Autoduality of WQSym, the Hopf algebra on packed words

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22 Octobre 2018

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Permutations

Definition

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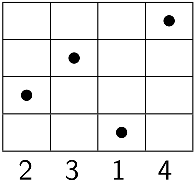
A representation :



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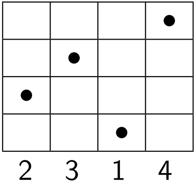
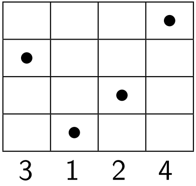
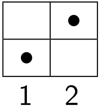


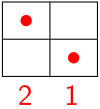
$\rightarrow$ transposition $\rightarrow$

$\rightarrow$ inversion $\rightarrow$



Shuffle product





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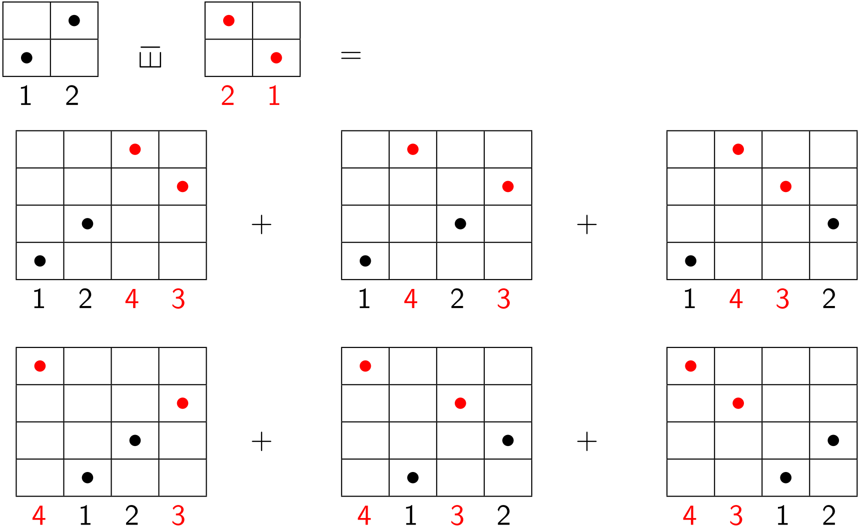

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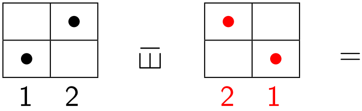
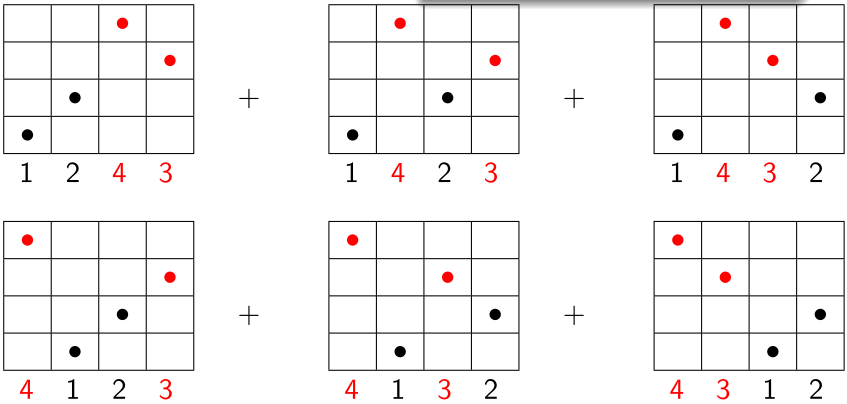


𝔽

$$\mathbb{F}_{12}\mathbb{F}_{21} = \mathbb{F}_{1243} + \mathbb{F}_{1423} + \mathbb{F}_{1432} + \mathbb{F}_{4123} + \mathbb{F}_{4132} + \mathbb{F}_{4312}$$



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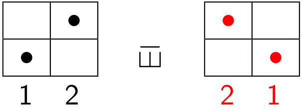
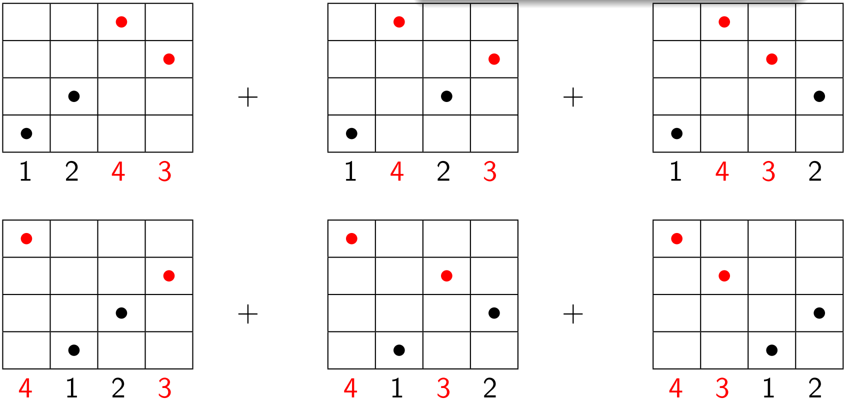


