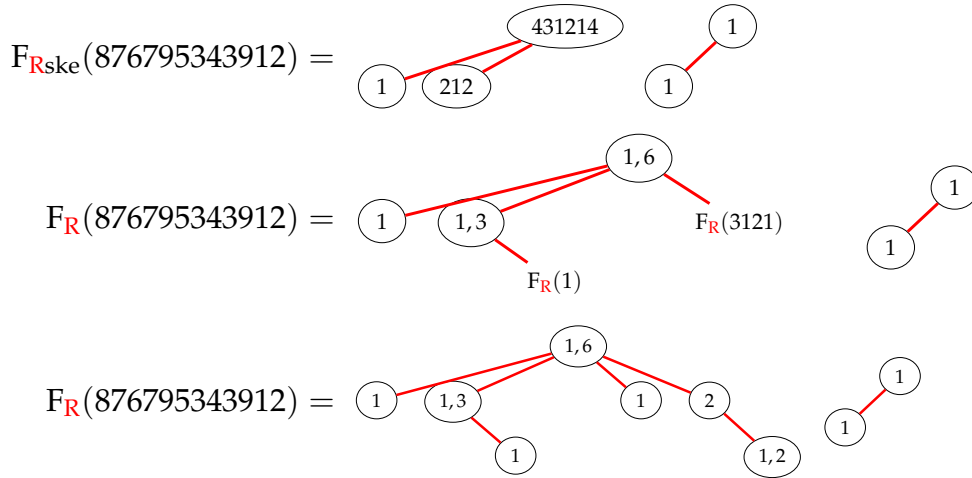
- $F_{\mathbf{R}}(\epsilon) = []$ (empty forest),
- for any packed word w , let $w_1/w_2/\dots/w_k$ be the global descent decomposition of w , then $F_{\mathbf{R}}(w) := [T_{\mathbf{R}}(w_1), T_{\mathbf{R}}(w_2), \dots, T_{\mathbf{R}}(w_k)]$.



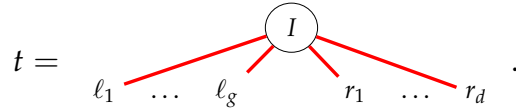
- for any irreducible packed word w , we define $T_{\mathbf{R}}(w) := \text{Node}_{\mathbf{R}}(I, F_{\mathbf{R}}(u), F_{\mathbf{R}}(v'))$ where $w = u \blacktriangleright \phi_I(v')$ is the red-factorization of w .



Example 64. Consider again $w = 876795343912$. We start from the red-skeleton forest from [Example 59](#).



Definition 65. Let t be a labeled biplane tree. We write $t = \text{Node}_{\mathbf{R}}(I, f_{\ell}, f_r)$ where $I = [i_1, \dots, i_p]$, $p > 0$, $(1 \leq i_1 < \dots < i_p)$, $f_{\ell} = [\ell_1, \dots, \ell_g]$ and $f_r = [r_1, \dots, r_d]$, which is depicted as follows:



The **weight** of t is recursively defined by $\omega(t) = p + \sum_{i=1}^g \omega(\ell_i) + \sum_{j=1}^d \omega(r_j)$. In particular, if t is a single node then $\omega(t) = p$. By extension, the **weight** of a forest is the sum of the weight of its trees.

Lemma 66. *The weight of a forest (resp. a tree) obtained by the functions $F_{\mathbf{R}}$ (resp. $T_{\mathbf{R}}$) is equal to the size of the word, i.e. For all $w \in \mathbf{PW}$ then $\omega(F_{\mathbf{R}}(w)) = |w|$ and for all $w \in \mathbf{PW}$ with w irreducible then $\omega(T_{\mathbf{R}}(w)) = |w|$.*

Proof. We prove by induction with the following hypothesis, for $n \in \mathbb{N}$:

$$\begin{aligned} \forall w \in \mathbf{PW}_n, \quad \omega(F_{\mathbf{R}}(w)) &= |w|, \\ \forall w \in \mathbf{PW}_n \text{ with } w \text{ irreducible, } \quad \omega(T_{\mathbf{R}}(w)) &= |w|. \end{aligned} \tag{2.4}$$

The base case is given by the first item of [Definition 63](#) as $F_{\mathbf{R}}(\epsilon) = []$ and $\omega([]) = |\epsilon| = 0$. Let us fix $n \geq 1$ and suppose that the hypothesis (2.4) holds. Let $w \in \mathbf{PW}_{n+1}$ and $w = w_1/w_2/\dots/w_k$ be the global descent decomposition of w .

• If $k = 1$, we have $F_{\mathbf{R}}(w) = [T_{\mathbf{R}}(w)]$. Let $w = u \blacktriangleright \phi_I(v)$ with $I = [i_1, \dots, i_p], p > 0$ be the red-factorization of w , then

$$\omega(T_{\mathbf{R}}(w)) = \omega(\text{Node}_{\mathbf{R}}(I, F_{\mathbf{R}}(u), F_{\mathbf{R}}(v))) = p + \omega(F_{\mathbf{R}}(u)) + \omega(F_{\mathbf{R}}(v)).$$

As $p > 0$, the sizes of u and v are at most n , by induction $\omega(T_{\mathbf{R}}(w)) = p + |u| + |v| = |w|$.

• If $k \geq 2$, by induction on each factors, we have that

$$\omega(F_{\mathbf{R}}(w)) = \omega(T_{\mathbf{R}}(w_1)) + \dots + \omega(T_{\mathbf{R}}(w_k)) = |w_1| + \dots + |w_k| = |w|. \quad \square$$

Definition 67. Using the same notations as in previous [Definition 65](#), we say that t is a **red-packed tree** if it satisfies:

$$\left\{ \begin{array}{l} d = 0, \\ i_k = k \text{ for all } k \leq p, \\ \ell_1, \dots, \ell_g \text{ are red-packed trees.} \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} d \geq 1, \\ 1 \leq i_1 \leq \omega(r_1), \\ 1 \leq p + \omega(f_r) + 1 - i_p \leq \omega(r_d), \\ \ell_1, \dots, \ell_g \text{ and } r_1, \dots, r_d \text{ are red-packed trees.} \end{array} \right.$$

An ordered list of red-packed trees is a **red-packed forest**.

Remark 68. Red-skeleton trees can be interpreted as flattened representations of red-packed trees. Symmetrically, red-packed trees can be interpreted as unfolded representations of red-skeleton trees. We use the operation ϕ_I to change between red-packed and red-skeleton trees.

Notations 69. From now on, we use these notations:

- $\mathfrak{F}_{\mathbf{R}n}$ the set of red-packed forests of weight n , ($\mathfrak{F}_{\mathbf{R}n} := \{F_{\mathbf{R}}(w)\}_{w \in \mathbf{PW}_n}$),
- $\mathfrak{T}_{\mathbf{R}n}$ the set of red-packed trees of weight n , ($\mathfrak{T}_{\mathbf{R}n} := \{T_{\mathbf{R}}(w)\}_{w \in \mathbf{PW}_n}$ with w irreducible),
- $\mathfrak{N}_{\mathbf{R}n}$ the set of red-packed trees of weight n such that the left forest of the root is empty, ($\mathfrak{N}_{\mathbf{R}n} := \{T_{\mathbf{R}}(w)\}_{w \in \mathbf{PW}_n}$ with w red-irreducible). In particular, the red-skeleton of a tree of $\mathfrak{N}_{\mathbf{R}n}$ consist of a single node labeled by a red-irreducible word.

Remark 70. The set $\mathfrak{N}_{\mathbf{R}n}$ can be described as a disjointed union of sets depending on $I = [i_1, \dots, i_p]$. Let $\mathfrak{F}_{\mathbf{R}n}^I$ denote the set of red-packed forests of weight n that can be right children of a node labeled by I (see [Definition 67](#) for conditions), we have the following description:

$$\mathfrak{N}_{\mathbf{R}n} = \bigsqcup_I \{\text{Node}_{\mathbf{R}}(I, [], f_r) \mid f_r \in \mathfrak{F}_{\mathbf{R}n-p}^I\}. \quad (2.5)$$

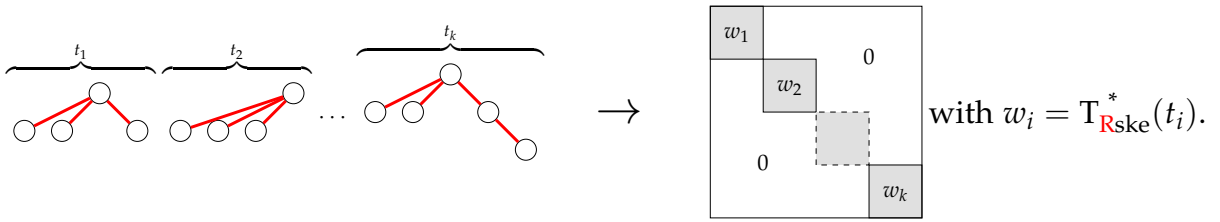
Analogously, we use $\mathfrak{F}_{R_{ske}}$, $\mathfrak{T}_{R_{ske}}$ and $\mathfrak{N}_{R_{ske}}$ for red-skeleton forests, trees and trees with only one node.

We can remark that for $n = 1$ we have $\mathfrak{N}_{R_1} = \mathfrak{T}_{R_1} = \mathfrak{F}_{R_1}$ and $\forall n > 1, \mathfrak{N}_{R_n} \subsetneq \mathfrak{T}_{R_n} \subsetneq \mathfrak{F}_{R_n}$.

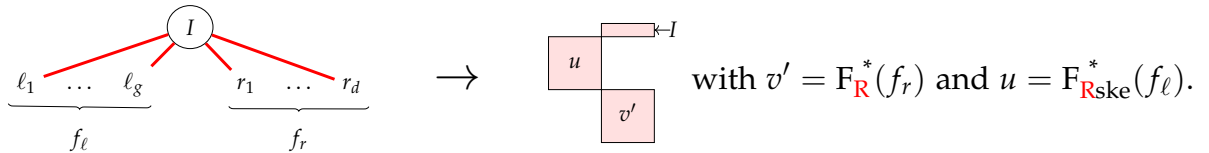
As with [Definition 61](#), we want to prove that the functions F_R and T_R are bijections. To do that we first define the two inverse functions.

Definition 71. We define here the functions F_R^* (resp. T_R^*) that transform a red-packed forest f (resp. tree t) into a packed word. We reverse all instructions of [Definition 63](#) as follows:

- $F_R^*([]) = \epsilon$,
- for any non empty red-packed forest $f = [t_1, t_2, \dots, t_k]$, then $F_R^*(f) = T_R^*(t_1) / T_R^*(t_2) / \dots / T_R^*(t_k)$.



- for any non empty red-packed tree $t = \text{Node}_R(I, f_\ell, f_r)$, then $T_R^*(t) = F_R^*(f_\ell) \blacktriangleright \phi_I(F_R^*(f_r))$.



There might be a problem with this definition since $\phi_I(F_R^*(f_r))$ is only defined if $i_p \leq |F_R^*(f_r)| + p$ (see [Definition 36](#)). We prove in the following [Lemma 72](#) that the inequality holds if $t \in \mathfrak{T}_R$.

Lemma 72. For any red-packed forest f , $F_R^*(f)$ is a well defined word of size $\omega(f)$. For any red-packed tree t , $T_R^*(t)$ is a well defined word of size $\omega(t)$.

Proof. We prove by induction with the following hypothesis, for $n \in \mathbb{N}$:

$$\begin{aligned} \forall f \in \mathfrak{F}_{R_{\leq n}}, \quad & F_R^*(f) \text{ is well defined and } |F_R^*(f)| = \omega(f), \\ \forall t \in \mathfrak{T}_{R_{\leq n}}, \quad & T_R^*(t) \text{ is well defined and } |T_R^*(t)| = \omega(t). \end{aligned} \tag{2.6}$$

The base case is given by the first item of [Definition 71](#) as $F_R^*([]) = \epsilon$ and $\omega([]) = |\epsilon| = 0$. Let us fix $n \geq 1$ and suppose that the hypothesis (2.6) holds. Let $f = [t_1, \dots, t_k] \in \mathfrak{F}_{R_{n+1}}$.

• If $k = 1$, it is sufficient to prove the second item of (2.6). Let $t = \text{Node}_{\mathbf{R}}(I, f_\ell, f_r) \in \mathfrak{T}_{\mathbf{R}n+1}$. According to Definition 67 with notations of Definition 65 there are two cases:

• $d = 0$ and $I = [1, \dots, p]$. We have that $i_p = p$ and $|\mathbf{F}_{\mathbf{R}}^*(f_r)| = |\epsilon| = 0$ so $i_p \leq 0 + p$ and $\phi_I(\epsilon) = 11\dots 11$ of size p . Now by induction on f_ℓ , we have that $\mathbf{F}_{\mathbf{R}}^*(f_\ell)$ is a well defined word of size $\omega(f_\ell)$. Finally $\mathbf{T}_{\mathbf{R}}^*(t) = \mathbf{F}_{\mathbf{R}}^*(f_\ell) \blacktriangleright \phi_I(\epsilon) = \mathbf{F}_{\mathbf{R}}^*(f_\ell) \setminus \phi_I(\epsilon)$ is a well defined word of size $|\mathbf{F}_{\mathbf{R}}^*(f_\ell)| + p = \omega(t)$.

• $d \geq 1$. As $p > 0$ we can apply the hypothesis (2.6) on f_r and f_ℓ . According to Definition 67 we have that

$$\begin{aligned} 1 &\leq p + \omega(f_r) + 1 - i_p, \\ i_p &\leq p + |\mathbf{F}_{\mathbf{R}}^*(f_r)|. \end{aligned}$$

So $\mathbf{T}_{\mathbf{R}}^*(t)$ is well defined. Moreover

$$\begin{aligned} |\mathbf{T}_{\mathbf{R}}^*(t)| &= |\mathbf{F}_{\mathbf{R}}^*(f_\ell) \blacktriangleright \phi_I \mathbf{F}_{\mathbf{R}}^*(f_r)| \\ &= |\mathbf{F}_{\mathbf{R}}^*(f_\ell)| + p + |\mathbf{F}_{\mathbf{R}}^*(f_r)| \\ &= \omega(f_\ell) + p + \omega(f_r) = \omega(t). \end{aligned}$$

• If $k \geq 2$, the weight of trees are at least 1 so we can apply (2.6) on trees of f . □

Theorem 73. *The functions $\mathbf{F}_{\mathbf{R}}$ and $\mathbf{F}_{\mathbf{R}}^*$ (resp. $\mathbf{T}_{\mathbf{R}}$ and $\mathbf{T}_{\mathbf{R}}^*$) are two converse bijections between packed words of size n and red-packed forests (resp. irreducible packed words and red-packed trees) of weight n . That is to say $\mathbf{F}_{\mathbf{R}}^{-1} = \mathbf{F}_{\mathbf{R}}^*$ and $\mathbf{T}_{\mathbf{R}}^{-1} = \mathbf{T}_{\mathbf{R}}^*$.*

Proof. The proof is very similar to the one of Lemma 62. Indeed, we start to prove that domain and codomain are as announced (see Items (a) and (b) bellow), then we prove that the functions $\mathbf{F}_{\mathbf{R}}$ and $\mathbf{F}_{\mathbf{R}}^*$ (resp. $\mathbf{T}_{\mathbf{R}}$ and $\mathbf{T}_{\mathbf{R}}^*$) are inverse to each other (see Items (c) and (d) bellow).

We now give the differences with the proof of Lemma 62 and we advise the reader to read the two proofs in parallel. While for Item (a) in Lemma 62 it was simple, we need to do an induction here to prove that conditions on labels are respected. For Items (b), (c) and (d), the same inductions are done with one additional argument, so only the different argument of the induction is explicited here.

(a) We prove by a mutual induction that $\mathbf{F}_{\mathbf{R}}$ returns a red-packed forest and that $\mathbf{T}_{\mathbf{R}}$ returns a red-packed tree. Indeed, we do an induction on the size of the word w . Here is our induction hypothesis for $n \in \mathbb{N}$:

$$\begin{aligned} \forall v \in \mathbf{PW}, \text{irreducible packed word of size } \leq n, \quad \mathbf{T}_{\mathbf{R}}(v) \text{ is a red-packed tree,} \\ \forall w, \text{packed word of size } \leq n, \quad \mathbf{F}_{\mathbf{R}}(w) \text{ is a red-packed forest.} \end{aligned} \tag{2.7}$$

The base case ($n = 0$) is given by the first item of Definition 63.

Now let us fix $n \geq 1$ and suppose that the hypothesis (2.7) holds. Let $w \in \mathbf{PW}$ be a packed word of size $n + 1$ and let $w_1 / \cdots / w_k$ be the global descent decomposition of w .

- If $k = 1$ (w is irreducible), then $F_{\mathbf{R}}(w)$ is reduced to a single tree $T_{\mathbf{R}}(w)$. We need to prove that $T_{\mathbf{R}}(w)$ is a red-packed tree (which also gives that $F_{\mathbf{R}}(w)$ is a red-packed forest). Let $w = u \blacktriangleright v$ be the red-factorization of w . With $\phi_I(v') = v$, $f_\ell = F_{\mathbf{R}}(u)$ and $f_r = F_{\mathbf{R}}(v')$ we have that $T_{\mathbf{R}}(w) = \text{Node}_{\mathbf{R}}(I, f_\ell, f_r)$. The inequalities on I and v' in Lemma 42 are the same as the inequalities on I and f_r in Definition 67. Therefore by Lemma 66 and (2.7) on f_r , we have that $T_{\mathbf{R}}(w)$ belongs to $\mathfrak{T}_{\mathbf{R}}$.

- If $k \geq 2$, the hypothesis (2.7) can be applied to each factors.

(b) Compared to the proof of Lemma 62 we use the same general arguments to prove that $F_{\mathbf{R}}^*$ and $F_{\mathbf{R}_{\text{ske}}}^*$ return a packed word. First of all the base case and the case where the size of the forest is $k \geq 2$ are dealt with by a similar argumentation. It remains to prove that $T_{\mathbf{R}}^*$ returns an irreducible packed word. We thus suppose that the induction hypothesis (2.8) holds for a given $n \in \mathbb{N}$:

$$\begin{aligned} \forall t, \text{red-packed tree of size } \leq n, \quad T_{\mathbf{R}}^*(t) \text{ is an irreducible packed word,} \\ \forall f, \text{red-packed forest of size } \leq n, \quad F_{\mathbf{R}}^*(f) \text{ is a packed word.} \end{aligned} \quad (2.8)$$

Let $t = \text{Node}_{\mathbf{R}}(I, f_\ell, f_r)$ be a red-packed tree of size $n + 1$. By induction we have that $F_{\mathbf{R}}^*(f_\ell)$ and $F_{\mathbf{R}}^*(f_r)$ are packed words. Moreover either $f_r = \epsilon$ and $I = [1, \dots, p]$ or $1 \leq p + \omega(f_r) + 1 - i_p \leq \omega(r_d)$ with $I = [i_1, \dots, i_p]$ and $f_r = [r_1, \dots, r_d]$. In both cases, $\phi_I(F_{\mathbf{R}}^*(f_r))$ is an red-irreducible packed word. Indeed, we recognize the two cases of Definition 45 with the same inequalities. Finally $T_{\mathbf{R}}^*(t) = F_{\mathbf{R}}^*(f_\ell) \blacktriangleright \phi_I(F_{\mathbf{R}}^*(f_r))$ is an irreducible packed word according to Corollary 49.