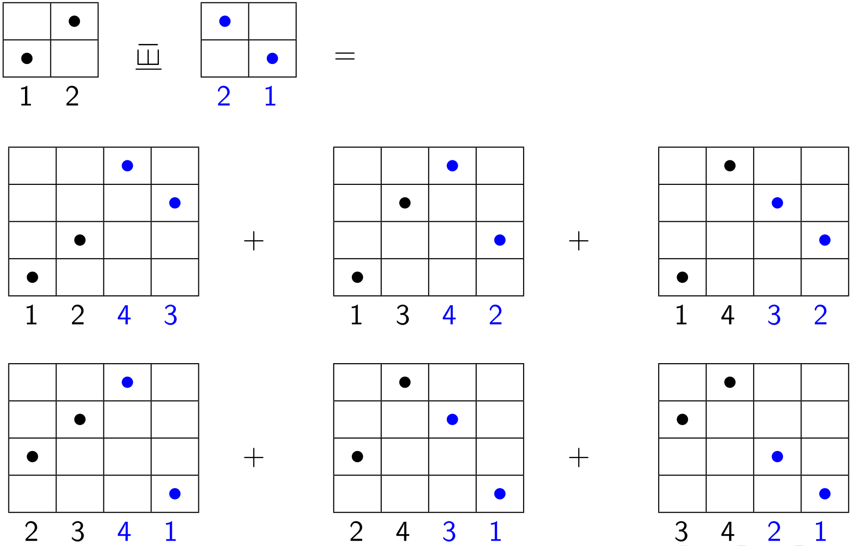
$\overline{\sqcup\!\sqcup}$ =

$\mathbb{F}$

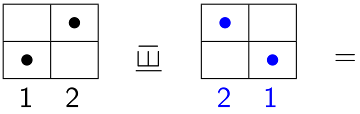
$$\mathbb{F}_\sigma \mathbb{F}_\mu := \sum_{\nu \in \sigma \overline{\sqcup\!\sqcup} \mu} \mathbb{F}_\nu$$



Shuffle product on values

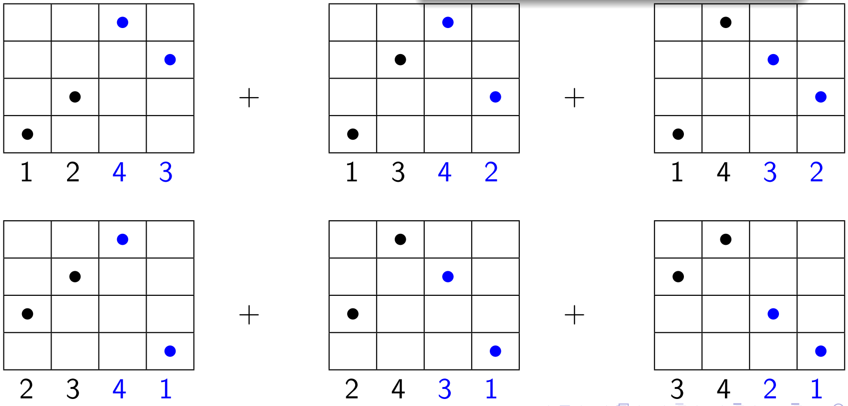


Shuffle product on values



$\mathbb{G}$

$\mathbb{G}_{12}\mathbb{G}_{21} = \mathbb{G}_{1243} + \mathbb{G}_{1342} + \mathbb{G}_{1432} + \mathbb{G}_{2341} + \mathbb{G}_{2431} + \mathbb{G}_{3421}$



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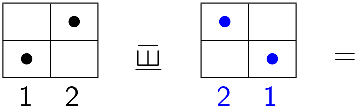

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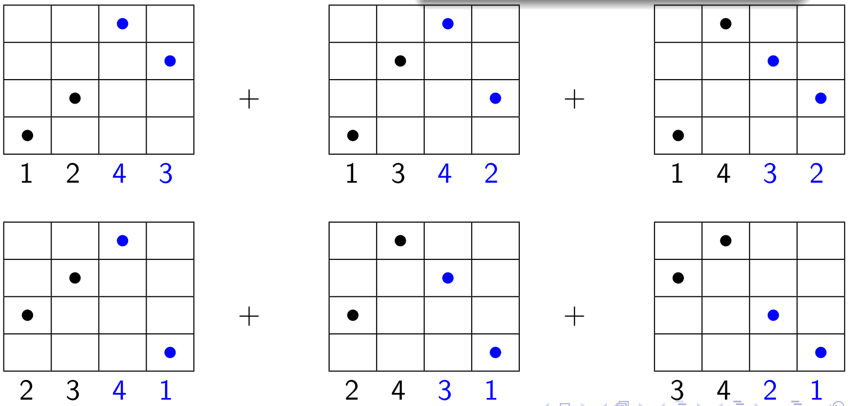
$$\mathbb{G}_\sigma \mathbb{G}_\mu := \sum_{\substack{\nu = uv, \\ std(u)=\sigma, \\ std(v)=\mu}} \mathbb{G}_\nu$$



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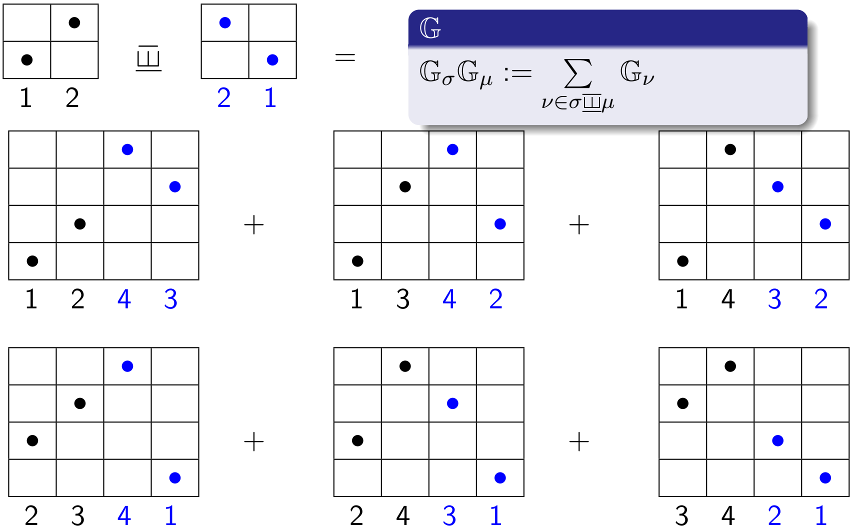
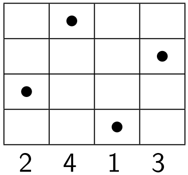


Vertical disassembly



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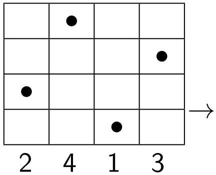

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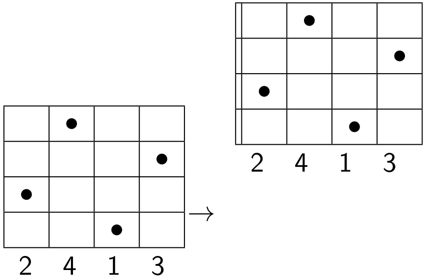

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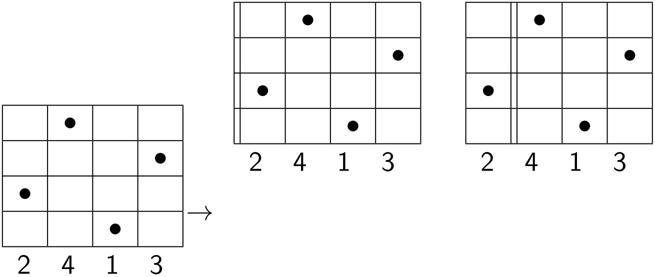

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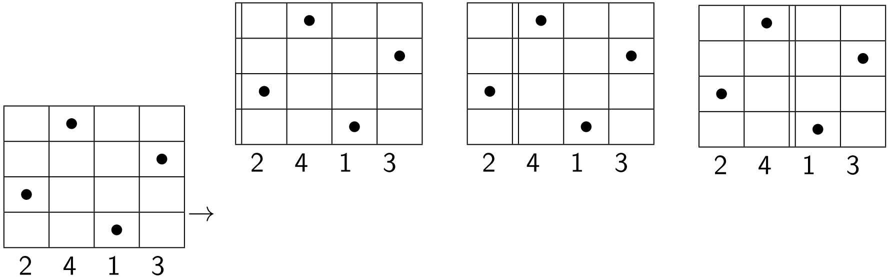

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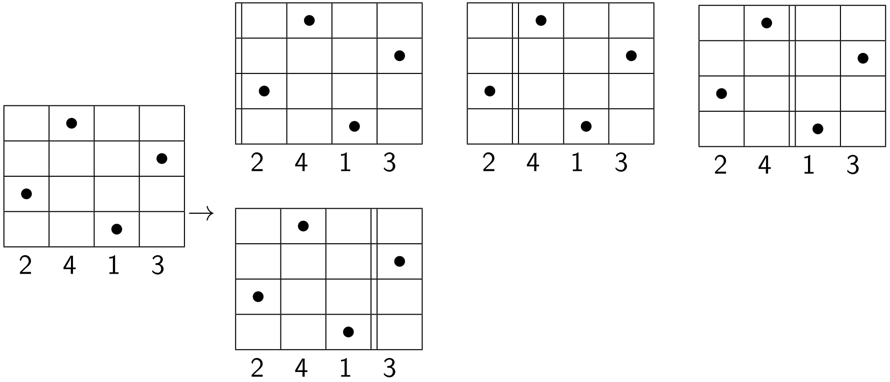

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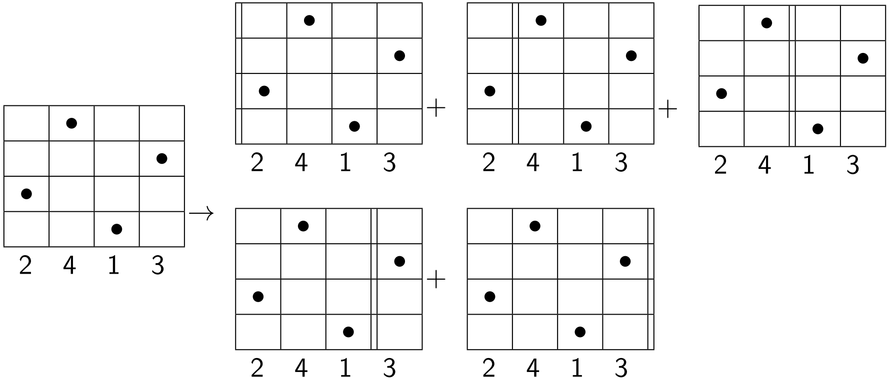
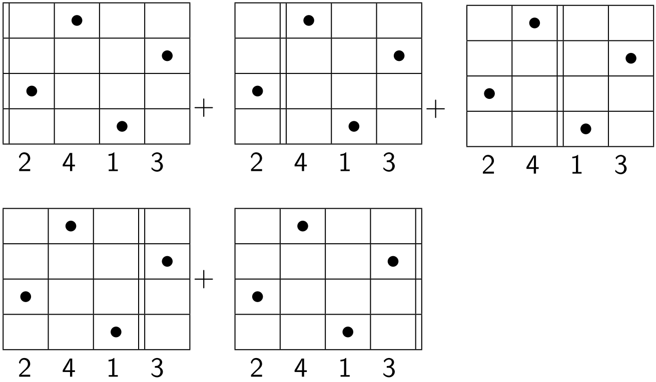


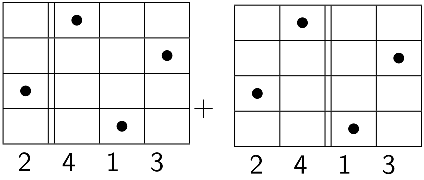
Vertical disassembly

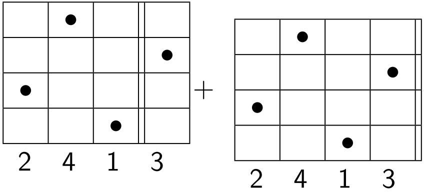
$\mathbb{F}_{2413} \xrightarrow{\Delta}$



Vertical disassembly

$\mathbb{F}_{2413}$ $\xrightarrow{\Delta}$ $\mathbb{F}_{\epsilon} \otimes \mathbb{F}_{2413}$ +





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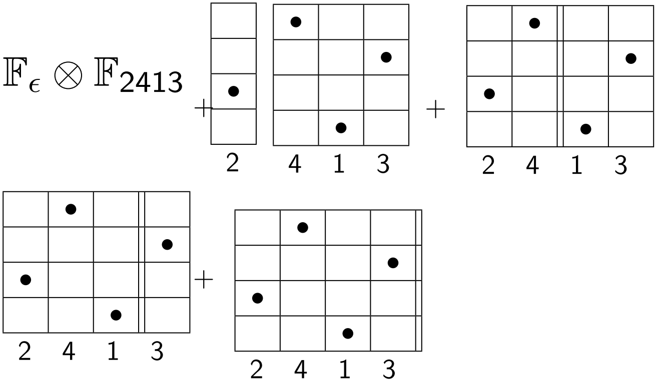