(c) We now want to prove that for any forest f (resp. t), $F_{\mathbf{R}}(F_{\mathbf{R}}^*(f)) = f$ (resp. $T_{\mathbf{R}}(T_{\mathbf{R}}^*(t)) = t$). As in Item (b), the arguments are the same as is the proof of Lemma 62 for $F_{\mathbf{R}}$ and $F_{\mathbf{R}_{\text{ske}}}$. In the case of $T_{\mathbf{R}}$ the new arguments are the same as in Item (b) (i.e. $\phi_I(F_{\mathbf{R}}^*(f_r))$ is a red-irreducible packed word).

(d) Finally, we want to prove that for any packed word w (resp. irreducible packed word v), $F_{\mathbf{R}}^*(F_{\mathbf{R}}(w)) = w$ (resp. $T_{\mathbf{R}}^*(T_{\mathbf{R}}(v)) = v$). Once again, the only difference with Lemma 62 is the former second case. It remains to prove that point and we thus suppose that the induction hypothesis (2.9) holds for $n \in \mathbb{N}$:

$$\begin{aligned} \forall v \in \mathbf{PW}, \text{irreducible packed word of size } \leq n, \quad T_{\mathbf{R}}^*(T_{\mathbf{R}}(v)) = v, \\ \forall w \in \mathbf{PW}, \text{packed word of size } \leq n, \quad F_{\mathbf{R}}^*(F_{\mathbf{R}}(w)) = w. \end{aligned} \quad (2.9)$$

Let v be an irreducible packed word of size $n + 1$. Let $v = v' \blacktriangleright \phi_I(v'')$ be the red-factorization of v . We have by Definition 63 that $T_{\mathbf{R}}(v) = \text{Node}_{\mathbf{R}}(I, F_{\mathbf{R}}(v'), F_{\mathbf{R}}(v''))$. As $|I| > 0$, the sizes of v' and v'' are smaller than n so we can apply (2.9). We have:

$$\begin{aligned} T_{\mathbf{R}}^*(T_{\mathbf{R}}(v)) &= T_{\mathbf{R}}^*(\text{Node}_{\mathbf{R}}(I, F_{\mathbf{R}}(v'), F_{\mathbf{R}}(v''))) \\ &= F_{\mathbf{R}}^*(F_{\mathbf{R}}(v')) \blacktriangleright \phi_I(F_{\mathbf{R}}^*(F_{\mathbf{R}}(v''))) \\ &= v' \blacktriangleright \phi_I(v'') = v. \end{aligned} \quad \square$$

Example 74. There is a unique forest in \mathfrak{F}_{R1} , namely $\textcircled{1}$, here are the 3 forests of \mathfrak{F}_{R2} with the associated packed word: $F_R(12) = \textcircled{1} \textcircled{1}$, $F_R(21) = \textcircled{1} \textcircled{1}$, $F_R(11) = \textcircled{1,2}$. We show below the 13 forests of \mathfrak{F}_{R3} with the corresponding packed word:

$$\begin{aligned}
 F_R(123) &= \textcircled{1} \textcircled{1} \textcircled{1}, & F_R(132) &= \textcircled{2} \textcircled{1} \textcircled{1}, & F_R(213) &= \textcircled{1} \textcircled{1} \textcircled{1}, & F_R(231) &= \textcircled{1} \textcircled{1} \textcircled{1}, \\
 F_R(312) &= \textcircled{1} \textcircled{1} \textcircled{1}, & F_R(321) &= \textcircled{1} \textcircled{1} \textcircled{1}, & F_R(122) &= \textcircled{1} \textcircled{1,2}, & F_R(212) &= \textcircled{1,3} \textcircled{1}, \\
 F_R(221) &= \textcircled{1,2} \textcircled{1}, & F_R(112) &= \textcircled{1,2} \textcircled{1}, & F_R(121) &= \textcircled{2} \textcircled{1,2}, & F_R(211) &= \textcircled{1} \textcircled{1,2}, \\
 F_R(111) &= \textcircled{1,2,3}.
 \end{aligned}$$

More examples can be found in the annexes section with [Tables 7 to 12](#).

We conclude by the main theorem of this subsection. It is a generalization of the construction of [\[Foi11\]](#) for **FQSym** and permutations to **WQSym** and packed words. Indeed, if we restrict the construction on permutations and we consider right children of a node as label of this node, we have the same construction as in [\[Foi11\]](#) with a shift of 1 for labels. Here are some examples of trees in [\[Foi11\]](#) and the equivalent red-packed tree.

\cdot_{τ} with $T = (\vee, 1)$		\cdot_{τ} with $T = (\cdot_{\tau}, 1)$ with $T' = (\uparrow, 1)$	
\cdot_{τ} with $T = (\vee, 2)$		\cdot_{τ} with $T = (\cdot_{\tau}, 2)$ with $T' = (\uparrow, 1)$	
\cdot_{τ} with $T = (\uparrow, 1)$		\uparrow_{τ} with $T = (\uparrow, 1)$	
\cdot_{τ} with $T = (\uparrow, 2)$		\uparrow_{τ} with $T = (\uparrow, 1)$	

Table 5: Equivalence between trees of [\[Foi11\]](#) and red-packed trees.

All the constructions with red-packed forests have been done in order to have this

theorem.

Theorem 75. For all $n \in \mathbb{N}$ we have the three following equalities :

$$\dim(\mathbf{WQSym}_n^*) = \#\mathfrak{F}_{R_n} \quad \text{and} \quad \dim(\mathbf{Prim}_n^*) = \#\mathfrak{T}_{R_n} \quad \text{and} \quad \dim(\mathbf{TPrim}_n^*) = \#\mathfrak{N}_{R_n}$$

Proof. **Theorem 73** proves the first equality. It also gives a relation between $\#\mathfrak{F}_R$ and $\#\mathfrak{T}_R$. Indeed a red-packed forest of weight n is an ordered sequence of red-packed trees of weight (n_k) such that $\sum_k(n_k) = n$. This relation is the same between $\dim(\mathbf{WQSym}_n^*)$ and $\dim(\mathbf{Prim}_n^*)$ according to **Proposition 1** (i.e. $\mathcal{A} = \mathcal{P}/(1 - \mathcal{P})$).

Red-skeleton trees are equivalent to ordered trees decorated by red-irreducible words as said in **Remark 68**. Recall that a basis of primitive elements is given by **Theorem 2** as ordered trees decorated by totally primitive elements. Elements of \mathfrak{N}_R are by definition in bijections with red-irreducible words, labels of red-skeleton trees. \square

2.2 Primal (Blue)

Now we do the same work for the primal side: **WQSym**. This subsection follows the same structure of statements as the previous one. Recall that in **Section 2.1** we constructed a bijection between packed words and red-forests by recursively decomposing packed words using global descent and removal of maximums. In this section we follow the same path: inserting the last letter using ψ_{i^α} (lowercase i designates the integer value) instead of new maximums using ϕ_I (uppercase I designates the list of their positions). We define a blue-factorization of packed words. When used recursively, blue-factorization and global descent decomposition construct a bijection between PW and so-called blue-packed forest. Since the general structure of proofs are the same as in the previous section, we will mostly focus on the differences between combinatorial arguments.

2.2.1 Decomposition of packed words through last letter

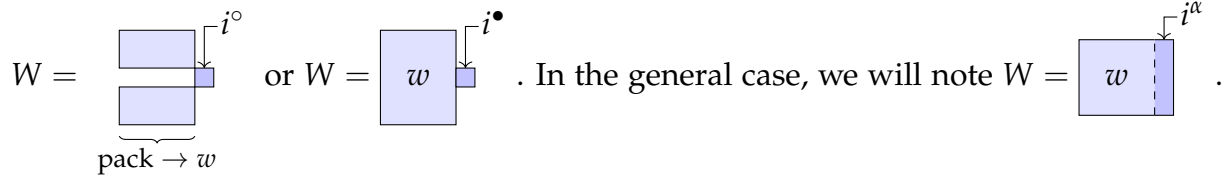
In this section, we define two combinatorial operations on packed words (ψ_{i^α} and \blacktriangle) and the blue-factorization that use them. The unary operation ψ_{i^α} insert the new value i at the end of a given word. A word that cannot be factorized $u \blacktriangle v$ in a non trivial way is called blue-irreducible. Blue-irreducible words will index a new basis of **TPrim**.

Definition 76. Fix $n \in \mathbb{N}$ and $w \in \mathbf{PW}_n$. For any $1 \leq i \leq \max(w) + 1$ (with the convention $\max(\epsilon) = 0$), we denote by $\psi_{i^\circ}(w) = u_1 \cdots u_n \cdot i$ the packed word defined by $u_k = w_k$ if $w_k < i$ and $u_k = w_k + 1$ otherwise. We also define $\psi_{i^\bullet}(w) = w \cdot i$ for any $1 \leq i \leq \max(w)$.

Example 77. $\psi_{2^\circ}(1232) = 13432$, $\psi_{2^\bullet}(1232) = 12322$, $\psi_{4^\circ}(1232) = 12324$ and $\psi_{1^\circ}(\epsilon) = 1$.

Lemma 78. For any $W \in \mathbf{PW}_\ell$ where $\ell > 0$ there exists a unique triplet (i, α, w) where $i \in [1 \dots \ell + 1]$, $\alpha \in \{\circ, \bullet\}$ and w is a packed word, such that $W = \psi_{i^\alpha}(w)$.

Depending on α , the box diagram can be represented as



Proof. Let $W \in \mathbf{PW}_\ell$ with $\ell > 0$ and i the value of the last letter of W .

- If i appears multiple times in W , then let $w = W_1 \dots W_{\ell-1}$, we only remove the last letter i of W . We have $W = \psi_{i^\bullet}(w)$.
- Otherwise, i appears only as the last letter, then let $w = \text{pack}(W_1 \dots W_{\ell-1})$, we remove the last letter i of W and pack the word. We have $W = \psi_{i^\circ}(w)$.

If $\psi_{i^\alpha}(u) = \psi_{j^\beta}(v)$ then the last letter is the same so $i = j$, the multiplicity of this letter is the same so $\alpha = \beta$ and the prefix are the same $u = v$. □

Definition 79. Let $u, v \in \mathbf{PW}$ with $v \neq \epsilon$. By **Lemma 78**, there is a unique triplet (i, α, v') such that $v = \psi_{i^\alpha}(v')$. Let $i' = i + \max(u)$, we define $u \blacktriangle v := \psi_{i'^\alpha}(v'/u)$. In other words, we remove the last letter of the right word, perform a reversed left shifted concatenation and adding back the last letter also shifted.

Example 80. $2123 \blacktriangle 312312 = 2123 \blacktriangle \psi_{(2)^\bullet}(31231) = \psi_{(2+3)^\bullet}(645642123) = 6456421235$.

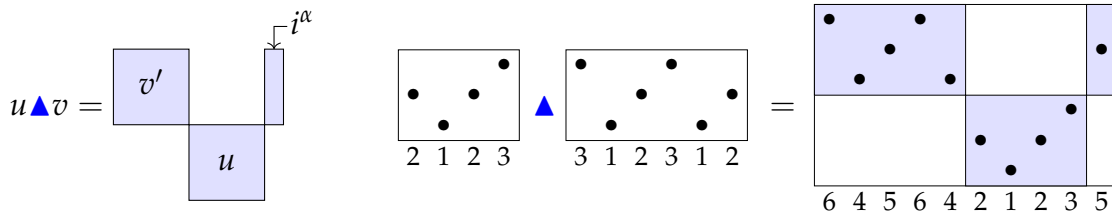


Figure 4: Box digrams: the operation \blacktriangle

Lemma 81. Let w be an irreducible packed word. There exists a unique factorization of the form $w = u \blacktriangle v$ which maximizes the size of u . In this factorization, let v' and i^α such that $v = \psi_{i^\alpha}(v')$,

- either $v' = \epsilon$ and $i^\alpha = 1^\circ$,
- or v' is irreducible and $1 \leq i \leq \max(v')$.

We call it the **blue-factorization** of a word.

Example 82. Here is a first detailed example of a blue-factorization of an irreducible packed word:

Consider the irreducible packed word $w = 654623314$.

- The first step is to remove the last letter $i = 4$. Here there are multiple occurrences of the last letter in w , then $\alpha = \bullet$, we get $w' = 65462331$ which is a packed word, but is not irreducible.
- The second step is to set w'_1 as the first irreducible factor of w' and u the rest of w' . This way $w' = w'_1 / u$ and the size of u is maximized. Here $w'_1 = 3213$ and $u = 2331$. Let $i' = i - \max(u) = 4 - 3 = 1$.

- Finally, we get the following decomposition of w (see [Definition 76](#) for ψ and [Definition 79](#) for \blacktriangle):

$$w = 654623314 = u \blacktriangle \psi_{i'\alpha}(w'_1) = (2331) \blacktriangle \psi_1 \bullet (3213) = 2331 \blacktriangle 32131.$$

Example 83. Here are some other blue-factorizations:

$$\begin{aligned} 234313 &= 1 \blacktriangle \psi_2 \bullet (1232) = 1 \blacktriangle 12322 & 245413 &= 1 \blacktriangle \psi_{2^\circ} (1232) = 1 \blacktriangle 13432 \\ 11 &= \epsilon \blacktriangle \psi_1 \bullet (1) = \epsilon \blacktriangle 11 & 112 &= 11 \blacktriangle \psi_{1^\circ} (\epsilon) = 11 \blacktriangle 1 \end{aligned}$$

Proof. Let w be irreducible and let (i, α, w') be the unique triplet such that $w = \psi_{i\alpha}(w')$ according to [Lemma 78](#).

If $i = \max(w)$ and it appears only one time (*i.e.* $\alpha = \circ$ and $i = \max(w') + 1$) then the blue-factorizations is $w = w' \blacktriangle \psi_{1^\circ} (\epsilon)$.

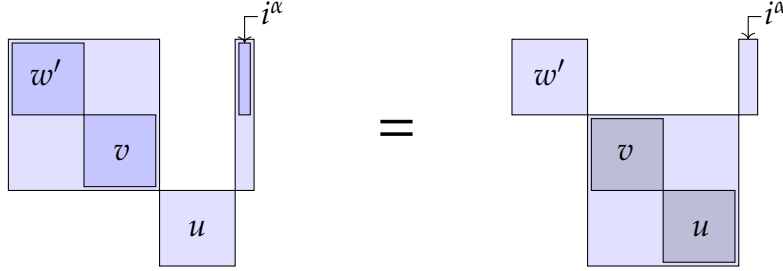
In any other case, we write $w' = w'_1 / w'_2 / \dots / w'_k$, the decomposition into irreducibles. Let $u = w'_2 / \dots / w'_k$ and $i' = i - \max(u)$. We have that $i' \leq \max(w'_1)$ otherwise w wouldn't be packed and $1 \leq i'$ otherwise w wouldn't be irreducible. If $\alpha = \circ$ then $1 \neq i'$ otherwise w wouldn't be irreducible. Then we have $w = u \blacktriangle \psi_{i'\alpha}(w'_1)$ where the size of u is maximized. \square

Remark 84. When restricted to permutations, blue-factorization is equal to a red-factorization applies to the inverse. Let σ be a permutation and $\sigma = \mu \blacktriangleright v$ be the red-factorization of σ , then $\sigma^{-1} = \mu^{-1} \blacktriangle v^{-1}$ is the blue-factorization of σ^{-1} .

Definition 85. A packed word w is **blue-irreducible** if w is irreducible and $w = u \blacktriangle v$ implies that $u = \epsilon$ (and $w = v$).

Here are some useful lemmas on the operation \blacktriangle . There are some similarities with [Lemmas 46](#) and [47](#), [Corollary 49](#), and [Proposition 50](#).

Lemma 86. For any $u, v, w \in \mathbf{PW}$ with $w \neq \epsilon$, we have $u \blacktriangle (v \blacktriangle w) = (v/u) \blacktriangle w$.

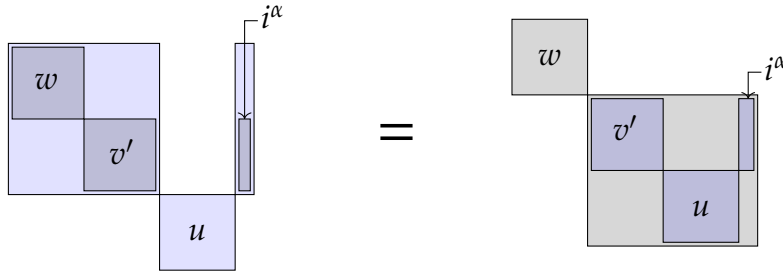


Proof. Let $u, v, w \in \mathbf{PW}$ with $w \neq \epsilon$, and let w' and i^α such that $w = \psi_{i^\alpha}(w')$.

$$\begin{aligned}
 u \blacktriangle (v \blacktriangle w) &= u \blacktriangle (\psi_{(i+\max(v))^\alpha}(w'/v)) \\
 &= \psi_{(i+\max(v)+\max(u))^\alpha}((w'/v)/u) \\
 &= \psi_{(i+\max(v)+\max(u))^\alpha}(w'/(v/u)) \\
 &= (v/u) \blacktriangle \psi_{i^\alpha}(w') \\
 &= (v/u) \blacktriangle w.
 \end{aligned}$$

□

Lemma 87. For any $u, v, w \in \mathbf{PW}$ with $v \neq \epsilon$, we have $u \blacktriangle (w/v) = w/(u \blacktriangle v)$.



Proof. Let $u, v, w \in \mathbf{PW}$ with $v \neq \epsilon$ and let v' and i^α such that $v = \psi_{i^\alpha}(v')$.

$$\begin{aligned}
 u \blacktriangle (w/v) &= u \blacktriangle (w/\psi_{i^\alpha}(v')) \\
 &= u \blacktriangle \psi_{i^\alpha}(w/v') \\
 &= \psi_{(i+\max(u))^\alpha}((w/v')/u) \\
 &= \psi_{(i+\max(u))^\alpha}(w/(v'/u)) \\
 &= w/\psi_{(i+\max(u))^\alpha}(v'/u) \\
 &= w/(u \blacktriangle v).
 \end{aligned}$$

□

Remark 88. These relations are the same up to symmetry as the one with \blacktriangleright (Lemmas 46 and 47). So adding the associativity of shifted concatenation $u/(v/w) = (u/v)/w$, the two operations \blacktriangle and $/$ verify relations of the *skew-duplicial operad* [BDO20].