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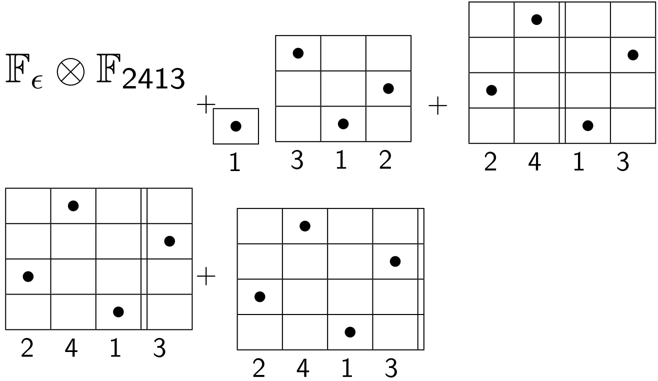
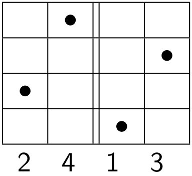
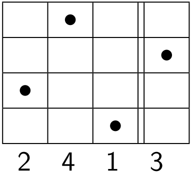


Vertical disassembly

$$\mathbb{F}_{\epsilon} \otimes \mathbb{F}_{2413} + \mathbb{F}_{1} \otimes \mathbb{F}_{312} +$$



$$\mathbb{F}_{2413} \xrightarrow{\Delta}$$



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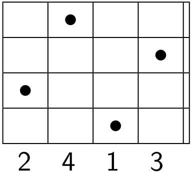
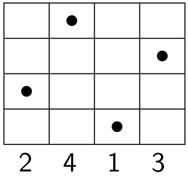


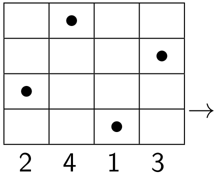
Vertical disassembly

$$\mathbb{F}_{2413} \overset{\Delta}{\rightarrow} \mathbb{F}_\epsilon \otimes \mathbb{F}_{2413} + \mathbb{F}_1 \otimes \mathbb{F}_{312} + \mathbb{F}_{12} \otimes \mathbb{F}_{12} + \mathbb{F}_{231} \otimes \mathbb{F}_1 + \mathbb{F}_{2413} \otimes \mathbb{F}_\epsilon$$

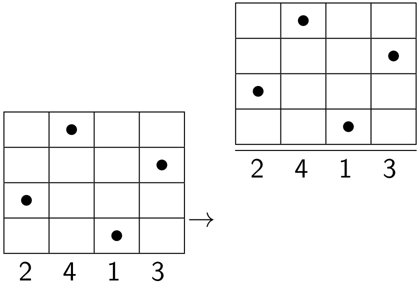
Horizontal disassembly



Horizontal disassembly



Horizontal disassembly



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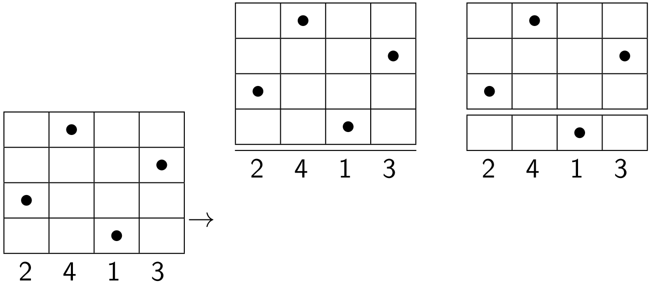
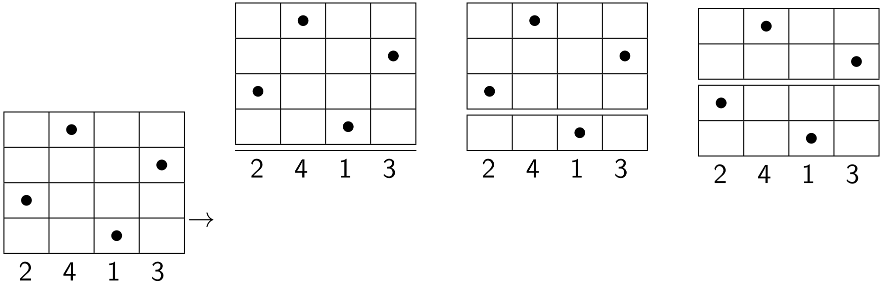
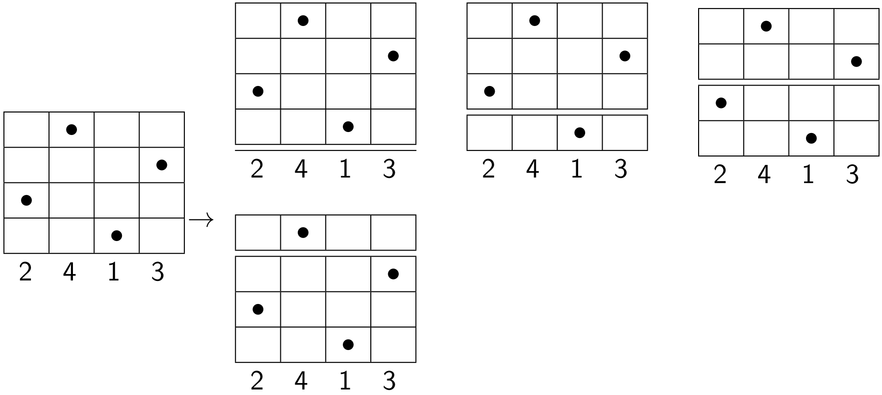


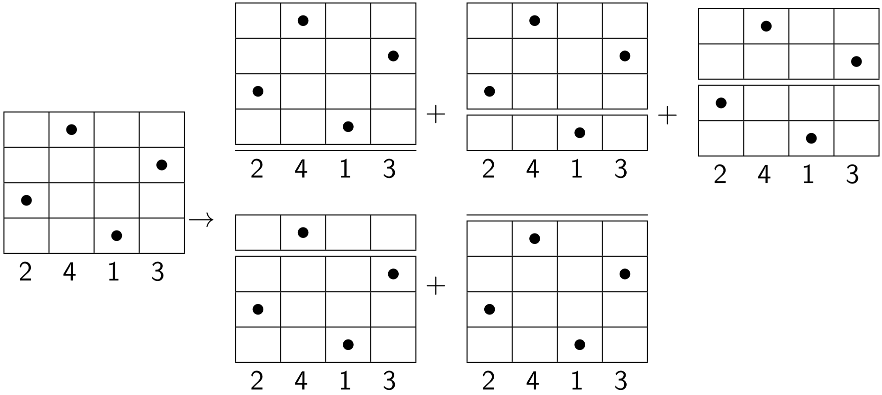
Horizontal disassembly



Horizontal disassembly

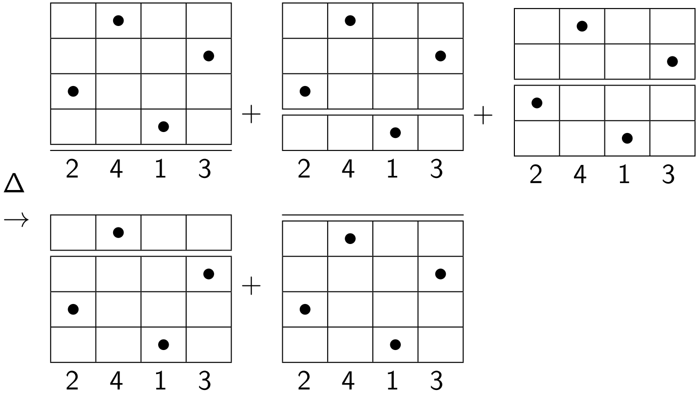


Horizontal disassembly



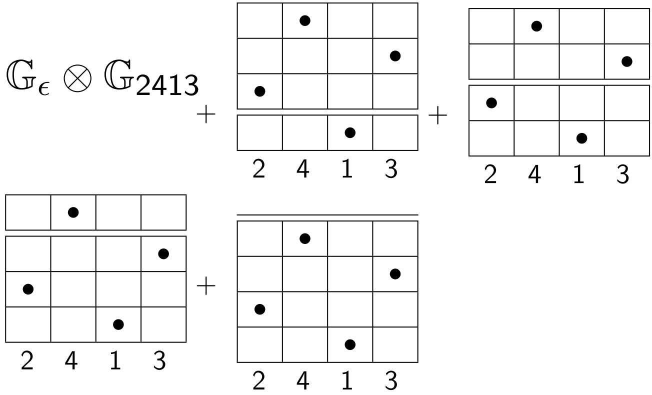
Horizontal disassembly

$\mathbb{G}_{2413}$

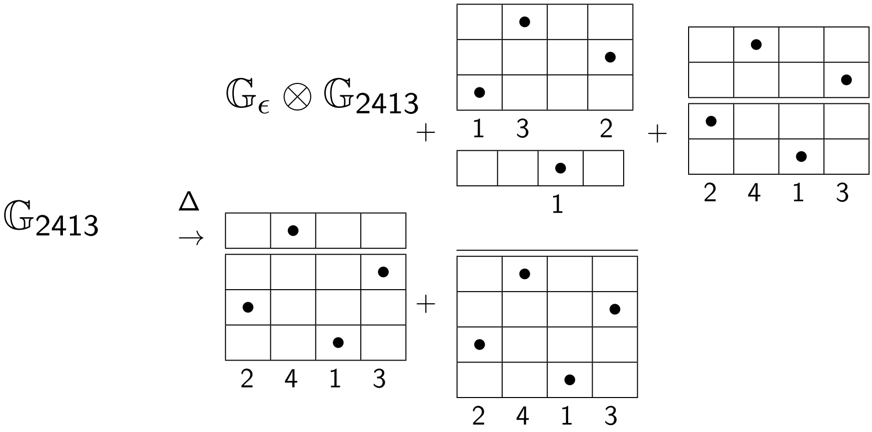


Horizontal disassembly

$\mathbb{G}_{2413} \xrightarrow{\Delta}$



Horizontal disassembly



Horizontal disassembly

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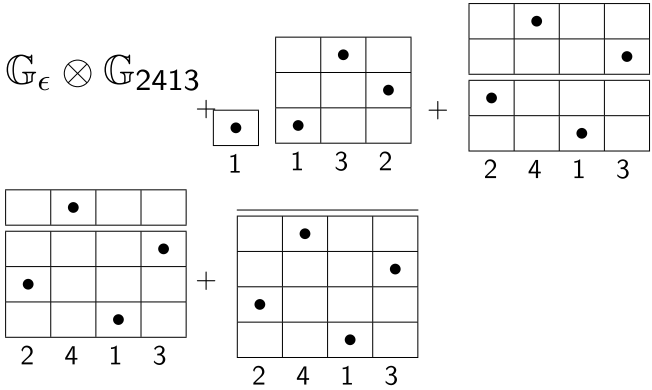

Horizontal disassembly

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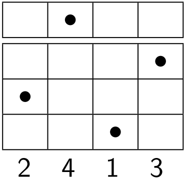

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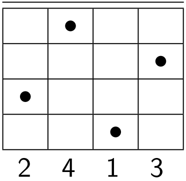

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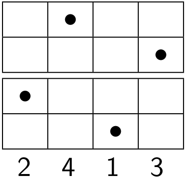
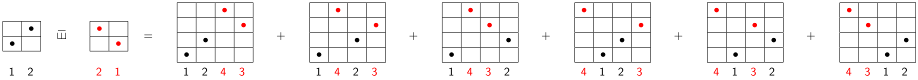


$\mathbb{G}_{\epsilon} \otimes \mathbb{G}_{2413} + \mathbb{G}_{1} \otimes \mathbb{G}_{132} + \ldots$