Corollary 89. For any $u, v \in \mathbf{PW}$, we have that $u \blacktriangle v$ is irreducible if and only if v is irreducible.

Proof. By contradiction, if $v = v_1/v_2$ then by [Lemma 87](#) $u \blacktriangle v = v_1/(u \blacktriangle v_2)$. Now if $u \blacktriangle v = w_1/w_2$ as the value of the last letter of $u \blacktriangle v$ is greater than $\max(u)$ we have that $w_2 = w'_2 \cdot w''_2 \cdot i$ such that $\text{pack}(w''_2) = u$. We also have that $w_1/\text{pack}(w'_2 \cdot i) = v$. \square

Proposition 90. For any word w , $w = u \blacktriangle v$ is the blue-factorization of w if and only if v is blue-irreducible.

Proof. Let $w \in \mathbf{PW}$ and let $u \blacktriangle v$ be the blue-factorization of w . Let v_1 and v_2 such that $v = v_1 \blacktriangle v_2$, then $(v_1/u) \blacktriangle v_2 = w$ by [Lemma 86](#), but in the blue-factorization the size of u is maximized so $|(v_1/u)| \leq |u|$ and then we have that $v_1 = \epsilon$ so v is blue-irreducible.

Let $w \in \mathbf{PW}$ and let u and v such that $w = u \blacktriangle v$ and v is blue-irreducible. By contradiction, suppose that there exists u', v' such that $w = u' \blacktriangle v'$ with $|u| < |u'|$ and $v' \neq \epsilon$. Then necessarily u is a suffix of u' . Let u'' such that $u' = u'' \cdot u$, then $\text{pack}(u'') \blacktriangle v' = v$. But v is blue-irreducible. So the size of u is maximal if v is blue-irreducible. \square

Thanks [Remark 84](#) the following proposition is immediate.

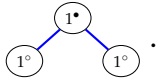
Proposition 91. A permutation σ is blue-irreducible if and only if σ^{-1} is red-irreducible.

2.2.2 Blue-forests from decomposed packed words using ψ

As in [Section 2.1.2](#) we will apply recursively the blue-factorization of the former section to construct a bijection between packed words and a certain kind of labeled biplane trees.

In this construction, the labels can be a blue-irreducible word for skeleton, or an integer with a sign $\alpha \in \{\circ, \bullet\}$. In order to differentiate the trees from the one of the previous section, we will draw them in blue. As before, for a labeled biplane tree, we denote the trees by $\text{Node}_{\mathbf{B}}(x, f_l, f_r)$.

Example 92. $\text{Node}_{\mathbf{B}}(1^\circ, [], []) = \textcircled{1^\circ}$, and $\text{Node}_{\mathbf{B}}(1^\bullet, [\text{Node}_{\mathbf{B}}(1^\circ, [], [])], [\text{Node}_{\mathbf{B}}(1^\circ, [], [])]) =$

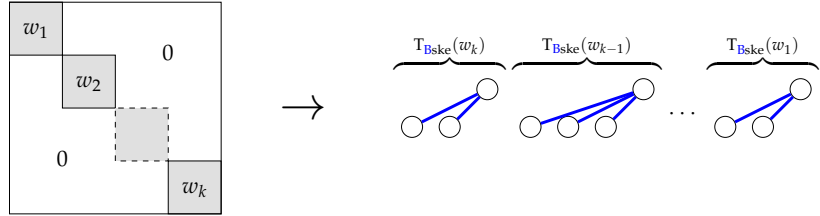


We apply recursively the global descent decomposition and the blue-factorization of [Lemma 81](#). We obtain an algorithm which takes a packed word and returns a biplane forest where nodes are decorated by blue-irreducible words:

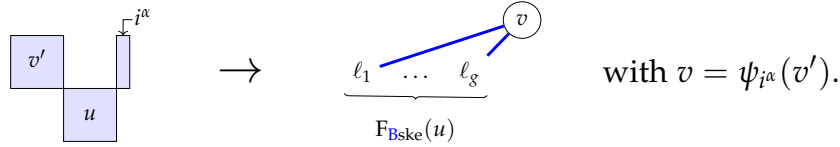
Definition 93. Exactly as [Definition 58](#) of $F_{\mathbf{R}\text{ske}}$ and $T_{\mathbf{R}\text{ske}}$, we now define two functions $F_{\mathbf{B}\text{ske}}$ and $T_{\mathbf{B}\text{ske}}$. These functions transform respectively a packed word and an irreducible packed word into respectively a biplane forest and a biplane tree. These functions are defined in a mutual recursive way as follow:

- $F_{\mathbf{B}\text{ske}}(\epsilon) = []$ (empty forest),

- for any packed word w , let $w_1/w_2/\dots/w_k$ be the global descent decomposition of w , then $F_{\text{Bske}}(w) := [T_{\text{Bske}}(w_k), T_{\text{Bske}}(w_{k-1}), \dots, T_{\text{Bske}}(w_1)]$ (notice the inversion compared to [Definition 58](#)).

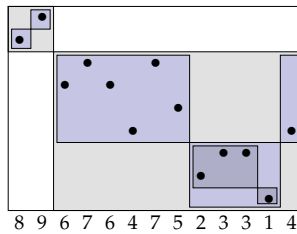


- for any irreducible packed word w , we define $T_{\text{Bske}}(w) := \text{Node}_B(v, F_{\text{Bske}}(u), [])$ where $w = u \blacktriangle v$ is the blue-factorization of w .



Example 94. Let $w = 8967647523314$, here is the global descent decomposition $w = w_1/w_2$ with $w_1 = 12$ and $w_2 = 67647523314$. Now, we have the blue-factorization of w_1 and w_2 using [Lemma 81](#) as

$$w_1 = 1 \blacktriangle \psi_1^\circ(\epsilon). \quad \text{and} \quad w_2 = 2331 \blacktriangle \psi_1^\bullet(343142) = (122/1) \blacktriangle 3431421,$$



It gives the following forest:

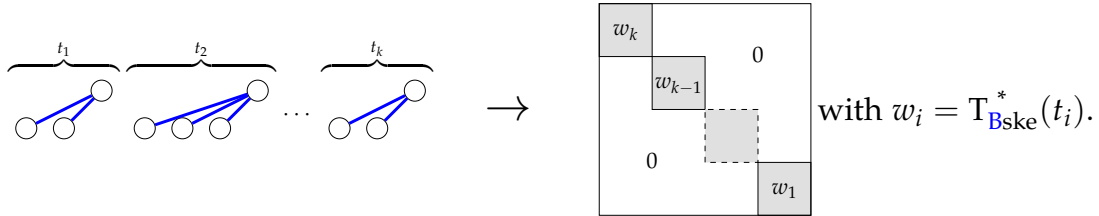
$$\begin{aligned} F_{\text{Bske}}(8967647523314) &= [T_{\text{Bske}}(12), T_{\text{Bske}}(67647523314)] \\ &= F_{\text{Bske}}(2331) \quad F_{\text{Bske}}(1) \\ &= \begin{array}{c} \text{Tree with root } 3431421 \text{ and children } 1, 122 \\ \text{Tree with root } 1 \end{array} \end{aligned}$$

Definition 95. A labeled biplane forest (resp. tree) is a **blue-skeleton forest** (resp. **tree**) if and only if it is labeled by blue-irreducible words and no node has a right child.

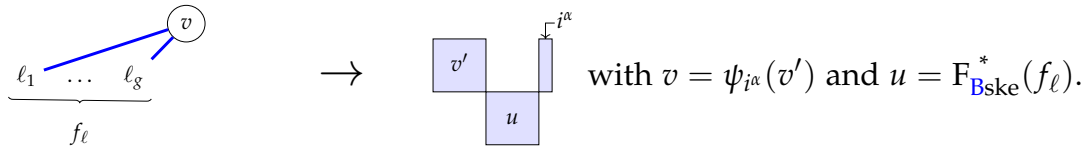
We want to prove that the functions F_{Bske} and T_{Bske} are bijections. To do that, as for F_{Rske} and T_{Rske} , we first define the two inverse functions.

Definition 96. We now define two functions F_{Bske}^* and T_{Bske}^* that transform respectively a blue-skeleton forest and tree into packed words. These functions are defined in a mutual recursive way as follow:

- $F_{\text{Bske}}^*(\emptyset) = \epsilon$,
- for any blue-skeleton forest $f = [t_1, \dots, t_k]$, we define $F_{\text{Bske}}^*(f) := T_{\text{Bske}}^*(t_k) / \dots / T_{\text{Bske}}^*(t_1)$.
(notice the inversion compared to [Definition 61](#))



- for any blue-skeleton tree $t = \text{Node}_{\text{B}}(v, f_r, \emptyset)$, we define $T_{\text{Bske}}^*(t) := F_{\text{Bske}}^*(f_r) \blacktriangle v$.



Lemma 97. The functions F_{Bske} and F_{Bske}^* (resp. T_{Bske} and T_{Bske}^*) are two converse bijections between packed words and blue-skeleton forests (resp. irreducible packed words and blue-skeleton trees). That is to say $F_{\text{Bske}}^{-1} = F_{\text{Bske}}^*$ and $T_{\text{Bske}}^{-1} = T_{\text{Bske}}^*$.

Proof. The proof structure is the same as the one of [Lemma 62](#) with use of statements coming from this subsection. We can see in this table some of the main statements that are exchanged for this dual part:

Lemma 42	Lemma 81	red-factorization and blue-factorization.
Definition 45	Definition 85	red-irreducible words and blue-irreducible words.
Corollary 49	Corollary 89	$u \blacktriangleright v$ irreducible $\iff v$ irreducible $u \blacktriangle v$ irreducible $\iff v$ irreducible.
Proposition 50	Proposition 90	$u \blacktriangleright v$ red-factorization $\iff v$ red-irreducible $u \blacktriangle v$ blue-factorization $\iff v$ blue-irreducible.
Definition 58	Definition 93	$F_{\text{Rske}}, T_{\text{Rske}}$ and $F_{\text{Bske}}, T_{\text{Bske}}$.
Definition 60	Definition 95	red-skeleton forest and blue-skeleton forest.
Definition 61	Definition 96	$F_{\text{Rske}}^*, T_{\text{Rske}}^*$ and $F_{\text{Bske}}^*, T_{\text{Bske}}^*$.

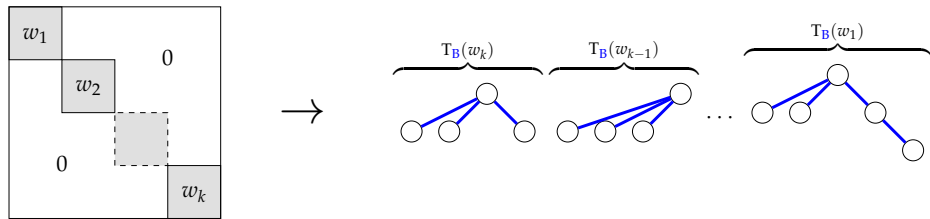
□

Now that we have the blue-skeleton, we will add right forests to every nodes to have biplane trees. For every nodes, if v is the blue-irreducible word in the label, then with **Lemma 78** $v = \psi_{i^\alpha}(v')$, i^α is the new label and $F_B(v')$ is the new right forest.

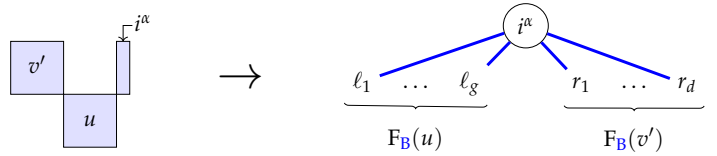
Here is the formal definition of $F_B(w)$ and $T_B(w)$ which is very similar to **Definition 93**, only the third item is different. The labels are now pairs of an integer and a sign $\alpha \in \{\circ, \bullet\}$.

Definition 98. The forest $F_B(w)$ (resp. tree $T_B(w)$) associated to a packed word (resp. irreducible packed word) w are defined in a mutual recursive way as follows:

- $F_B(\epsilon) = []$ (empty forest),
- for any packed word w , let $w_1/w_2/\dots/w_k$ be the global descent decomposition of w , then $F_B(w) := [T_B(w_k), T_B(w_{k-1}), \dots, T_B(w_1)]$ (notice the inversion compared to **Definition 63**).

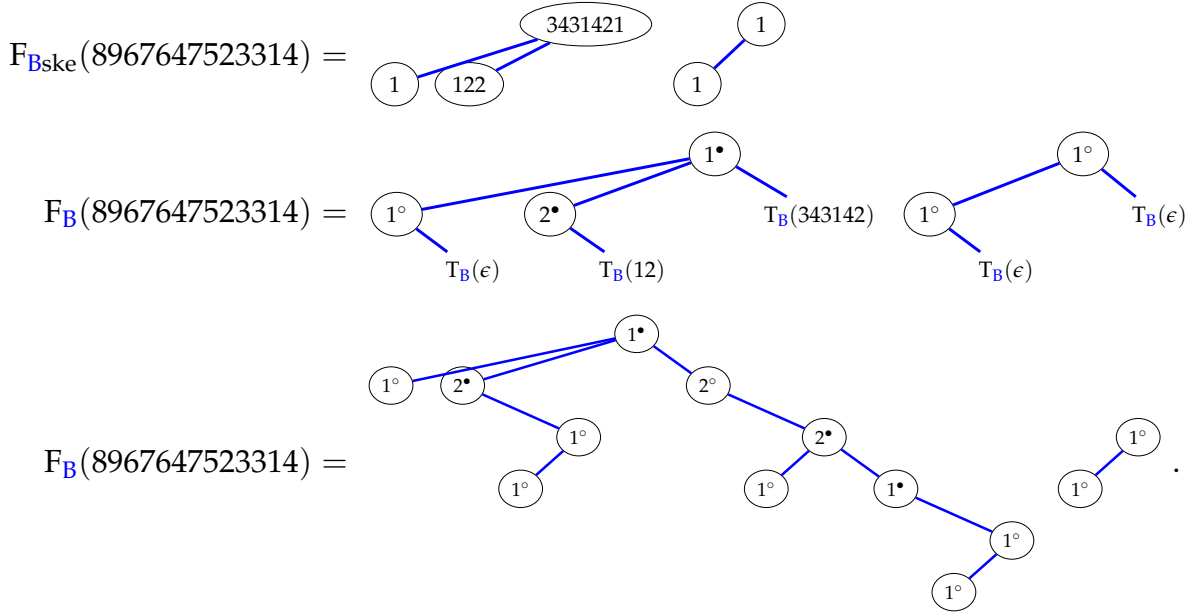


- for any irreducible packed word w , we define $T_B(w) := \text{Node}_B(i^\alpha, F_B(u), F_B(v'))$ where $w = u \blacktriangle \psi_{i^\alpha}(v')$ is the blue-factorization of w .

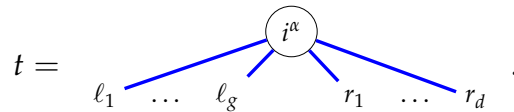


Example 99. Consider again $w = 8967647523314$. We start from the blue-skeleton forest

from **Example 94**.



Definition 100. Let t be a labeled biplane tree. We write $t = \text{Node}_B(i^\alpha, f_\ell, f_r)$ where $i \in \mathbb{N}_{>0}$, $\alpha \in \{\circ, \bullet\}$, $f_\ell = [\ell_1, \dots, \ell_g]$, and $f_r = [r_1, \dots, r_d]$, which is depicted as follows:



The **weight** of t ($\omega(t)$) is the number of nodes with \circ in t . By extension, the **weight** of a forest is the sum of the weight of its trees.

Lemma 101. The weight of a forest (resp. a tree) obtain by the functions F_B (resp. T_B) is equal to the maximum value of the word. i.e. $\forall w \in \mathbf{PW}, \omega(F_B(w)) = \max(w)$, $\forall w \in \mathbf{PW}$ with w irreducible, $\omega(T_B(w)) = \max(w)$.

Proof. We prove by induction with the following hypothesis, for $n \in \mathbb{N}$:

$$\begin{aligned} \forall w \in \mathbf{PW}_n, \quad \omega(F_B(w)) &= \max(w), \\ \forall w \in \mathbf{PW}_n \text{ with } w \text{ irreducible, } \omega(T_B(w)) &= \max(w). \end{aligned} \tag{2.10}$$

The base case is given by the first item of **Definition 98** as $F_B(\epsilon) = []$ and $\omega([]) = \max(\epsilon) = 0$ by convention.

Let us fix $n \geq 1$ and suppose that the hypothesis (2.10) holds. Let $w \in \mathbf{PW}_{n+1}$ and $w = w_1/w_2/\dots/w_k$ be the global descent decomposition of w .

• If $k = 1$, we have $F_B(w) = [T_B(w)]$. Let $w = u \blacktriangle \psi_{i^\alpha}(v)$ with $i \in \mathbb{N}_{>0}$ and $\alpha \in \{\circ, \bullet\}$ be the blue-factorization of w , then, depending of α the node is counted or not:

$\omega(\mathbf{T}_{\mathbf{B}}(w)) = \omega(\text{Node}_{\mathbf{B}}(i^\alpha, \mathbf{F}_{\mathbf{B}}(u), \mathbf{F}_{\mathbf{B}}(v))) = \overbrace{(1+)}^{\text{if } \alpha=0} \omega(\mathbf{F}_{\mathbf{B}}(u)) + \omega(\mathbf{F}_{\mathbf{B}}(v))$. The sizes of u and v are at most n so by induction $\omega(\mathbf{T}_{\mathbf{B}}(w)) = (1+) \max(u) + \max(v) = \max(w)$.

• If $k \geq 2$, by induction on each factors, we have that

$$\omega(\mathbf{F}_{\mathbf{B}}(w)) = \omega(\mathbf{T}_{\mathbf{B}}(w_1)) + \cdots + \omega(\mathbf{T}_{\mathbf{B}}(w_k)) = \max(w_1) + \cdots + \max(w_k) = \max(w). \quad \square$$

Definition 102. Using the same notations as in previous [Definition 100](#), we say that t is a **blue-packed tree** if it satisfies:

$$\left\{ \begin{array}{l} d = 0, \\ i^\alpha = 1^\circ, \\ \ell_1, \dots, \ell_g \text{ are blue-packed trees.} \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} d = 1, \\ i^\alpha \neq 1^\circ, \\ 1 \leq i \leq \omega(r_1), \\ \ell_1, \dots, \ell_g \text{ and } r_1 \text{ are blue-packed trees.} \end{array} \right. \quad (2.11)$$

An ordered list of blue-packed trees is a **blue-packed forest**.

Remark 103. The same remark as [Remark 68](#) can be done with blue-skeleton trees that can be interpreted as flattened representations of blue-packed trees. Symmetrically, blue-packed trees can be interpreted as unfolded representations of blue-skeleton trees. We use the operation ψ_{i^α} to change between blue-packed and blue-skeleton trees.