Horizontal disassembly

$$\mathbb{G}_{2413} \xrightarrow{\Delta} \mathbb{G}_\epsilon \otimes \mathbb{G}_{2413} + \mathbb{G}_1 \otimes \mathbb{G}_{132} + \mathbb{G}_{21} \otimes \mathbb{G}_{21} + \mathbb{G}_{213} \otimes \mathbb{G}_1 + \mathbb{G}_{2413} \otimes \mathbb{G}_\epsilon$$

Duality of FQSym

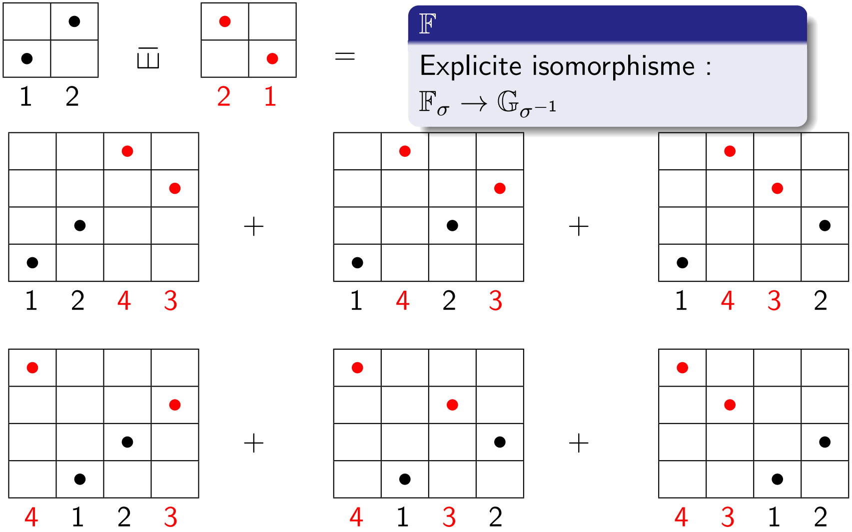


Duality

If H is a Hopf algebra,

$$< \Delta(z), x \otimes y > = < z, x.y > \qquad \forall x, y \in H, z \in H^*,$$

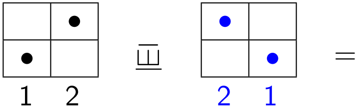
$$< y.z, x > = < y \otimes z, \Delta(x) > \qquad \forall x \in H, y, z \in H^*$$



$\mathbb{F}$

Explicite isomorphisme :

$\mathbb{F}_\sigma \rightarrow \mathbb{G}_{\sigma^{-1}}$



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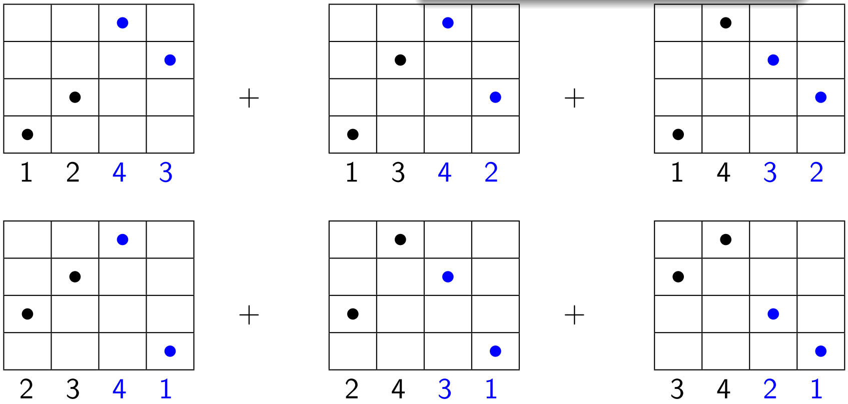

Packed words

Definition

A word w with letters in $\{1, ..., n\}$ is a packed word if for each number $k > 1$ appearing in w, the number $k - 1$ appears in w too.

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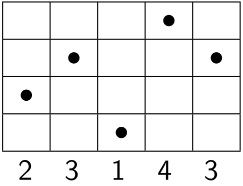
With the same representation : $\#$lines $\leq$ $\#$columns



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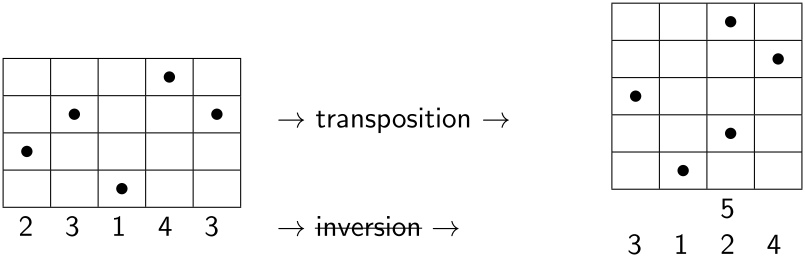
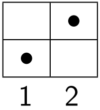
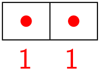
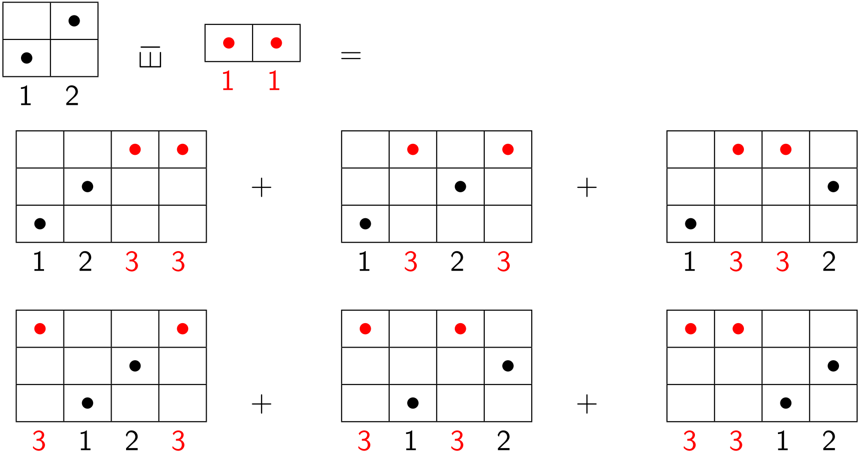


Shuffle product on packed words

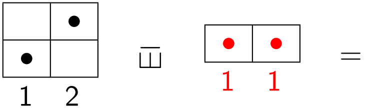




Shuffle product on packed words

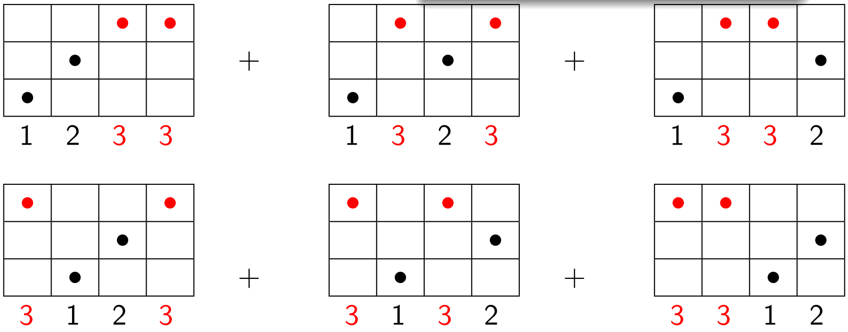


Shuffle product on packed words

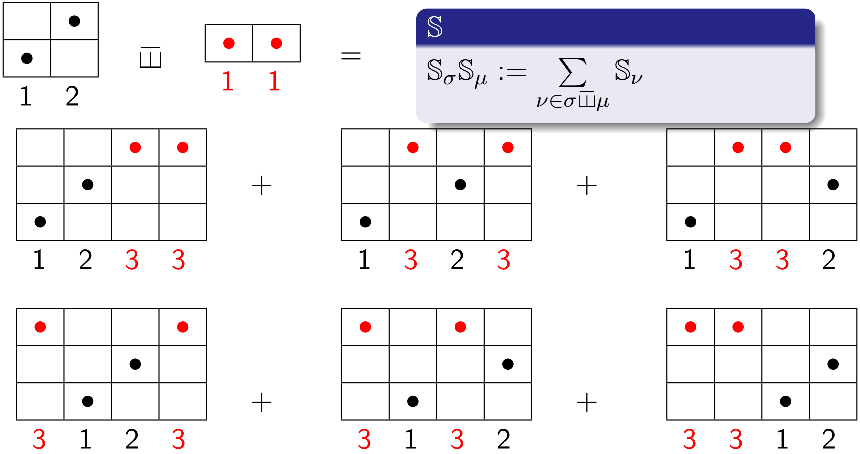


$\mathbb{S}$

$$\mathbb{S}_{12}\mathbb{S}_{11} = \mathbb{S}_{1233} + \mathbb{S}_{1323} + \mathbb{S}_{1332} + \mathbb{S}_{3123} + \mathbb{S}_{3132} + \mathbb{S}_{3312}$$



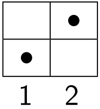
Shuffle product on packed words

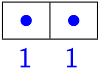


$\mathbb{S}$

$$\mathbb{S}_\sigma \mathbb{S}_\mu := \sum_{\nu \in \sigma \overline{\sqcup\!\sqcup} \mu} \mathbb{S}_\nu$$

Quasi shuffle product on values





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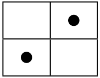
$$\mathbb{M}_\sigma \mathbb{M}_\mu := \sum_{\substack{\nu = uv,\\ pack(u)=\sigma,\\ pack(v)=\mu}} \mathbb{M}_\nu$$

Quasi shuffle product on values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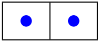
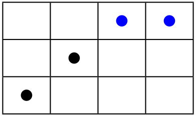
$\overline{\underline{\sqcup\!\sqcup}}$


1 1

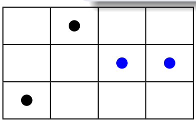
=

$\mathbb{M}$

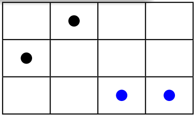
$$\mathbb{M}_\sigma \mathbb{M}_\mu := \sum_{\substack{\nu = uv, \\ pack(u)=\sigma, \\ pack(v)=\mu}} \mathbb{M}_\nu$$

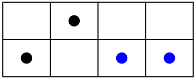

1 2 3 3

+

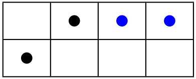

1 3 2 2

+

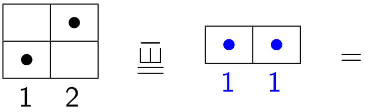

2 3 1 1


1 2 1 1

+

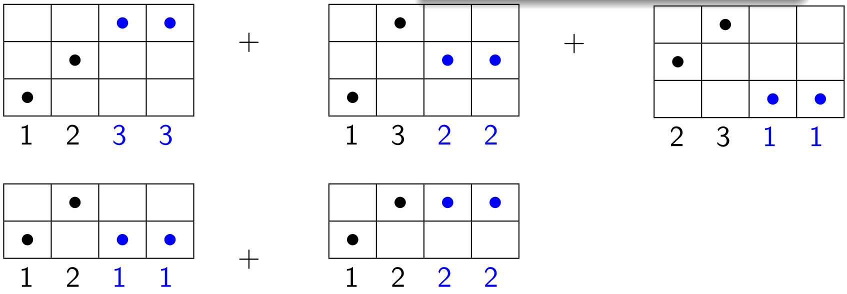

1 2 2 2

Quasi shuffle product on values

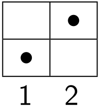


M

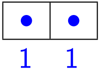
$$\mathbb{M}_{12}\mathbb{M}_{11} = \mathbb{M}_{1233} + \mathbb{M}_{1322} + \mathbb{M}_{2311} + \mathbb{M}_{1211} + \mathbb{M}_{1222}$$