Notations 104. In the same way as [Notations 69](#) we add the following notations:

- \mathfrak{F}_{B_n} the set of blue-packed forests of size n , ($\mathfrak{F}_{B_n} := \{\mathbf{F}_{\mathbf{B}}(w)\}_{w \in \mathbf{PW}_n}$),
- \mathfrak{T}_{B_n} the set of blue-packed trees of size n , ($\mathfrak{T}_{B_n} := \{\mathbf{T}_{\mathbf{B}}(w)\}_{w \in \mathbf{PW}_n}$ with w irreducible),
- \mathfrak{N}_{B_n} the set of blue-packed trees of size n such that the left forest of the root is empty, ($\mathfrak{N}_{B_n} := \{\mathbf{T}_{\mathbf{B}}(w)\}_{w \in \mathbf{PW}_n}$ with w blue-irreducible). In particular, the blue-skeleton of a tree of \mathfrak{N}_{B_n} consist of a single node labeled by a blue-irreducible word.

Remark 105. The set \mathfrak{N}_{B_n} can be described as a disjointed union of sets depending on i and α . Let $\mathfrak{F}_{B_n}^{i^\alpha}$ denote the set of blue-packed forests of weight n that can be right children of a node labeled by i^α (see [Definition 102](#) for conditions), we have the following description:

$$\mathfrak{N}_{B_n} = \bigsqcup_{i, \alpha} \{\text{Node}_{\mathbf{B}}(i^\alpha, [], f_r) \mid f_r \in \mathfrak{F}_{B_{n-p}}^{i^\alpha}\} \quad (2.12)$$

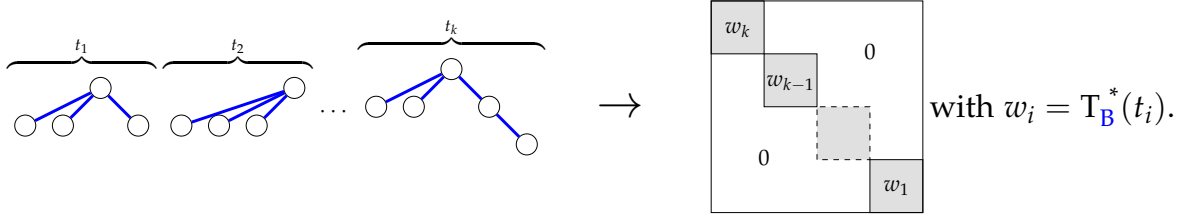
Analogously, we use $\mathfrak{F}_{B_{ske}}$, $\mathfrak{T}_{B_{ske}}$ and $\mathfrak{N}_{B_{ske}}$ for blue-skeleton forests, trees and trees with only one node.

We can remark that for $n = 1$ we have $\mathfrak{N}_{B_1} = \mathfrak{T}_{B_1} = \mathfrak{F}_{B_1}$ and $\forall n > 1, \mathfrak{N}_{B_n} \subsetneq \mathfrak{T}_{B_n} \subsetneq \mathfrak{F}_{B_n}$.

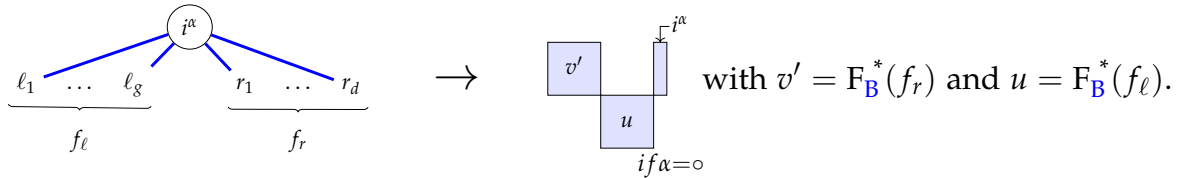
Once again we define two functions in order to prove that $\mathbf{F}_{\mathbf{B}}$ and $\mathbf{T}_{\mathbf{B}}$ are bijections.

Definition 106. We define here the functions F_B^* (resp. T_B^*) that transform blue-packed forest f (resp. tree t) into a packed word. We reverse all instructions of [Definition 98](#) as follows:

- $F_B^*([\])=\epsilon$,
- for any non empty blue-packed forest $f = [t_1, t_2, \dots, t_k]$, then $F_B^*(f) = T_B^*(t_k) / T_B^*(t_{k-1}) / \dots / T_B^*(t_1)$ (notice the inversion compared to [Definition 71](#)).



- for any non empty blue-packed tree $t = \text{Node}_B(i^\alpha, f_\ell, f_r)$, then $T_B^*(t) = F_B^*(f_\ell) \blacktriangle \psi_{i^\alpha}(F_B^*(f_r))$.



As $\psi_{i^\alpha}(F_B^*(f_r))$ is only defined if $i \leq \max(F_B^*(f_r)) \widehat{(+1)}$ (see [Definition 76](#)), there might be a problem with this definition. We prove in the following [Lemma 107](#) that this is the case if $t \in \mathfrak{T}_B$.

Lemma 107. For any blue-packed forest f , $F_B^*(f)$ is a well defined word and its maximum value is $\omega(f)$. For any blue-packed tree t , $T_B^*(t)$ is a well defined word and its maximum is $\omega(t)$.

Proof. We prove by induction with the following hypothesis, for $n \in \mathbb{N}$:

$$\begin{aligned} \forall f \in \mathfrak{F}_{B \leq n}, \quad & F_B^*(f) \text{ is well defined and } \max(F_B^*(f)) = \omega(f), \\ \forall t \in \mathfrak{T}_{B \leq n}, \quad & T_B^*(t) \text{ is well defined and } \max(T_B^*(t)) = \omega(t). \end{aligned} \quad (2.13)$$

The base case is given by the first item of [Definition 106](#) as $F_B^*([\])=\epsilon$ and $\omega([\])=\max(\epsilon)=0$.

Let us fix $n \geq 1$ and suppose that the hypothesis (2.13) holds. Let $f = [t_1, \dots, t_k] \in \mathfrak{F}_{B_{n+1}}$.

- If $k = 1$, it is sufficient to prove the second item of (2.13). Let $t = \text{Node}_B(i^\alpha, f_\ell, f_r) \in \mathfrak{T}_{B_{n+1}}$. According to [Definition 102](#) with notations of [Definition 100](#) there are two cases:
 - $d = 0$ and $i^\alpha = 1^\circ$. We have that $\max(F_B^*(f_r)) = \max(\epsilon) = 0$ so $i \leq 0 + 1$ and $\psi_{i^\alpha}(\epsilon) = 1$. Now by induction on f_ℓ , we have that $F_B^*(f_\ell)$ is a well defined word and

its maximum value is $\omega(f_\ell)$. Finally $T_B^*(t) = F_B^*(f_\ell) \blacktriangle \psi_{i^\alpha}(\epsilon) = F_B^*(f_\ell) \setminus \psi_{i^\alpha}(\epsilon)$ is a well defined word and its maximum is the last value: $\max(F_B^*(f_\ell)) + 1 = \omega(t)$.

• $d = 1$. In this case, we can directly apply the hypothesis (2.13) on f_r and f_ℓ . According to Definition 102 we have that

$$\begin{aligned} i &\leq \omega(r_1), \\ i &\leq \max(F_B^*(f_r)). \end{aligned}$$

So $T_B^*(t)$ is well defined. Moreover

$$\begin{aligned} \max(T_B^*(t)) &= \max(F_B^*(f_\ell) \blacktriangle \psi_{i^\alpha} F_B^*(f_r)) \\ &= \max(F_B^*(f_\ell)) + \max(F_B^*(f_r))(+1) \\ &= \omega(f_\ell) + \omega(f_r)(+1) = \omega(t). \end{aligned}$$

• If $k \geq 2$, the weight of trees are at least 1 so we can apply (2.13) on trees of f . □

Theorem 108. *The functions F_B and F_B^* (resp. T_B and T_B^*) are two converse bijections between packed words of size n and blue-packed forests (resp. irreducible packed words and blue-packed trees) of size n . That is to say $F_B^{-1} = F_B^*$ and $T_B^{-1} = T_B^*$.*

Proof. The proof structure is the same as the one of Theorem 73 which is similar to the one of Lemmas 62 and 97. But with use of statements coming from this subsection. We can see in this table some of the main statements that are exchanged with their counterpart:

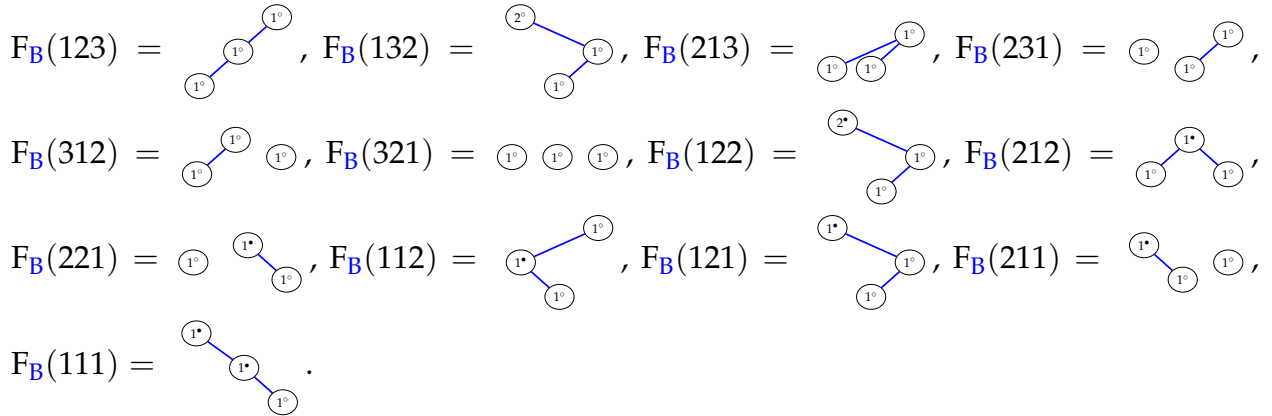
Lemma 42	Lemma 81	red-factorization and blue-factorization.
Definition 45	Definition 85	red-irreducible words and blue-irreducible words.
Corollary 49	Corollary 89	$u \blacktriangleright v$ irreducible $\iff v$ irreducible $u \blacktriangle v$ irreducible $\iff v$ irreducible.
Proposition 50	Proposition 90	$u \blacktriangleright v$ red-factorization $\iff v$ red-irreducible $u \blacktriangle v$ blue-factorization $\iff v$ blue-irreducible.
Definition 63	Definition 98	F_R, T_R and F_B, T_B .
Definition 65	Definition 100	weight of red-forests (\sum size of labels) weight of blue-forests (\sum nodes with \circ).
Lemma 66	Lemma 101	$\omega(F_R(w)) = w $ and $\omega(F_B(w)) = \max(w)$.
Definition 67	Definition 102	red-packed forest and blue-packed forest.
Definition 71	Definition 106	F_R^*, T_R^* and F_B^*, T_B^* .

□

Remark 109. As we can see in Sections 2.1 and 2.2, the role of size and weight are exchanged for red and blue-forests. For red forests, the size (number of nodes) is equal to the maximum letter of the word associated while the weight (Definition 65) is the

number of letter of the associated word. For blue-forests, it is the opposite, the number of letters of the associated word is equal to the size of the forest while the maximum letter is equal to the weight (Definition 100) of the forest. That is why we denote the set of red packed forests of **weight** n by \mathfrak{F}_{Rn} and the set of blue-packed forests of **size** n by \mathfrak{F}_{Bn} .

Example 110. There is a unique forest in \mathfrak{F}_{B1} , namely $\textcircled{1^\circ}$, here are the 3 forests of \mathfrak{F}_{B2} with the associated packed word: $F_B(12) = \textcircled{1^\circ} \textcircled{1^\circ}$, $F_B(21) = \textcircled{1^\circ} \textcircled{1^\circ}$, $F_B(11) = \textcircled{1^\circ} \textcircled{1^\circ}$. We show below the forests of \mathfrak{F}_{B3} :



More examples can be found in the annexes section with Tables 7 to 12.

We conclude by the main theorem of this subsection. It is the dual of Theorem 75.

Theorem 111. For all $n \in \mathbb{N}$ we have the three following equalities :

$$\dim(\mathbf{WQSym}_n) = \#\mathfrak{F}_{Bn} \quad \text{and} \quad \dim(\mathbf{Prim}_n) = \#\mathfrak{T}_{Bn} \quad \text{and} \quad \dim(\mathbf{TPrim}_n) = \#\mathfrak{P}_{Bn}$$

Proof. The proof is similar to the one of Theorem 75 thanks to Theorem 108 instead of Theorem 73. □

3 Bases for totally primitive elements

In this section we construct two bases of primitive and totally primitive elements of \mathbf{WQSym} and \mathbf{WQSym}^* . Thanks to Theorems 75 and 111 we now have the combinatorial objects to index those bases and we know that their numbers agree with the dimensions. We therefore only need to show that they are linearly independent. We will proceed by showing that the decompositions through maximum and through last letter preserve the total primitivity.

As in Section 2, we start by working on \mathbf{WQSym}^* associated to the color red and do the same work on the primal \mathbf{WQSym} associated to the color blue.

3.1 Dual (Red)

3.1.1 Decomposition through maximums and totally primitive elements

Definition 112. Let $I = [i_1, \dots, i_p]$ with $0 < i_1 < \dots < i_p$. We define a linear map $\Phi_I : \mathbf{WQSym}^* \rightarrow \mathbf{WQSym}^*$ as follows: for all $n \in \mathbb{N}$ and $w = w_1 \cdot w_2 \cdots w_n \in \mathbf{PW}_n$,

$$\Phi_I(\mathbb{R}_w) := \begin{cases} \mathbb{R}_{\phi_I(w)} & \text{if } i_p \leq n + p, \\ 0 & \text{if } i_p > n + p. \end{cases} \quad (3.1)$$

Definition 113. Let $I = (i_1, \dots, i_p)$ with $0 < i_1 < \dots < i_p$. We define a projector $\tau_I : \mathbf{WQSym}^* \rightarrow \mathbf{WQSym}^*$ as follows: for all $n \in \mathbb{N}$ and $w = w_1 \cdot w_2 \cdots w_n \in \mathbf{PW}_n$,

$$\tau_I(\mathbb{R}_w) := \begin{cases} \mathbb{R}_w & \text{if } w_i = \max(w) \text{ if and only if } i \in I, \\ 0 & \text{else.} \end{cases} \quad (3.2)$$

These are orthogonal projectors in the sense that $\tau_I^2 = \tau_I$ and $\tau_I \circ \tau_J = 0$ ($I \neq J$).

Lemma 114. For any I , we have $\text{Im}(\Phi_I) = \text{Im}(\tau_I)$ where $\text{Im}(f)$ denotes the image of f .

Proof. For any I , the inclusion $\text{Im}(\Phi_I) \subset \text{Im}(\tau_I)$ is automatic by definition of Φ_I and τ_I . Indeed, for any $w \in \mathbf{PW}_n$ if $i_p \leq n + p$ then $\Phi_I(\mathbb{R}_w) = \mathbb{R}_{\phi_I(w)}$ and $\tau_I(\mathbb{R}_{\phi_I(w)}) = \mathbb{R}_{\phi_I(w)}$ and $\Phi_I(\mathbb{R}_w) = 0$ otherwise. By linearity $\text{Im}(\Phi_I) \subset \text{Im}(\tau_I)$.

For any I , the inclusion $\text{Im}(\Phi_I) \supset \text{Im}(\tau_I)$ is a consequence of **Lemma 39** and linearity. Indeed, for any $w \in \mathbf{PW}$, $\tau_I(\mathbb{R}_w) = \mathbb{R}_w \Leftrightarrow (w_i = \max(w) \Leftrightarrow i \in I)$. If $\tau_I(\mathbb{R}_w) = \mathbb{R}_w$ let w' be such that $\phi_I(w') = w$ using **Lemma 39**, then $\Phi_I(\mathbb{R}_{w'}) = \mathbb{R}_w = \tau_I(\mathbb{R}_w)$. By linearity $\text{Im}(\Phi_I) \supset \text{Im}(\tau_I)$. \square

Lemma 115. For any I , the projection by τ_I of a totally primitive element is still a totally primitive element, so that $\tau_I(\mathbf{TPrim}^*) = \text{Im}(\tau_I) \cap \mathbf{TPrim}^*$. Moreover,

$$\mathbf{TPrim}^* = \bigoplus_I \text{Im}(\tau_I) \cap \mathbf{TPrim}^* . \quad (3.3)$$

Proof. Let w a packed word. We have $\Delta_{\prec}(\tau_I(\mathbb{R}_w)) = (\tau_I \otimes \text{Id}) \circ \Delta_{\prec}(\mathbb{R}_w)$ by definition of τ_I and Δ_{\prec} . Indeed, in $\Delta_{\prec}(\mathbb{R}_w)$, the deconcatenations cannot be done before the last maximum letter of w . By linearity, for all $p \in \mathbf{TPrim}^*$, we have $\Delta_{\prec}(\tau_I(p)) = (\tau_I \otimes \text{Id}) \circ \Delta_{\prec}(p) = 0$. The same argument works on the right so that $\tau_I(p) \in \mathbf{TPrim}^*$. Moreover τ_I are orthogonal projectors so $\mathbf{TPrim}^* = \bigoplus_I \tau_I(\mathbf{TPrim}^*) = \bigoplus_I \text{Im}(\tau_I) \cap \mathbf{TPrim}^*$. \square

3.1.2 The new basis \mathbb{P}

Definition 116. Let t_1, \dots, t_k be k red-packed trees, $I = [i_1, \dots, i_p]$, $f_l = [\ell_1, \dots, \ell_g]$ be a red-packed forest and $f_r \in \mathfrak{F}_R^I$ be a red-packed forest that can be right children of a node

labeled by I ,

$$\mathbb{P}_{[]} := \mathbb{R}_\epsilon, \quad (3.4)$$

$$\mathbb{P}_{t_1, \dots, t_k} := \mathbb{P}_{t_k} \prec (\mathbb{P}_{t_{k-1}} \prec (\dots \prec \mathbb{P}_{t_1}) \dots), \quad (3.5)$$

$$\mathbb{P}_{\text{Node}_R(I, f_l = [\ell_1, \dots, \ell_g], f_r)} := \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \dots, \mathbb{P}_{\ell_g}; \mathbb{P}_{\text{Node}_R(I, [], f_r)} \rangle, \quad (3.6)$$

$$\mathbb{P}_{\text{Node}_R(I, [], f_r)} := \Phi_I(\mathbb{P}_{f_r}). \quad (3.7)$$

Example 117.

$$\begin{aligned} \mathbb{P}_{\begin{array}{c} \textcircled{1,3} \\ \diagdown \quad \diagup \\ \textcircled{1} \quad \textcircled{1} \end{array}} &= \mathbb{P}_{\begin{array}{c} \textcircled{1} \\ \diagdown \\ \textcircled{1} \end{array}} \prec \mathbb{P}_{\begin{array}{c} \textcircled{1,3} \\ \diagdown \\ \textcircled{1} \end{array}} = (\mathbb{P}_{\textcircled{1}} \succ \mathbb{P}_{\textcircled{1}} - \mathbb{P}_{\textcircled{1}} \prec \mathbb{P}_{\textcircled{1}}) \prec \Phi_{1,3}(\mathbb{P}_{\textcircled{1}}) \\ &= \mathbb{R}_{14342} + \mathbb{R}_{41342} + \mathbb{R}_{43142} + \mathbb{R}_{43412} - \mathbb{R}_{24341} - \mathbb{R}_{42341} - \mathbb{R}_{43241} - \mathbb{R}_{43421} \end{aligned}$$

More examples can be found in the annexes section with [Table 13](#) and [Figures 6](#) and [8](#).

Theorem 118. For all $n \in \mathbb{N}_{>0}$

1. $(\mathbb{P}_f)_{f \in \mathfrak{F}_{R_n}}$ is a basis of \mathbf{WQSym}_n^* ,
2. $(\mathbb{P}_t)_{t \in \mathfrak{T}_{R_n}}$ is a basis of \mathbf{Prim}_n^* ,
3. $(\mathbb{P}_t)_{t \in \mathfrak{N}_{R_n}}$ is a basis of \mathbf{TPrim}_n^* .

Proof. We do a mutually recursive induction on n to prove these three items. As $\dim(\mathbf{WQSym}_1^*) = \dim(\mathbf{Prim}_1^*) = \dim(\mathbf{TPrim}_1^*) = 1$ the base case is trivial. By [Proposition 1, Item 2](#) up to degree n implies [Item 1](#) up to degree n . Similarly, [Theorem 2](#) shows that [Item 3](#) up to degree n implies [Item 2](#) up to degree n . By induction it is sufficient to show that [Items 1](#) and [2](#) up to degree $n-1$ implies [Item 3](#) for n .

For all $k \in \mathbb{N}$, let π_k be the canonical projector on the homogeneous component of degree k of \mathbf{WQSym}^* . We define $\pi_{<k} := \sum_{i=0}^{k-1} \pi_i$. Fix $I = [i_1, \dots, i_p]$ with $p \leq n$ and u a packed word of size $n-p$. Notice that if $p = n$ we immediately have $u = \epsilon$ and $\Delta_{\prec}(\Phi_I(\mathbb{R}_\epsilon)) = 0$. We suppose now that $p < n$. By [Equation \(1.26\)](#), in the half coproduct $\Delta_{\prec}(\Phi_I(\mathbb{R}_u))$ all the maximums must be in the left tensor factor, which therefore must be at least of degree i_p . By linearity, for all $x \in \mathbf{WQSym}_{n-p}^*$

$$\Delta_{\prec}(\Phi_I(x)) = \left(\sum_{j=i_p}^{n-1} \Phi_I \circ \pi_j \otimes \pi_{n-1-j} \right) \circ \tilde{\Delta}(x). \quad (3.8)$$

Thanks to [Corollary 4](#) for $f = [r_1, \dots, r_d] \in \mathfrak{F}_{R_{n-p}}$, the coproduct $\tilde{\Delta}(\mathbb{P}_f)$ is computed by deconcatenation of forests. So if $f \in \mathfrak{F}_{R_{n-p}}^I$, in particular r_d is of weight at least $n - i_p + 1$ then for $j \geq i_p$ we have $n - 1 - j < \omega(r_d)$ so that all the terms in the previous sum vanishes. A similar reasoning applies to $\Delta_{\succ}(\Phi_I(x))$ using the fact that $f \in \mathfrak{F}_{R_{n-p}}^I$ imply $1 \leq i_1 \leq \omega(r_1)$.

So for all $t = \text{Node}_{\mathbf{R}}(I, [], f_r) \in \mathfrak{N}_{\mathbf{R}n}$, we have that $\Delta_{\prec}(\mathbb{P}_t) = \Delta_{\prec}(\Phi_I(\mathbb{P}_{f_r})) = 0$ and $\Delta_{\succ}(\mathbb{P}_t) = \Delta_{\succ}(\Phi_I(\mathbb{P}_{f_r})) = 0$.

Moreover, by induction we have that $\{\mathbb{P}_f \mid f \in \mathfrak{F}_{\mathbf{R}n-p}^I\}$ are linearity independent as $\{\mathbb{P}_f \mid f \in \mathfrak{F}_{\mathbf{R}n-p}\}$ is a basis of \mathbf{WQSym}_{n-p}^* . Recall the description of $\mathfrak{N}_{\mathbf{R}n}$ in [Remark 70](#) as a disjointed union of sets depending on I :

$$\mathfrak{N}_{\mathbf{R}n} = \bigsqcup_I \{\text{Node}_{\mathbf{R}}(I, [], f_r) \mid f_r \in \mathfrak{F}_{\mathbf{R}n-p}^I\} \quad (3.9)$$

Since Φ_I is injective on \mathbf{WQSym}_{n-p}^* then $\{\Phi_I(\mathbb{P}_f) \mid f \in \mathfrak{F}_{\mathbf{R}n-p}^I\}$ are linearly independent. According to [Lemma 114](#), for all $f \in \mathfrak{F}_{\mathbf{R}n-p}^I$ we have $\Phi_I(\mathbb{P}_f) \in \text{Im}(\tau_I) \cap \mathbf{TPrim}_n^*$. Moreover, thanks to the direct sum of [Equation \(3.3\)](#):

$$\mathbf{TPrim}_n^* = \bigoplus_I \text{Im}(\tau_I) \cap \mathbf{TPrim}_n^*$$

and by definition of \mathbb{P} , in particular $\mathbb{P}_{\text{Node}_{\mathbf{R}}(I, [], f_r)} := \Phi_I(\mathbb{P}_{f_r})$, the family $\{\mathbb{P}_t \mid t \in \mathfrak{N}_{\mathbf{R}n}\}$ are linearly independent. Finally, by cardinalities of [Theorem 75](#) it is a basis of \mathbf{TPrim}_n^* . \square

Remark 119. The basis \mathbb{P} is indexed by red-packed forests ($\mathfrak{F}_{\mathbf{R}}$). We will also use red-skeletons ($\mathfrak{F}_{\mathbf{R}ske}$) or packed words (\mathbf{PW}) as index thanks to the bijections of [Remark 68](#) and $F_{\mathbf{R}}$ of [Definition 63](#).

3.2 Primal (Blue)

3.2.1 Decomposition through last letter and totally primitive elements

Definition 120. Let $i \in \mathbb{N}_{>0}$ and $\alpha \in \{\circ, \bullet\}$. We define a linear map $\Psi_{i\alpha} : \mathbf{WQSym} \rightarrow \mathbf{WQSym}$ as follows: for all $n \in \mathbb{N}$ and $w \in \mathbf{PW}_n$,

$$\Psi_{i\alpha}(\mathbb{Q}_w) := \begin{cases} \mathbb{Q}_{\psi_{i\circ}(w)} & \text{if } \alpha = \circ \text{ and } 1 \leq i \leq \max(w) + 1, \\ \mathbb{Q}_{\psi_{i\bullet}(w)} & \text{if } \alpha = \bullet \text{ and } 1 \leq i \leq \max(w), \\ 0 & \text{else.} \end{cases} \quad (3.10)$$

Definition 121. Let $i \in \mathbb{N}_{>0}$ and $\alpha \in \{\circ, \bullet\}$. We define a projector $\tau_{i\alpha} : \mathbf{WQSym} \rightarrow \mathbf{WQSym}$ as follows: for all $n \in \mathbb{N}$ and $w = w_1 \cdot w_2 \cdots w_n \in \mathbf{PW}_n$,

$$\tau_{i\alpha}(\mathbb{Q}_w) := \begin{cases} \mathbb{Q}_w & \text{if } w_n = i \text{ and } \alpha = \bullet \text{ and } i \in [w_1, \dots, w_{n-1}], \\ \mathbb{Q}_w & \text{if } w_n = i \text{ and } \alpha = \circ \text{ and } i \notin [w_1, \dots, w_{n-1}], \\ 0 & \text{else.} \end{cases} \quad (3.11)$$

These are orthogonal projectors in the sense that $\tau_{i\alpha}^2 = \tau_{i\alpha}$ and $\tau_{i\alpha} \circ \tau_{j\beta} = 0$ ($i \neq j$ or $\alpha \neq \beta$).

Lemma 122. For any i and α , we have $Im(\Psi_{i^\alpha}) = Im(\tau_{i^\alpha})$ where $Im(f)$ denotes the image of f .

Proof. For any i and α , the inclusion $Im(\Psi_{i^\alpha}) \subset Im(\tau_{i^\alpha})$ is automatic by definition of Ψ_{i^α} and τ_{i^α} and linearity. Indeed, for any $w \in \mathbf{PW}_n$ $\tau_{i^\alpha}(\Psi_{i^\alpha}(\mathbf{Q}_w)) = \Psi_{i^\alpha}(\mathbf{Q}_w)$.

For any i and any α , the inclusion $Im(\Psi_{i^\alpha}) \supset Im(\tau_{i^\alpha})$ is a consequence of [Lemma 78](#) and linearity. Indeed, for any $w \in \mathbf{PW}$, if $\tau_{i^\alpha}(\mathbf{Q}_w) = \mathbf{Q}_w$ then $w_n = i$. With $w' = \text{pack}(w_1 \dots w_{n-1})$ we have $\Psi_{i^\alpha}(\mathbf{Q}_{w'}) = \mathbf{Q}_w = \tau_{i^\alpha}(\mathbf{Q}_w)$. \square

Lemma 123. For any i and α , the projection by τ_{i^α} of a totally primitive element is still a totally primitive element, so that $\tau_{i^\alpha}(\mathbf{TPrim}) = Im(\tau_{i^\alpha}) \cap \mathbf{TPrim}$. Moreover,

$$\mathbf{TPrim} = \bigoplus_{\alpha, i} Im(\tau_{i^\alpha}) \cap \mathbf{TPrim} . \quad (3.12)$$

Proof. Let w a packed word. We have $\Delta_{\preceq}(\tau_{i^\alpha}(\mathbf{Q}_w)) = (\tau_{i^\alpha} \otimes Id) \circ \Delta_{\preceq}(\mathbf{Q}_w)$ by definition of τ_{i^α} and Δ_{\preceq} . Indeed, in $\Delta_{\preceq}(\mathbf{Q}_w)$, the decomposition can't be done under the last letter of w . By linearity, for all $p \in \mathbf{TPrim}$, we have $\Delta_{\preceq}(\tau_{i^\alpha}(p)) = (\tau_{i^\alpha} \otimes Id) \circ \Delta_{\preceq}(p) = 0$. The same argument works on the right so that $\tau_{i^\alpha}(p) \in \mathbf{TPrim}$. Moreover τ_{i^α} are orthogonal projectors so $\mathbf{TPrim} = \bigoplus_{\alpha, i} \tau_{i^\alpha}(\mathbf{TPrim}) = \bigoplus_{\alpha, i} Im(\tau_{i^\alpha}) \cap \mathbf{TPrim}$. \square

3.2.2 The new basis \mathcal{O}

Definition 124. Let $t_1, \dots, t_k, r \in \mathfrak{T}_B$, $f_r \in \{\emptyset, [r]\}$ and $f_l = [\ell_1, \dots, \ell_g] \in \mathfrak{F}_B$,

$$\mathcal{O}_{\emptyset} := \mathbf{Q}_\varepsilon, \quad (3.13)$$

$$\mathcal{O}_{t_1, \dots, t_k} := \mathcal{O}_{t_k} \preceq (\mathcal{O}_{t_{k-1}} \preceq (\dots \preceq \mathcal{O}_{t_1}) \dots), \quad (3.14)$$

$$\mathcal{O}_{\text{Node}_B(i^\alpha, f_l = [\ell_1, \dots, \ell_g], f_r)} := \langle \mathcal{O}_{\ell_1}, \mathcal{O}_{\ell_2}, \dots, \mathcal{O}_{\ell_g}; \mathcal{O}_{\text{Node}_B(i^\alpha, \emptyset, f_r)} \rangle, \quad (3.15)$$

$$\mathcal{O}_{\text{Node}_B(i^\alpha, \emptyset, f_r)} := \Psi_{i^\alpha}(\mathcal{O}_{f_r}). \quad (3.16)$$

Example 125.

$$\begin{aligned} \mathcal{O} & \begin{array}{c} \textcircled{2^\bullet} \\ \diagdown \\ \textcircled{1^\circ} \\ \diagup \\ \textcircled{1^\circ} \end{array} \quad \begin{array}{c} \textcircled{1^\circ} \\ \diagup \\ \textcircled{1^\circ} \end{array} \\ &= \mathcal{O} \begin{array}{c} \textcircled{1^\circ} \\ \diagup \\ \textcircled{1^\circ} \end{array} \preceq \mathcal{O} \begin{array}{c} \textcircled{2^\bullet} \\ \diagdown \\ \textcircled{1^\circ} \\ \diagup \\ \textcircled{1^\circ} \end{array} \\ &= (\mathcal{O}_{\textcircled{1^\circ}} \succeq \mathcal{O}_{\textcircled{1^\circ}} - \mathcal{O}_{\textcircled{1^\circ}} \preceq \mathcal{O}_{\textcircled{1^\circ}}) \preceq \Psi_{2^\bullet}(\mathcal{O}_{\textcircled{1^\circ}}) \\ &= \mathbf{Q}_{34122} + \mathbf{Q}_{24133} + \mathbf{Q}_{14233} + \mathbf{Q}_{43212} + \mathbf{Q}_{42313} + \mathbf{Q}_{41323} \\ & \quad - \mathbf{Q}_{34212} - \mathbf{Q}_{24313} - \mathbf{Q}_{14323} - \mathbf{Q}_{43122} - \mathbf{Q}_{42133} - \mathbf{Q}_{41233} \end{aligned}$$

More examples can be found in the annexes section with [Table 13](#) and [Figure 9](#).

Theorem 126. For all $n \in \mathbb{N}_{>0}$

1. $(\mathbb{O}_f)_{f \in \mathfrak{F}_{B_n}}$ is a basis of \mathbf{WQSym}_n ,
2. $(\mathbb{O}_t)_{t \in \mathfrak{N}_{B_n}}$ is a basis of \mathbf{Prim}_n ,
3. $(\mathbb{O}_t)_{t \in \mathfrak{N}_{B_n}}$ is a basis of \mathbf{TPrim}_n .

Proof. The proof structure is the same as the one of [Theorem 118](#) except for some statements that are exchanged as we can see in this table:

Equation (1.26)	Equation (1.23)	left corproduct in basis \mathbb{R} and \mathbb{Q} .
Remark 70	Remark 105	$\mathfrak{N}_{R_n} = \bigsqcup_I \{\text{Node}_R(I, [], f_r) \mid f_r \in \mathfrak{F}_{R_{n-p}}^I\}$ $\mathfrak{N}_{B_n} = \bigsqcup_{i,\alpha} \{\text{Node}_B(i^\alpha, [], f_r) \mid f_r \in \mathfrak{F}_{B_{n-p}}^{i^\alpha}\}$.
Lemma 114	Lemma 122	$\text{Im}(\Phi_I) = \text{Im}(\tau_I)$ and $\text{Im}(\Psi_{i^\alpha}) = \text{Im}(\tau_{i^\alpha})$.
Lemma 115	Lemma 123	$\mathbf{TPrim}^* = \bigoplus_I \text{Im}(\tau_I) \cap \mathbf{TPrim}^*$. $\mathbf{TPrim} = \bigoplus_{\alpha,i} \text{Im}(\tau_{i^\alpha}) \cap \mathbf{TPrim}$.
Theorem 75	Theorem 111	$\dim(\mathbf{TPrim}_n^*) = \#\mathfrak{N}_{R_n}$ and $\dim(\mathbf{TPrim}_n) = \#\mathfrak{N}_{B_n}$.

□

Remark 127. The same remark as [Remark 119](#) can be done on the basis \mathbb{O} . Indeed, it is defined with blue-packed forests (\mathfrak{F}_B), it is nevertheless possible to use the blue-skeletons ($\mathfrak{F}_{B_{ske}}$) or packed words(PW) thanks to the bijections of [Remark 103](#) and F_B of [Definition 98](#).

4 Isomorphism between \mathbf{WQSym} and \mathbf{WQSym}^*

According to [Corollary 3](#) \mathbf{WQSym} (resp. \mathbf{WQSym}^*) is freely generated as a dendriform algebra by \mathbf{TPrim} (resp. \mathbf{TPrim}^*). Therefore, any linear isomorphism between \mathbf{TPrim} and \mathbf{TPrim}^* would lead to a bidendriform isomorphism between \mathbf{WQSym} and its dual. Thanks to the two bases \mathbb{P} and \mathbb{O} any graded bijection between red-irreducible and blue-irreducible packed words leads to such an isomorphism. We first make explicit how this is done. Then the bijection is actually obtained as the restriction to red-irreducibles of an involution on all packed words. The definition of the bijection requires a new kind of forest mixing red and blue factorizations, namely bicolored-packed forests.

4.1 A combinatorial solution to an algebraic problem

In this [Section 4.1](#), we use the skeleton representation for bases \mathbb{P} and \mathbb{O} as said in [Remarks 119](#) and [127](#). Moreover we fix a graded bijection μ between red-irreducible and blue-irreducible packed words.

Definition 128. Recall that $(\mathbb{P}_t)_{t \in \mathfrak{N}_{R_n}}$ is a basis of \mathbf{TPrim}_n^* ([Theorem 118](#)) and $(\mathbb{O}_t)_{t \in \mathfrak{N}_{B_n}}$ is a basis of \mathbf{TPrim}_n ([Theorem 126](#)). By linearity, setting

$$t' := \textcircled{\mu(v)} \in \mathfrak{N}_B, \quad \text{and} \quad M_\mu(\mathbb{P}_t) := \mathbb{O}_{t'}. \quad (4.1)$$

for all $t = \textcircled{v} \in \mathfrak{N}_{\mathbf{R}}$, defines a linear isomorphism between the vector spaces \mathbf{TPrim}_n^* and \mathbf{TPrim}_n

Definition 129. We define σ_μ as the extension of μ from red-skeleton to blue-skeleton forests by:

$$\forall f = [t_1, \dots, t_k] \in \mathfrak{F}_{\mathbf{R}}, \quad \sigma_\mu(f) := [\sigma_\mu(t_1), \dots, \sigma_\mu(t_k)] \quad (4.2)$$

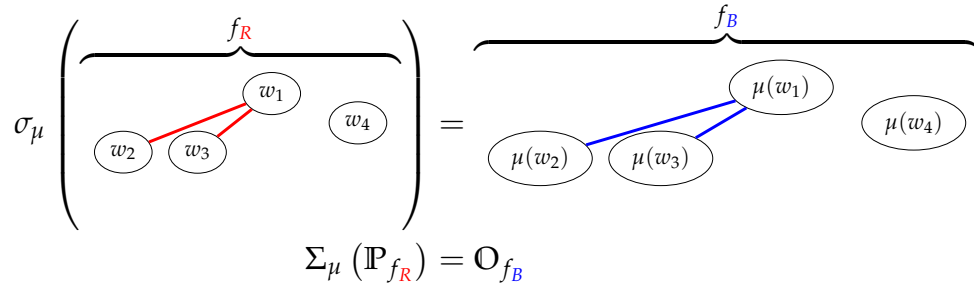
$$\forall t = \text{Node}_{\mathbf{R}}(v, f_\ell, []) \in \mathfrak{T}_{\mathbf{R}}, \quad \sigma_\mu(t) := \text{Node}_{\mathbf{B}}(\mu(v), \sigma_\mu(f_\ell), []). \quad (4.3)$$

Definition 130. We denote Σ_μ the unique bidendriform isomorphism from \mathbf{WQSym}^* to \mathbf{WQSym} which verify for all $f \in \mathfrak{F}_{\mathbf{R}}$:

$$\Sigma_\mu(\mathbb{P}_f) := \mathbb{O}_{\sigma_\mu(f)} \quad (4.4)$$

The existence and unicity are guaranteed by [Corollary 3](#).