Shuffle product on values



$\overline{\sqcup\!\sqcup}$

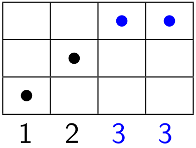


$=$

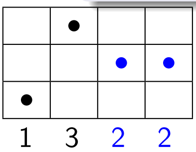
$\mathbb{L}$

$$\mathbb{L}_{12}\mathbb{L}_{11} = \mathbb{L}_{1233} + \mathbb{L}_{1322} + \mathbb{L}_{2311}$$

$$\mathbb{L}_{\sigma}\mathbb{L}_{\mu} := \sum_{\nu \in \sigma \overline{\sqcup\!\sqcup} \mu} \mathbb{L}_{\nu}$$



$+$



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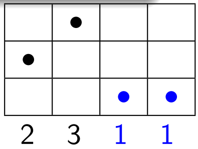
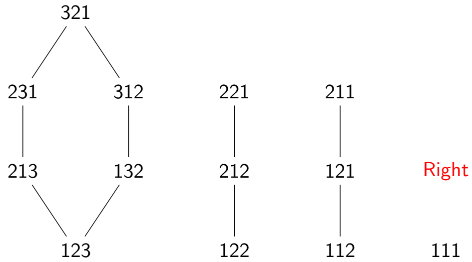
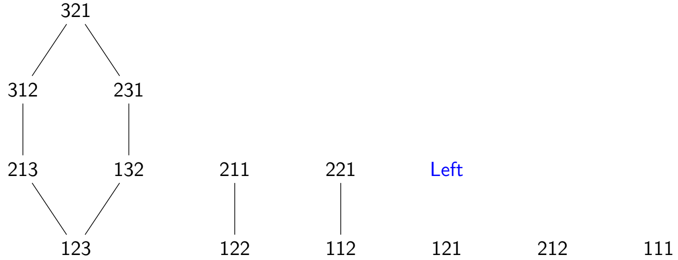


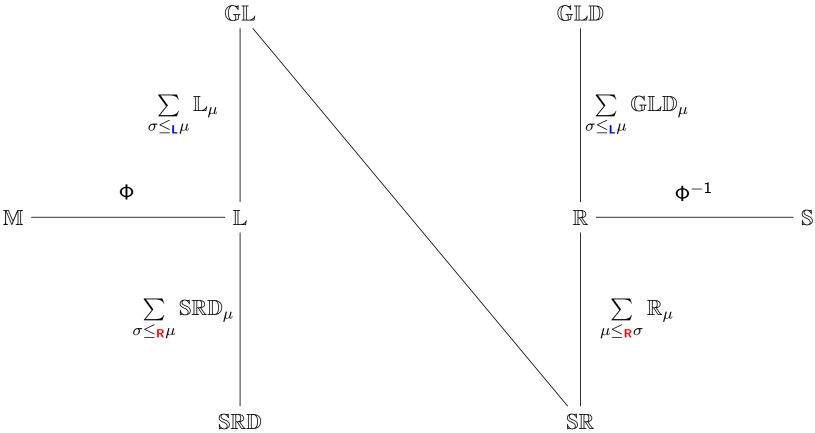
Poset





Poset :
Reflexivity,
Transitivity,
Antisymmetrie.

From $\mathbb{M}$ to $\mathbb{S}$ in **WQSym**



Some matrices

	123	132	213	231	312	321	122	212	221	112	121	211	111
123	.	.	.	.	.	1	.	.	.	.	.	.	.
132	.	-1	1	1	.	.	.	.	.	.	.	.	.
213	.	1	-1	.	1	.	.	.	.	.	.	.	.
231	.	1	.	.	.	.	.	.	.	.	.	.	.
312	.	.	1	.	.	.	.	.	.	.	.	.	.
321	1	.	.	.	.	.	.	.	.	.	.	.	.
122	.	.	.	.	.	.	1	1	1	-1	.	.	.
212	.	.	.	.	.	.	1	1	.	.	.	.	.
221	.	.	.	.	.	.	1	.	.	.	.	.	.
112	.	.	.	.	.	.	-1	.	.	1	1	1	.
121	.	.	.	.	.	.	.	.	.	1	1	.	.
211	.	.	.	.	.	.	.	.	.	1	.	.	.
111	.	.	.	.	.	.	.	.	.	.	.	.	1

Figure: Transformation matrix from the basis $\mathbb{L}$ to $\mathbb{R}$ over packed words of size 3.

Some matrices

	123	132	213	231	312	321	122	212	221	112	121	211	111
123	0	0	0	0	0	1	0	0	1/2	0	0	1/2	1/6
132	0	0	0	1	0	0	0	0	1/2	0	1/2	0	1/6
213	0	0	0	0	1	0	0	1/2	0	0	0	1/2	1/6
231	0	1	0	1	-1	0	1/2	-1/2	1/2	0	1	-1/2	1/6
312	0	0	1	-1	1	0	0	1	-1/2	1/2	-1/2	1/2	1/6
321	1	0	0	0	0	0	1/2	0	0	1/2	0	0	1/6
122	0	0	0	1/2	0	1/2	0	0	3/2	0	1/4	1/4	2/3
212	0	0	1/2	-1/2	1	0	0	7/4	-5/4	1/4	-1/4	1/2	1/6
221	1/2	1/2	0	1/2	-1/2	0	3/2	-5/4	1/4	1/4	1/2	3/4	2/3
112	0	0	0	0	1/2	1/2	0	1/4	1/4	0	0	3/2	2/3
121	0	1/2	0	1	-1/2	0	1/4	-1/4	1/2	0	7/4	-5/4	1/6
211	1/2	0	1/2	-1/2	1/2	0	1/4	1/2	3/4	3/2	-5/4	1/4	2/3
111	1/6	1/6	1/6	1/6	1/6	1/6	2/3	1/6	2/3	2/3	1/6	2/3	13/6

Figure: Transformation matrix from the basis $\mathbb{S}$ to $\mathbb{M}$ over packed words of size 3.

Contributions

- Implementing packed words in Sage, *#25916 implement Packed Words*.

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- Some conjecures
  - This describe an infinity of automorphisme of **WQSym**.
  - Generalization to **PQSym** (on parking functions).