Example 131.

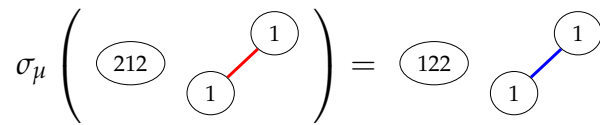


In the following example, we take $\mu(212) := 122$ and $\mu(w) := w$ for other words with a size less than 3 as they are simultaneously red-irreducible and blue-irreducible packed words.

Here are all red and blue-irreducible packed words of size 1, 2 and 3:

Red : 1, 11, 132 121 212 111, **Blue :** 1, 11, 132 121 122 111.

Example 132.



These two forests are the same as those used in [Examples 117](#) and [125](#). So we have here the first example of the isomorphism from the basis \mathbb{R} to the basis \mathbb{Q} :

$$\Sigma_\mu \left(\begin{array}{c} \mathbb{R}_{14342} + \mathbb{R}_{41342} + \mathbb{R}_{43142} + \mathbb{R}_{43412} \\ -\mathbb{R}_{24341} - \mathbb{R}_{42341} - \mathbb{R}_{43241} - \mathbb{R}_{43421} \end{array} \right) = \left(\begin{array}{c} \mathbb{Q}_{34122} + \mathbb{Q}_{24133} + \mathbb{Q}_{14233} + \mathbb{Q}_{43212} \\ +\mathbb{Q}_{42313} + \mathbb{Q}_{41323} - \mathbb{Q}_{34212} - \mathbb{Q}_{24313} \\ -\mathbb{Q}_{14323} - \mathbb{Q}_{43122} - \mathbb{Q}_{42133} - \mathbb{Q}_{41233} \end{array} \right)$$

We now have a construction of a bidendriform isomorphism for any graded bijection μ between red-irreducible and blue-irreducible packed words.

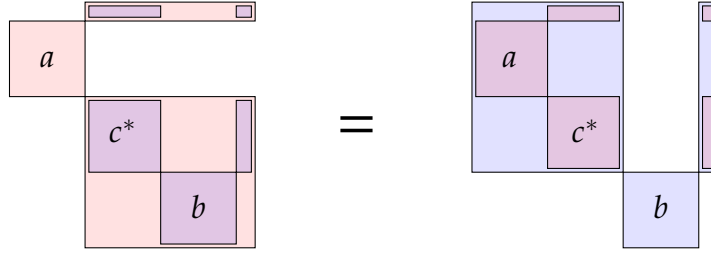
4.2 Full decomposition of packed words into bicolored forests

In order to define a bijection between red-irreducible and blue-irreducible packed words, we need a new kind of forests that mixes up the red and blue factorizations. More precisely, we will recursively alternate these factorizations. We start with an unexpected lemma which implies that starting by red or blue does not matter.

Lemma 133. *For all $a, b, c \in \mathbf{PW}$, with $c \neq \epsilon$, the following relations hold:*

$$a \blacktriangleright (b \blacktriangle c) = b \blacktriangle (a \blacktriangleright c) \quad \text{and} \quad a \blacktriangleright 1 = a \blacktriangle 1, \quad (4.5)$$

where 1 is the packed word of size 1.



Proof. Let $a \in \mathbf{PW}$ and 1 the packed word of size 1, by [Definitions 40](#) and [79](#) $a \blacktriangleright 1 = a \setminus 1$ and $a \blacktriangle 1 = a \setminus 1$.

Let $a, b, c \in \mathbf{PW}$, with $c = c_1 \cdots c_n$ of size $n > 0$. We start by assuming that $c_n \neq \max(c)$ which implies that $c = \phi_I(\psi_{i^\alpha}(c^*)) = \psi_{i^\alpha}(\phi_I(c^*))$ with I, i, α, c^* unique by [Lemmas 39](#) and [78](#). With this relation we can deduce:

$$\begin{aligned} a \blacktriangleright (b \blacktriangle c) &= a \blacktriangleright (b \blacktriangle \psi_{i^\alpha}(\phi_I(c^*))) \\ &= a \blacktriangleright (\psi_{i+\max(b)^\alpha}(\phi_I(c^*)/b)) \\ &= a \blacktriangleright (\psi_{i+\max(b)^\alpha}(\phi_I(c^*/b))) \\ &= a \blacktriangleright (\phi_I(\psi_{i+\max(b)^\alpha}(c^*/b))) \\ &= \phi_{I+|a|}(a/\psi_{i+\max(b)^\alpha}(c^*/b)) \\ &= \phi_{I+|a|}(\psi_{i+\max(b)^\alpha}(a/c^*/b)) \end{aligned}$$

$$\begin{aligned} b \blacktriangle (a \blacktriangleright c) &= b \blacktriangle (a \blacktriangleright \phi_I(\psi_{i^\alpha}(c^*))) \\ &= b \blacktriangle (\phi_{I+|a|}(a/\psi_{i^\alpha}(c^*))) \\ &= b \blacktriangle (\phi_{I+|a|}(\psi_{i^\alpha}(a/c^*))) \\ &= b \blacktriangle (\psi_{i^\alpha}(\phi_{I+|a|}(a/c^*))) \\ &= \psi_{i+\max(b)^\alpha}(\phi_{I+|a|}(a/c^*)/b) \\ &= \psi_{i+\max(b)^\alpha}(\phi_{I+|a|}(a/c^*/b)) \\ &= \phi_{I+|a|}(\psi_{i+\max(b)^\alpha}(a/c^*/b)). \end{aligned}$$

The case where $c_n = \max(c)$ can be decomposed into different particular cases. In each of these cases, it is possible to find a relation with two different writings of c that begin with ϕ or ψ just like $c = \phi_I(\psi_{i^\alpha}(c^*)) = \psi_{i^\alpha}(\phi_I(c^*))$. These cases with the associated relation are:

- the case where c is the packed word 1 then $c = \phi_1(\epsilon) = \psi_{1^\circ}(\epsilon)$,
- the case where c is of the form $c' \setminus 1$ then $c = \phi_1(c') = \psi_{1^\circ}(c')$,
- the more general case where there is more than 1 maximum including the one at the end then $c = \phi_{I \cdot c_n}(c^*) = \psi_{c_n^\bullet}(\phi_I(c^*))$.

In each of these cases it is possible to prove with a similar method that $a \blacktriangleright (b \blacktriangle c) = b \blacktriangle (a \blacktriangleright c)$. \square

Example 134. Here are some examples of this relation:

$$\begin{aligned} 1 \blacktriangleright (1 \blacktriangle 1) &= 213 &= 1 \blacktriangle (1 \blacktriangleright 1), \\ 11 \blacktriangleright (12 \blacktriangle 2111) &= 44533123 = 12 \blacktriangle (11 \blacktriangleright 2111), \\ 11 \blacktriangleright (21 \blacktriangle 123) &= 5534216 = 21 \blacktriangle (11 \blacktriangleright 123), \\ 1 \blacktriangleright (112 \blacktriangle 3132) &= 56361124 = 112 \blacktriangle (1 \blacktriangleright 3132). \end{aligned}$$

Definition 135. Let w be an irreducible packed word. Let $w = x \blacktriangleright u$ be the red-factorization of w and let $u = y \blacktriangle z$ be the blue-factorization of u . Then $w = x \blacktriangleright (y \blacktriangle z)$ is called the **red-blue-factorization** of w . Symmetrically we define $w = y' \blacktriangle (x' \blacktriangleright z')$ the **blue-red-factorization** of w .

Lemma 136. Let $w = x \blacktriangleright (y \blacktriangle z)$ be the red-blue-factorization and let $w = y' \blacktriangle (x' \blacktriangleright z')$ be the blue-red-factorization of an irreducible packed word w .

With these two factorizations, we have that $z = z'$ and it is both red-irreducible and blue-irreducible packed word. Moreover,

- either $z = z' = 1$, $y = x' = \epsilon$ and $x = y'$
- or $x = x'$, $y = y'$.

Example 137. Here are some examples of red-blue-factorization and blue-red-factorization:

$$\begin{aligned} 12 \blacktriangleright (\epsilon \blacktriangle 1) &= 213 &= 12 \blacktriangle (\epsilon \blacktriangleright 1), \\ 11 \blacktriangleright (12 \blacktriangle 1211) &= 44353123 = 12 \blacktriangle (11 \blacktriangleright 1211), \\ 553421 \blacktriangleright (\epsilon \blacktriangle 1) &= 5534216 = 553421 \blacktriangle (\epsilon \blacktriangleright 1), \\ 1 \blacktriangleright (112 \blacktriangle 3132) &= 56361124 = 112 \blacktriangle (1 \blacktriangleright 3132). \end{aligned}$$

Proof. We start by proving the case where $z = z' = 1$, $y = x' = \epsilon$ and $x = y'$. Let w' be an irreducible packed word and $w = w' \setminus 1$. We have that $w = w' \blacktriangleright 1$ is the red-factorization of w and $w = w' \blacktriangle 1$ is the blue-factorization of w . In this case we immediately have that $w = w' \blacktriangleright (\epsilon \blacktriangle 1)$ is the red-blue-factorization of w and that $w = w' \blacktriangle (\epsilon \blacktriangleright 1)$ is the blue-red-factorization of w .

Now let w be an irreducible packed word of size n that cannot be written as $w' \setminus 1$. In other words, there is a maximum strictly before the last letter of w ($\exists i < n, w_i = \max(w)$).

We define the two sets of triplet of packed words that verify equations of the factorizations for w :

$$\begin{aligned} S_{RB}(w) &:= \{(a, b, c) \in \mathbf{PW}, c \neq \epsilon, w = a \blacktriangleright (b \blacktriangle c)\}, \\ S_{BR}(w) &:= \{(a, b, c) \in \mathbf{PW}, c \neq \epsilon, w = b \blacktriangle (a \blacktriangleright c)\}. \end{aligned}$$

Thanks to [Lemma 133](#) these two sets are equal, we define $S(w) := S_{RB}(w) = S_{BR}(w)$.

In the red-blue-factorization $w = x \blacktriangleright (y \blacktriangle z)$, we maximize the size of x , then we maximize the size of y in the remaining word. In the blue-red-factorization we commute the order of maximizations. We will characterize $S(w)$ and see the limit of the two maximizations to prove that they can commute.

Let w^* be the packed word coming from w where the last letter and all occurrences of the maximum are removed and let $I = [i_1, \dots, i_p], i, \alpha$ such that $w = \psi_{i^\alpha}(\phi_I(w^*))$. By hypothesis, we have that $I \neq \emptyset$. Let $w^* = w_1 / \dots / w_k$ be the global descent decomposition of w^* . Let ℓ be the maximum such that $|w_1 / \dots / w_\ell| \leq i_1$, i_1 being the position of the first maximum of w . Let r be the minimum such that $|w_r / \dots / w_k| \leq n - i_p - 1$, i_p being the position of the last maximum of w before the last letter. As $i_1 \leq i_p$ by definition, we have that $\ell < r$. We can characterize the set $S(w)$:

$$\begin{aligned} S(w) = \{ & (a = w_1 / \dots / w_{r^0}, \\ & b = w_{\ell^0} / \dots / w_k, \\ & c = \psi_{(i - \max(b))^\alpha}(\phi_{I - |a|}(w_{r^0+1} / \dots / w_{\ell^0-1}))), \\ & \text{with } r^0 \leq r \text{ and } \ell \leq \ell^0 \}. \end{aligned} \quad \square$$

Here are all packed words that are both red-irreducible and blue-irreducible of size less than 4:

$$\begin{aligned} & 1, \quad 11, \quad 111 \ 121 \ 132, \\ & 1111 \ 1121 \ 1132 \ 1211 \ 1212 \ 1221 \ 1231 \ 1232 \ 1243 \ 1312 \ 1321 \\ & 1322 \ 1323 \ 1332 \ 1342 \ 1423 \ 1432 \ 2121 \ 2122 \ 2132 \ 2143 \ 3132 \end{aligned}$$

Recall our notations for Hilbert series of an algebra A , $\mathcal{A}(z) := \sum_{n=1}^{+\infty} \dim(A_n)z^n$, $\mathcal{P}(z) := \sum_{n=1}^{+\infty} \dim(\text{Prim}(A_n))z^n$ and $\mathcal{T}(z) := \sum_{n=1}^{+\infty} \dim(\text{TPrim}(A_n))z^n$.

n	1	2	3	4	5	6	7	8	9
$i_n \in \mathbf{PW}_n$	1	1	3	22	196	2 008	23 184	297 456	4 199 216
$i_n \in \mathfrak{S}_n$	1	0	1	5	32	236	1 951	17 827	178 418

Table 6: Number of both red-and-blue-irreducible packed words and permutations.

Recall the relations between these series:

$$\mathcal{P} = \mathcal{A}/(1 + \mathcal{A}) \text{ or equivalently } \mathcal{A} = \mathcal{P}/(1 - \mathcal{P}) \text{ (see Proposition 1),}$$

$$\mathcal{T} = \mathcal{A}/(1 + \mathcal{A})^2 \text{ or equivalently } \mathcal{P} = \mathcal{T}(1 + \mathcal{A}) \text{ (see Theorem 2).}$$

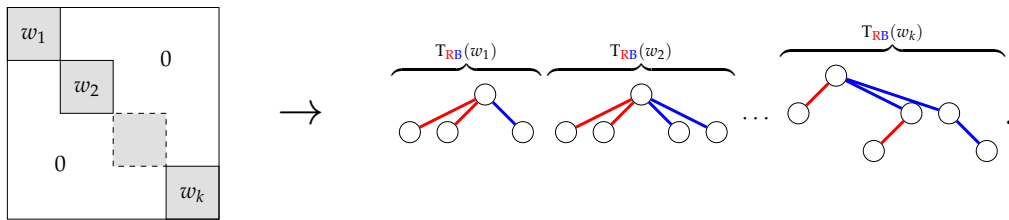
If we define the serie $\mathcal{I} = \sum_{n=1}^{+\infty} i_n z^n$ where i_n is the number of both red-and-blue-irreducible words of size n , then we have the following relation:

$$\mathcal{I} = \mathcal{A}/(1 + \mathcal{A})^3 + z\mathcal{P} \text{ or equivalently } \mathcal{T} = (\mathcal{I} - z)(1 + \mathcal{A}) + z.$$

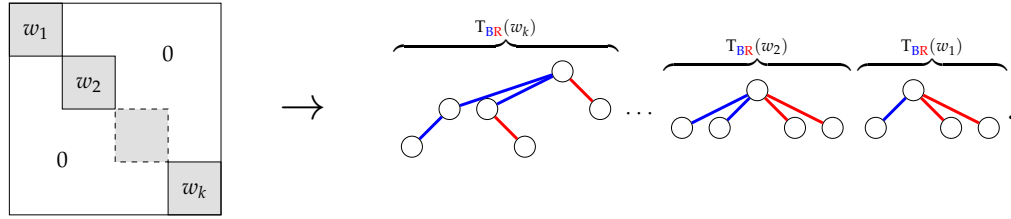
So far we have seen red-biplane trees and blue-biplane trees. In this section we define red-blue-biplane trees and blue-red-biplane trees, the edges of these trees are of two different colors and the labels are red-and-blue-irreducible packed words. We denote by $\text{Node}_{\mathbf{RB}}(x, f_\ell, f_r)$ (resp. $\text{Node}_{\mathbf{BR}}(x, f_\ell, f_r)$) the biplane tree whose edges between the root and the left forest f_ℓ are red (resp. blue) and edges between the root and the right forest f_r are blue (resp. red).

Definition 138. The bicolored forests $F_{\mathbf{RB}}(w)$ and $F_{\mathbf{BR}}(w)$ (resp. trees $T_{\mathbf{RB}}(w)$ and $T_{\mathbf{BR}}(w)$) associated to a packed word (resp. irreducible packed word) w are defined in a mutual recursive way as follows:

- $F_{\mathbf{RB}}(\epsilon) = F_{\mathbf{BR}}(\epsilon) = []$ (empty forest),
- for any packed word w , let $w = w_1/w_2/\dots/w_k$ be the global descent decomposition, then $F_{\mathbf{RB}}(w) := [T_{\mathbf{RB}}(w_1), T_{\mathbf{RB}}(w_2), \dots, T_{\mathbf{RB}}(w_k)]$,

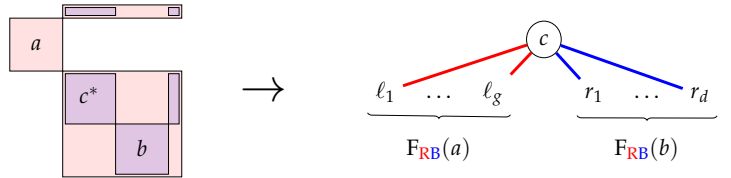


- for any packed word w , let $w = w_1/w_2/\dots/w_k$ be the global descent decomposition, then $F_{\mathbf{BR}}(w) := [T_{\mathbf{BR}}(w_k), T_{\mathbf{BR}}(w_{k-1}), \dots, T_{\mathbf{BR}}(w_1)]$.

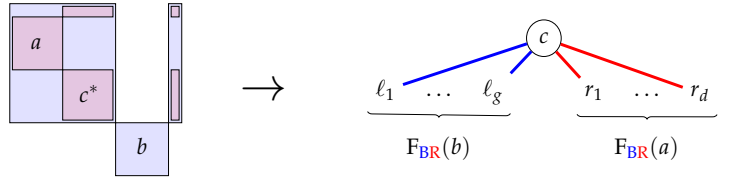


(notice the same inversion as in **Definitions 93** and **98** for F_{BR} .)

- for any irreducible packed word w , let $w = a \blacktriangleright (b \blacktriangleleft c)$ be the red-blue-factorization, then $T_{RB}(w) := \text{Node}_{RB}(c, F_{RB}(a), F_{RB}(b))$.



- for any irreducible packed word w , let $w = b \blacktriangleleft (a \blacktriangleright c)$ be the blue-red-factorization, then $T_{BR}(w) := \text{Node}_{BR}(c, F_{BR}(b), F_{BR}(a))$.

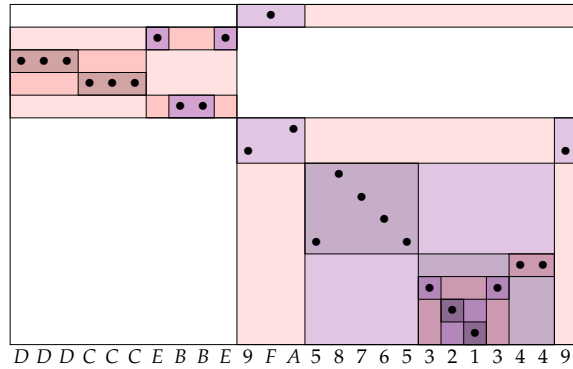


Example 139. For this example we write the word w in hexadecimal in order to have a big and clear example, let $w = \text{DDCCCEBBE9FA587653213449}$. The word w is irreducible so there is only one tree in the forest. To have $T_{RB}(w)$, we start by the blue-red-factorization,

$$w = \underline{\text{DDCCCEBBE9FA587653213449}} = \underline{\text{3332224114}} \blacktriangleright (\underline{\text{58765321344}} \blacktriangleleft \underline{\text{1321}}).$$

Then we decompose each sub-word according to their global descents and do blue-red-factorizations recursively until we have only both red-irreducible and blue-irreducible packed words:

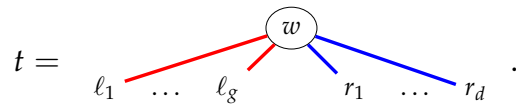
$$\begin{aligned} w &= \underline{\text{3332224114}} \blacktriangleright (\underline{\text{58765321344}} \blacktriangleleft \underline{\text{1321}}), \\ &= \underline{\text{3332224114}} \blacktriangleright ((\underline{\text{14321}} / \underline{\text{321344}}) \blacktriangleleft \underline{\text{1321}}), \\ &[\dots] \\ &= ((\underline{\text{111}} / \underline{\text{111}}) \blacktriangleright (\underline{\text{11}} \blacktriangleleft \underline{\text{11}})) \blacktriangleright ((\underline{\text{14321}} / (((\underline{\text{1}} / \underline{\text{1}}) \blacktriangleleft \underline{\text{11}}) \blacktriangleright \underline{\text{11}})) \blacktriangleleft \underline{\text{1321}}) \end{aligned}$$



$$\begin{aligned}
 T_{\text{RB}}(w) &= \begin{array}{c} \textcircled{1321} \\ \text{F}_{\text{RB}}(3332224114) \quad \text{F}_{\text{RB}}(58765321344) \end{array} = \begin{array}{c} \textcircled{1321} \\ T_{\text{RB}}(3332224114) \quad T_{\text{RB}}(14321) \quad T_{\text{RB}}(321344) \end{array} \\
 &[\dots] \\
 &= \begin{array}{c} \textcircled{1321} \\ \textcircled{11} \quad \textcircled{14321} \quad \textcircled{11} \\ \textcircled{111} \quad \textcircled{111} \quad \textcircled{11} \quad \textcircled{11} \quad \textcircled{1} \quad \textcircled{1} \end{array}
 \end{aligned}$$

More examples can be found in the annexes section with [Tables 7 to 12](#).

Definition 140. There are two types of bicolored trees, the only difference is that colors **red** and **blue** are inverted. Let t be a labeled biplane tree. We write $t = \text{Node}_{\text{RB}}(w, f_\ell, f_r)$ where $w \in \mathbf{PW}$, $f_\ell = [\ell_1, \dots, \ell_g]$ is the left forest of t and $f_r = [r_1, \dots, r_d]$ is the right forest of t . We depict t as follows:



We say that t is a **red-blue-packed tree** if it satisfies:

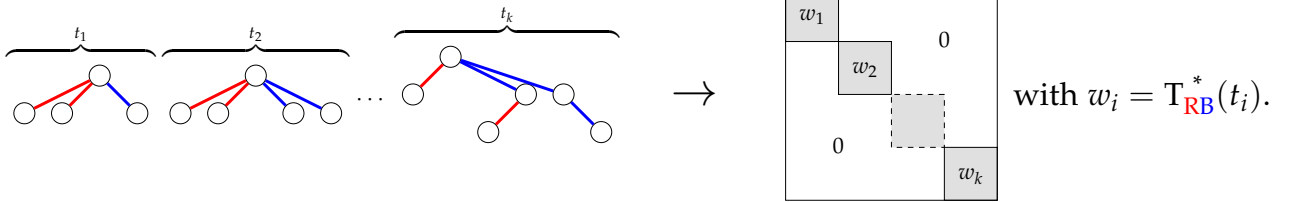
$$\begin{cases} w = 1, \\ f_r = \square, \\ f_\ell \text{ is a red-blue-packed forest.} \end{cases} \quad \text{or} \quad \begin{cases} w \neq 1 \text{ is red-irreducible and blue-irreducible,} \\ f_r \text{ and } f_\ell \text{ are red-blue-packed forest.} \end{cases}$$

Definition 141. The **weight** of a bicolored-packed tree is the sum of the size of packed words in the nodes.

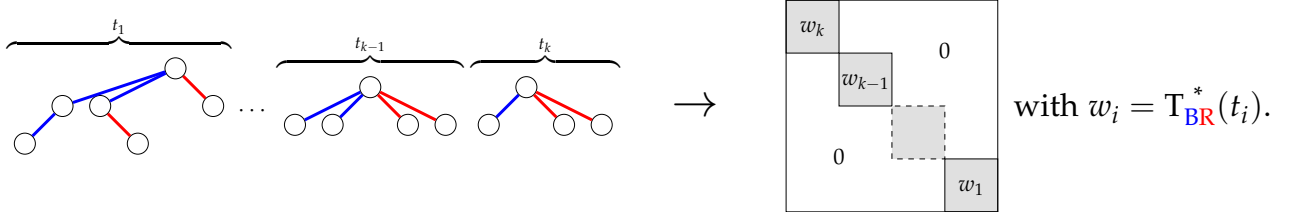
We have already done it four times (Definitions 61, 71, 96 and 106) and we will do it one last time, to prove that $F_{RB}, T_{RB}, F_{BR}, T_{BR}$ are bijections, we define $F_{RB}^*, T_{RB}^*, F_{BR}^*, T_{BR}^*$ and prove that they are the inverse maps.

Definition 142. We define here the maps $F_{RB}^*, T_{RB}^*, F_{BR}^*, T_{BR}^*$ that transforms bicolored-packed forests and trees into packed words. We reverse all instructions of Definition 140 as follows:

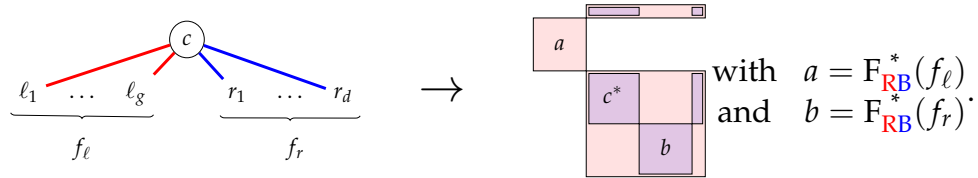
- $F_{RB}^*([\])=F_{BR}^*([\])=\epsilon$ (empty packed word),
- for any red-blue-packed forest $f=[t_1, t_2, \dots, t_k]$, we have $F_{RB}^*(f):=[T_{RB}^*(t_1), T_{RB}^*(t_2), \dots, T_{RB}^*(t_k)]$.



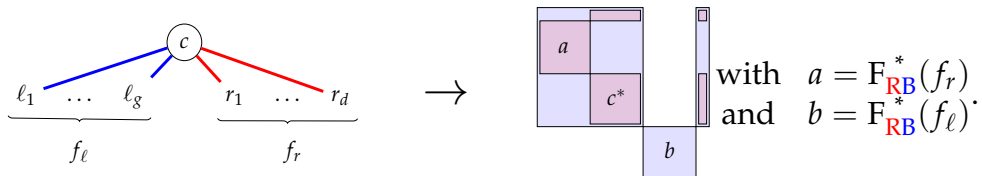
- for any blue-red-packed forest $f=[t_1, t_2, \dots, t_k]$, we have $F_{BR}^*(f):=[T_{BR}^*(t_k), T_{BR}^*(t_{k-1}), \dots, T_{BR}^*(t_1)]$.



- for any red-blue-packed tree $t=\text{Node}_{RB}(c, f_\ell, f_r)$, we have $T_{RB}^*(t):=F_{RB}^*(f_\ell)\blacktriangleright(F_{RB}^*(f_r)\blacktriangleleft c)$.



- for any blue-red-packed tree $t=\text{Node}_{BR}(c, f_\ell, f_r)$, we have $T_{BR}^*(t):=F_{BR}^*(f_r)\blacktriangleleft(F_{BR}^*(f_\ell)\blacktriangleright c)$.



Theorem 143. *The maps F_{RB} and F_{RB}^* (resp. T_{RB} and T_{RB}^*) are two converse bijections between packed words of size n and red-blue-packed forests (resp. irreducible packed words and red-blue-packed trees) of weight n . That is to say $F_{RB}^{-1} = F_{RB}^*$ and $T_{RB}^{-1} = T_{RB}^*$. We have the same result with inversions of red and blue.*

Proof. It is simple to prove by induction on the size of the trees that domain and codomain are as announced and that the functions are inverse to each other. The proof is similar to the proofs of [Lemmas 62](#) and [97](#) and [Theorems 73](#) and [108](#) using [Definition 138](#) of $F_{RB}, T_{RB}, F_{BR}, T_{BR}$, [Definition 140](#) of bicolored-packed forests and trees and [Definition 142](#) of $F_{RB}^*, T_{RB}^*, F_{BR}^*, T_{BR}^*$. \square

Remark 144. We now have two new families of forests \mathfrak{F}_{RB} and \mathfrak{F}_{BR} that are in bijection with packed words and therefore in bijection with red-packed and blue-packed forests. As in [Remarks 119](#) and [127](#), this gives us two other way to index bases \mathcal{O} and \mathcal{IP} of \mathbf{WQSym} and \mathbf{WQSym}^* .

4.3 An involution on packed words

We are now in position to define a bijection between red-irreducible and blue-irreducible packed words. This bijection is actually the restriction of an involution defined on all packed words. Precisely, we will define two transformations on bicolored forests

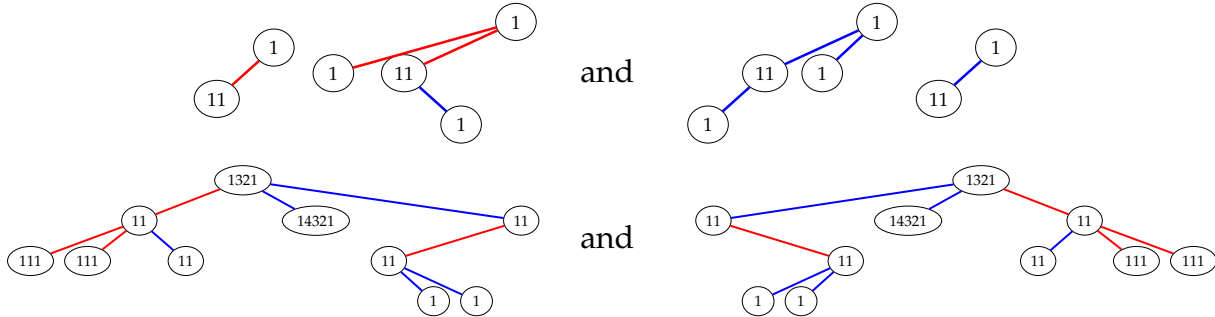
We need to define the notion of mirror transformation of bicolored-packed forests and trees. This transformation is defined from a red-blue to blue-red or from blue-red to red-blue, so in the notations we will use XY instead of RB or BR to point out where the swap is made.

Definition 145. The mirror transformation of a bicolored-packed forest $f = [t_1, \dots, t_k]$ is given by $\tilde{f} := [\tilde{t}_k, \dots, \tilde{t}_1]$ where \tilde{t}_i is the mirror transformation of t_i recursively defined as follows. For any $t = \text{Node}_{XY}(z, f_\ell, f_r)$ then

$$\tilde{t} := \begin{cases} \text{Node}_{YX}(z, \tilde{f}_r, \tilde{f}_\ell) & \text{if } z \neq 1, \\ \text{Node}_{YX}(1, \tilde{f}_\ell, []) & \text{if } z = 1. \end{cases}$$

Note that when $z \neq 1$, the left and right forests are swapped whereas they are not when $z = 1$. But in the latter case, we have necessarily $f_r = \tilde{f}_r = []$. These two cases correspond to the two cases of [Definition 140](#) so the mirror transformation of a red-blue-packed forest is indeed a blue-red-packed forest.

Example 146. Here are two examples of mirror transformations.



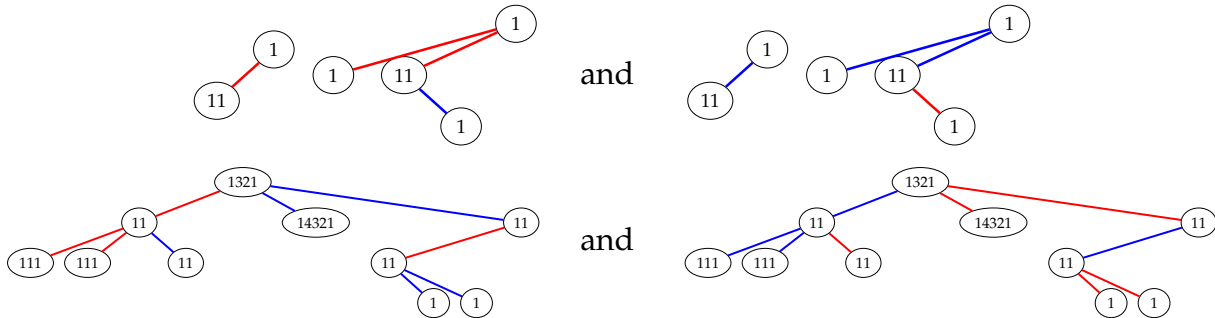
Proposition 147. For all packed words w , both associated bicolored-packed forests $F_{RB}(w)$ and $F_{BR}(w)$ are mirror image of each other.

Proof. The proof is a computation of mirror transformation (Definition 145) on each items of Definition 138 of F_{RB} . For the first three items the computation of $\tilde{f} := [\tilde{t}_k, \dots, \tilde{t}_1]$ is sufficient. Thanks to the relation of Lemma 136 ($x \blacktriangleright (y \blacktriangle z) = y \blacktriangle (x \blacktriangleright z)$ in the case $z \neq 1$) the two remaining items are also simple computation of \tilde{t} . \square

Definition 148. The color swap of a bicolored-packed forest $f = [t_1, \dots, t_k]$ is given by $\hat{f} := [\hat{t}_k, \dots, \hat{t}_1]$ where \hat{t}_i is the color swap of t_i recursively defined as follows. For any $t = \text{Node}_{XY}(z, f_\ell, f_r)$ then $\hat{t} := \text{Node}_{YX}(z, \hat{f}_\ell, \hat{f}_r)$.

In other words, it is a recoloration of each edges using the other color. Every blue edges become red and vice versa.

Example 149. Here are two examples of color swaps.



More examples can be found in the annexes section with Table 13.

When we focus on the packed words associated to these forest, the color swap correspond to the swap of the two operations \blacktriangleright and \blacktriangle in a bicolored-factorization. More precisely, if w is an irreducible packed word and $w = x \blacktriangleright (y \blacktriangle z)$ is the red-blue-factorization of w , then the color swap on the associated forest correspond to $w' = x \blacktriangle (y \blacktriangleright z)$.

Lemma 150. *Mirror transformation and color swap commute. It means that for all bicolored-packed forest f , we have $\widehat{\widetilde{f}} = \widetilde{\widehat{f}}$.*

Proof. The proof is immediate. Indeed, the definition of mirror transformation is independant of color swap and symmetrically, the color swap is independant of the tree shape. \square

Corollary 151. *The diagram on Figure 5 is commutative. So $\widehat{w} := F_{RB}^{-1}(\widehat{F_{BR}(w)})$ is an involution*

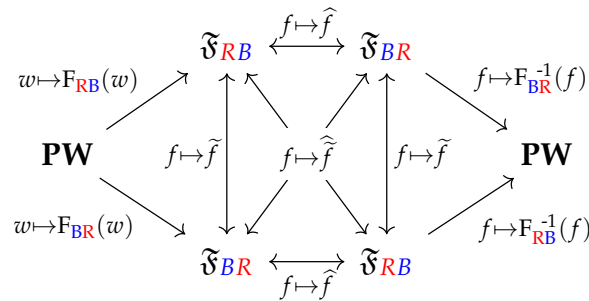


Figure 5: Commutative diagram of maps on bicolored-packed forests

on packed words.

Proof. Thanks to Proposition 147 and Lemma 150 the diagram is immediately commutative. The mirror transformation and the color swap are independant involutions so the conjunction is an involution. \square

Corollary 152. *The application $w \mapsto \widehat{w}$ send blue (resp. red) irreducibles packed words to red (resp. blue) irreducibles packed words.*

Some examples can be found in th annexes section with Tables 14 and 15.

Proof. If w is a red-irreducible packed word, then the red-blue-factorization of w is of the form $w = \epsilon \blacktriangleright (y \blacktriangle z)$. Then the color swap correspond to the words $w' = \epsilon \blacktriangle (y \blacktriangleright z)$ which is blue-irreducible. \square

4.4 Main theorem

In Section 4.1 we fixed a graded bijection μ between red-irreducible and blue-irreducible packed words. After that, we extend it to all red-skeleton forests as σ_μ . We finished by defining Σ_μ as a bidendriform isomorphism from \mathbf{WQSym}^* to \mathbf{WQSym} . Now we can set $\mu : w \mapsto \widehat{w}$ as a graded bijection. The extension σ_μ correspond to the color swap on red-packed forests (i.e. $\sigma_\mu : f \mapsto \widehat{f}$). Finally we have the following theorem:

Theorem 153. *The linear map $\Sigma : \mathbf{WQSym}^* \rightarrow \mathbf{WQSym}$ defined as for all packed forest f ,*

$$\Sigma(\mathbb{P}_f) := \mathbb{O}_{\hat{f}}$$

is a bidendriform isomorphism between \mathbf{WQSym}^ and \mathbf{WQSym} .*

Proof. This theorem is a direct consequence of [Corollaries 3](#) and [152](#). □

Conclusion

The main contribution of this paper is the combinatorial construction of biplane trees. They are the combinatorial ingredient which completes the algebraic theory of Foissy [\[Foi11\]](#) and allows us to describe the explicit isomorphism. Besides, they are also an innovative combinatorial family and open promising research perspective.

Generalization of the inversion of permutation to packed words

The inherent difficulty of finding an explicit isomorphism between \mathbf{WQSym} and its dual lies in the fact that there is no “inversion” operation on packed words. Indeed, in the case of \mathbf{FQSym} , the Hopf algebra indexed by permutations, the isomorphism is given by the inversion of permutations. The solution we offer, using biplane trees, is actually not a generalization of \mathbf{FQSym} in this sense. Indeed, even though permutations are a subset of packed words, the restriction of our involution on packed words to permutation is not the inversion. In particular, if a permutation σ is both red-irreducible and blue-irreducible, its image is itself and not its inverse. This is the case for all packed words which are both red-irreducible and blue-irreducible. Nevertheless, our involution is somehow “compatible” with the inversion in the sense that if we arbitrary decide that the image of σ is σ^{-1} for all σ such that σ is a red-blue-irreducible permutation, then the rest of construction ensures that the image of σ is σ^{-1} for all permutations (not necessarily irreducible anymore). But we don’t know how to define the inversion on red-blue-irreducible elements which are not permutations, which is why to stick with the identity in all case, including permutations.

Stays the open question: is there a generalization of the inversion of permutations on packed words? In other words, one would want an involution on packed words which restricts to the inversion on permutations and gives a bidendriform isomorphism between \mathbf{WQSym} and its dual. A consequence of our work is that it is sufficient to find such an involution on red-blue-irreducible packed words.

Generalization of the biplane trees to parking functions

A long term goal would be to somehow generalize the structure of biplane trees to all bidendriform Hopf algebra. The first step would be to look at the Hopf algebra indexed

by parking functions **PQSym**. Indeed **PQSym** is also a bidendriform bialgebra and parking functions are a superset of the packed words. The question of generalizing the structure to parking functions involves both combinatorics and algebra. The first thing is to compute bases of **PQSym** in which the shuffle product is not shifted. It can be done with a generalization of [Definition 29](#)[[BZ05](#)].

The lines of research induced by this work are the following:

- How to generalize biplane tree structure to **PQSym**?
- We will then look for what are the necessary and sufficient ingredients to develop biplane tree structures and obtain bidendriform automorphisms on all bidendriform bialgebras.

Link between bidendriform bialgebras and skew-duplicial operad

As said in [Remarks 48](#) and [88](#) the operations \blacktriangleright and $/$ (resp. \blacktriangle and $/$) unexpectedly verify relations of the skew-duplicial operad [[BDO20](#)]. These relations reveal a new application of the skew-duplicial operad applied on packed words.

- Can we find a skew-duplicial structure on **WQSym** which is linked to the bidendriform structure?
- More generally, is there a link between bidendriform bialgebra and skew-duplicial?

Link between bidendriform bialgebras and L -algebras

As said in [Remark 56](#), the sequence that count unlabeled biplane trees is the dimensions of the free L -algebra on one generator (see [[Phi07](#)]). It would be interesting to investigate the link between L -algebra and bidendriform bialgebras through the use of biplane trees.

The study of the operad on the three operations $\{\blacktriangleright, \blacktriangle, /\}$ is a start in order to study the link between bidendriform bialgebras and the skew-duplicial operade or L -algebras.

References

- [AS05] Marcelo Aguiar and Frank Sottile. Structure of the malvenuto–reutenauer hopf algebra of permutations. *Advances in Mathematics*, 191(2):225–275, 2005. URL: <https://www.sciencedirect.com/science/article/pii/S0001870804000787>, doi:<https://doi.org/10.1016/j.aim.2004.03.007>.
- [BDO20] Emily Burgunder and B er enice Delcroix-Oger. Structure theorems for dendriform and tridendriform algebras. *IRMA Lectures in Mathematics and Theoretical Physics Vol. 31, Algebraic Combinatorics, Resurgence, Moulds and Applications (CARMA), Volume 1*, 2020. URL: <https://www.ems-ph.>

org/books/book.php?proj_nr=251&srch=series%7Cirma, doi:<https://www.irif.fr/~bdelcroix/Recherche/stationnementFinal.pdf>.

- [BZ05] Nantel Bergeron and Mike Zabrocki. The Hopf algebras of symmetric functions and quasisymmetric functions in non-commutative variables are free and cofree, 2005. [arXiv:math/0509265](https://arxiv.org/abs/math/0509265).
- [Cha02] Frédéric Chapoton. Un théorème de Cartier–Milnor–Moore–Quillen pour les bigèbres dendrifformes et les algèbres braces. *J. of Pure and Applied Algebra*, 168(1):1 – 18, 2002. URL: <http://www.sciencedirect.com/science/article/pii/S0022404901000524>, doi:[https://doi.org/10.1016/S0022-4049\(01\)00052-4](https://doi.org/10.1016/S0022-4049(01)00052-4).
- [DHT01] G. Duchamp, F. Hivert, and J. Y. Thibon. Noncommutative symmetric functions vi: Free quasi-symmetric functions and related algebras, 2001. [arXiv:math/0105065](https://arxiv.org/abs/math/0105065).
- [Foi02a] L. Foissy. Les algèbres de hopf des arbres enracinés décorés, i. *Bulletin des Sciences Mathématiques*, 126(3):193–239, 2002. URL: <https://www.sciencedirect.com/science/article/pii/S0007449702011089>, doi:[https://doi.org/10.1016/S0007-4497\(02\)01108-9](https://doi.org/10.1016/S0007-4497(02)01108-9).
- [Foi02b] L. Foissy. Les algèbres de Hopf des arbres enracinés décorés, II. *Bulletin des Sciences Mathématiques*, 126(4):249 – 288, 2002. URL: <http://www.sciencedirect.com/science/article/pii/S0007449702011132>, doi:[https://doi.org/10.1016/S0007-4497\(02\)01113-2](https://doi.org/10.1016/S0007-4497(02)01113-2).
- [Foi07] Loïc Foissy. Bidendriform bialgebras, trees, and free quasi-symmetric functions. *Journal of Pure and Applied Algebra*, 209(2):439–459, 2007. doi: [10.1016/j.jpaa.2006.06.005](https://doi.org/10.1016/j.jpaa.2006.06.005).
- [Foi11] L. Foissy. Primitive elements of the Hopf algebra of free quasi-symmetric functions. *Contemporary Mathematics Combinatorics and Physics*, page 79–88, 2011. doi:[10.1090/conm/539/10629](https://doi.org/10.1090/conm/539/10629).
- [Ges83] Ira M. Gessel. Multipartite p-partitions and inner products of skew schur functions. volume 34, 1983.
- [GKL⁺95] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, and J.Y. Thibon. Noncommutative Symmetrical Functions. *Adv. in Math.*, 112(2):218 – 348, 1995. URL: <http://www.sciencedirect.com/science/article/pii/S0001870885710328>, doi:<https://doi.org/10.1006/aima.1995.1032>.

- [Hiv99] Florent Hivert. Combinatoire des fonctions quasi-symétriques. PhD thesis, Informatique fondamentale Université de Marne-la-Vallée, 1999. URL: <http://www.theses.fr/1999MARN0051>.
- [HNT04] F. Hivert, J. C. Novelli, and J. Y. Thibon. The algebra of binary search trees. 2004. URL: <https://arxiv.org/abs/math/0401089>, doi:10.48550/ARXIV.MATH/0401089.
- [Lod01] Jean-Louis Loday. Dialgebras, 2001. [arXiv:math/0102053](https://arxiv.org/abs/math/0102053).
- [LR98] Jean-Louis Loday and María O. Ronco. Hopf Algebra of the Planar Binary Trees. Adv. in Math., 139(2):293 – 309, 1998. URL: <http://www.sciencedirect.com/science/article/pii/S000187089817595>, doi:<https://doi.org/10.1006/aima.1998.1759>.
- [MR95] C. Malvenuto and C. Reutenauer. Duality between Quasi-Symmetrical Functions and the Solomon Descent Algebra. J. of Algebra, 177(3):967 – 982, 1995. URL: <http://www.sciencedirect.com/science/article/pii/S0021869385713361>, doi:<https://doi.org/10.1006/jabr.1995.1336>.
- [NPT13] Jean-Christophe Novelli, Frédéric Patras, and Jean-Yves Thibon. Natural endomorphisms of quasi-shuffle Hopf algebras. Bulletin de la Société Mathématique de France, 141(1):107–130, 2013. URL: <http://www.numdam.org/articles/10.24033/bsmf.2644/>, doi:10.24033/bsmf.2644.
- [NT06] J. C. Novelli and J. Y. Thibon. Polynomial realizations of some trialgebras, 2006. [arXiv:math/0605061](https://arxiv.org/abs/math/0605061).
- [NT07] Jean-Christophe Novelli and Jean-Yves Thibon. Hopf algebras and dendriform structures arising from parking functions. Fundamenta Mathematicae, 193(1):189–241, 2007. URL: <https://hal-upec-upem.archives-ouvertes.fr/hal-00622717>.
- [Phi07] Leroux Philippe. L-algebras, triplicial-algebras, within an equivalence of categories motivated by graphs, 2007. [arXiv:0709.3453](https://arxiv.org/abs/0709.3453).
- [Poi96] Stéphane Poirier. Fonctions symétriques, ensembles de descentes et classes de conjugaison. PhD thesis, Université du Québec à Montréal, 1996.
- [Ron00] María Ronco. Primitive elements in a free dendriform algebra. Contemp. Math., 267, 01 2000. doi:10.1090/conm/267/04274.
- [Ron02] María Ronco. Eulerian idempotents and Milnor–Moore theorem for certain non-cocommutative Hopf algebras. J. of Algebra, 254:152–172, 08 2002. doi:10.1016/S0021-8693(02)00097-2.

- [SI20] Neil J. A. Sloane and The OEIS Foundation Inc. The on-line encyclopedia of integer sequences, 2020. URL: <http://oeis.org/>.
- [Var19] Y. Vargas. Algèbres de Hopf de mots tassés et de fonctions motifs. PhD thesis, Université du Québec à Montréal, 2019.

Annexes

In [Tables 7 to 12](#) we have red-packed forests, blue-packed forest and bicolored-packed forests associated to all packed words of size smaller than 4.

In [Table 13](#) we have the isomorphism between **WQSym** (bases \mathcal{O} and \mathcal{Q}) and its dual (bases \mathcal{P} and \mathcal{R}) for size smaller than 3. Basis \mathcal{O} and \mathcal{P} are indexed by bicolored-packed forests. This illustrates the main theorem of [Section 4.4](#).

In [Table 14](#) we have the involution of [Corollary 152](#) for all packed words of size 4. They are organized by evaluations. Red-irreducible (resp. blue-irreducible) packed words are underlined in [red](#) (resp. [blue](#)) in the first (resp. second) column.

In [Table 15](#) we have the involution of [Corollary 152](#) for all red-irreducible packed words that are not blue-irreducible. It correspond to words underlined in red in front of a word underlined in blue in [Table 14](#).

The matrix of [Figure 6](#) is redundant with the column \mathcal{R} and \mathcal{P} of [Table 13](#). Note that even though the matrix of [Figure 7](#) is symmetric, it is not the case anymore on [Figure 10](#). Even if we restrict to permtuations, the matrix is not symmetric for size 5.

	123	132	213	231	312	321	122	212	221	112	121	211	111
123	1
132	.	1	-1	.	1
213	-1	.	1
231	-1	-1	1	1	-1
312	.	.	-1	.	1
321	1	.	.	-1	-1	1
122	1
212	1	1
221	1	-1	.	.	.
112	1	.	.	.
121	-1	.	.	.	1	1	.
211	-1	1	.
111	1

Figure 6: Change-of-basis matrix from \mathcal{P}_3 to \mathcal{R}_3 .

	123	132	213	231	312	321	122	212	221	112	121	211	111
123	1
132	.	1
213	.	.	1
231	1
312	.	.	.	1
321	1
122	1	1
212	1
221	1	1	.
112	1	.	.	.
121	1	.	1	.	.
211	1
111	1

Figure 7: Change-of-basis matrix from Q_3 to \mathbb{R}_3 .

w	$F_R(w)$	$F_B(w)$	$F_{RB}(w)$	$F_{BR}(w)$
1	(1)	(1 ^o)	(1)	(1)

w	$F_R(w)$	$F_B(w)$	$F_{RB}(w)$	$F_{BR}(w)$
12				
21	(1) (1)	(1 ^o) (1 ^o)	(1) (1)	(1) (1)
11	(1,2)		(11)	(11)

w	$F_R(w)$	$F_B(w)$	$F_{RB}(w)$	$F_{BR}(w)$
123				
132			(132)	(132)
213				
231				
312				
321	(1) (1) (1)	(1 ^o) (1 ^o) (1 ^o)	(1) (1) (1)	(1) (1) (1)
122				
212				
221	(1,2) (1)		(11) (1)	(1) (11)
112				
121			(121)	(121)
211	(1) (1,2)		(1) (11)	(11) (1)
111	(1,2,3)		(111)	(111)

Table 7: All packed words of size smaller than 3 and forests associated to it.

w	$T_R(w)$	$T_B(w)$	$T_{RB}(w)$	$T_{BR}(w)$
1234				
1243				
1324				
1342				
1423				
1432				
2134				
2143				
2314				
2413				
3124				
3142				
3214				

Table 8: Packed words of size 4 and associated forests (part 1).

w	$T_R(w)$	$T_B(w)$	$T_{RB}(w)$	$T_{BR}(w)$
1233				
1323				
1332				
2133				
2313				
3123				
3132				
3213				

Table 9: Packed words of size 4 and associated forests (part 2).

w	$T_R(w)$	$T_B(w)$	$T_{RB}(w)$	$T_{BR}(w)$
1223				
1232				
1322				
2123				
2132				
2213				
2312				









Table 10: Packed words of size 4 and associated forests (part 3).

w	$T_R(w)$	$T_B(w)$	$T_{RB}(w)$	$T_{BR}(w)$
1123				
1132				
1213				
1231				
1312				
1321				
2113				
2131				

Table 11: Packed words of size 4 and associated forests (part 4).

w	$T_R(w)$	$T_B(w)$	$T_{RB}(w)$	$T_{BR}(w)$
1222				
2122				
2212				
1122				
1212				
1221				
2112				
2121				
1112				
1121				
1211				
1111				

Table 12: Packed words of size 4 and associated forests (part 5).

\mathbb{R}_1	\mathbb{P} 	\mathbb{O} 	\mathbb{Q}_1
$\mathbb{R}_{12} - \mathbb{R}_{21}$	\mathbb{P} 	\mathbb{O} 	$\mathbb{Q}_{12} - \mathbb{Q}_{21}$
\mathbb{R}_{21}	\mathbb{P} 	\mathbb{O} 	\mathbb{Q}_{21}
\mathbb{R}_{11}	\mathbb{P} 	\mathbb{O} 	\mathbb{Q}_{11}

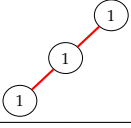
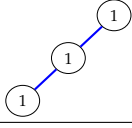


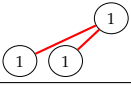
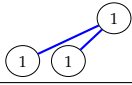
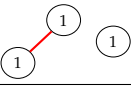
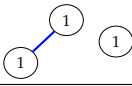
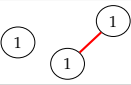
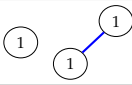

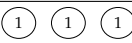














\mathbb{R}	\mathbb{P}	\mathbb{O}	\mathbb{Q}
$123 - 213 - 231 + 321$			$123 - 213 - 312 + 321$
$132 - 231$			$132 - 312$
$213 - 312 + 231 - 132$			$213 + 312 - 231 - 132$
$231 - 321$			$312 - 321$
$132 + 312 - 231 - 321$			$231 + 132 - 321 - 312$
321			321
$122 - 121 + 212 - 211$			$122 - 221$
212			$122 - 212$
221			211
$112 - 221$			$112 - 211$
121			$121 - 211$
$121 + 211$			221
111			111

Table 13: The automorphism of $\mathbf{WQSym}_{\leq 3}$.

1234	1234
<u>1243</u>	<u>1243</u>
<u>1324</u>	<u>1324</u>
1342	1342
<u>1423</u>	<u>1423</u>
<u>1432</u>	<u>1432</u>
2134	2134
<u>2143</u>	<u>2143</u>
2314	3124
2341	4123
<u>2413</u>	<u>3142</u>
2431	4132
3124	2314
3142	2413
3214	3214
3241	4213
3412	3412
3421	4312
4123	2341
4132	2431
4213	3241
4231	4231
4312	3421
4321	4321

<u>1233</u>	<u>3123</u>
<u>1323</u>	<u>1323</u>
<u>1332</u>	<u>1332</u>
2133	3213
2313	2313
2331	3212
<u>3123</u>	<u>1233</u>
<u>3132</u>	<u>3132</u>
3213	2133
3231	3122
3312	2311
3321	3211

1223	2123
<u>1232</u>	<u>1232</u>
<u>1322</u>	<u>1322</u>
2123	1223
<u>2132</u>	<u>2132</u>
2213	2113
2231	3112
<u>2312</u>	<u>2131</u>
2321	3121
3122	3231
3212	2331
3221	3221

1123	1123
<u>1132</u>	<u>1132</u>
1213	1213
<u>1231</u>	<u>1231</u>
<u>1312</u>	<u>1312</u>
<u>1321</u>	<u>1321</u>
2113	2213
<u>2131</u>	<u>2312</u>
2311	3312
3112	2231
3121	2321
3211	3321

<u>1122</u>	<u>2112</u>
<u>1212</u>	<u>1212</u>
<u>1221</u>	<u>1221</u>
<u>2112</u>	<u>1122</u>
<u>2121</u>	<u>2121</u>
2211	2211

<u>1222</u>	<u>2212</u>
<u>2122</u>	<u>2122</u>
<u>2212</u>	<u>1222</u>
2221	2111

1112	1112
<u>1121</u>	<u>1121</u>
<u>1211</u>	<u>1211</u>
2111	2221

<u>1111</u>	<u>1111</u>
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Table 14: The involution $w \mapsto \hat{w}$ on packed words of size 4.

23514	41253	24314	31424	23413	31242	22413	31142
24513	41352	24413	31442	24313	31422	23412	31241
25314	41523	41234	12344	32413	32142	24213	31412
25413	41532	41324	13244	34123	23141	24312	31421
32514	42153	42134	21344	34213	32141		
35124	34152	42314	31244	32313	23133		
35214	43152	42413	34142	33123	12333		
		43124	23144	33213	21333		
		43214	32144				
23213	21313	22312	21131	22212	12222	24113	33142
23312	21331	23212	21311				
31223	21233						
32123	12233	31123	11233	23112	22131	22112	11222
32213	21133	31213	12133				
32312	23131	32113	22133	21112	11122		

Table 15: The involution $w \mapsto \hat{w}$ on red-irreducible packed words that are not blue-irreducible of size 5.

