

Autoduality of WQSym, the Hopf algebra on packed words

Hugo Mlodecki

Supervisors :

Florent Hivert

Viviane Pons

22 Octobre 2018

Table of contents

- 1 FQSym / permutations
- 2 WQSym / packed words
- 3 Contributions

Permutations

Definition

A permutation of size n is a word on the alphabet $\{1, 2, \dots, n\}$ where each letter appears exactly one time.

Permutations

Definition

A permutation of size n is a word on the alphabet $\{1, 2, \dots, n\}$ where each letter appears exactly one time.

A representation :

			•
	•		
•			
		•	
2	3	1	4

Permutations

Definition

A permutation of size n is a word on the alphabet $\{1, 2, \dots, n\}$ where each letter appears exactly one time.

A representation :

			•
	•		
•			
		•	

2 3 1 4

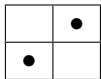
→ transposition →

			•
•			
		•	
	•		

→ inversion →

3 1 2 4

Shuffle product



1 2



2 1

Shuffle product

$$\begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \\ \hline \end{array} \quad \boxplus \quad \begin{array}{|c|c|} \hline \bullet & \\ \hline & \bullet \\ \hline \end{array} =$$

1 2
2 1

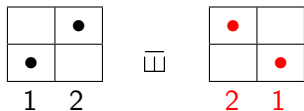
$$\begin{array}{|c|c|c|c|} \hline & & \bullet & \\ \hline & & & \bullet \\ \hline & \bullet & & \\ \hline \bullet & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & & \bullet \\ \hline & & \bullet & \\ \hline \bullet & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & \bullet & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline \end{array}$$

1 2 4 3
1 4 2 3
1 4 3 2

$$\begin{array}{|c|c|c|c|} \hline \bullet & & & \\ \hline & & & \bullet \\ \hline & & \bullet & \\ \hline & \bullet & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \bullet & & & \\ \hline & & \bullet & \\ \hline & & & \bullet \\ \hline & \bullet & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \bullet & & & \\ \hline & \bullet & & \\ \hline & & & \bullet \\ \hline & & \bullet & \\ \hline \end{array}$$

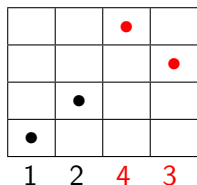
4 1 2 3
4 1 3 2
4 3 1 2

Shuffle product

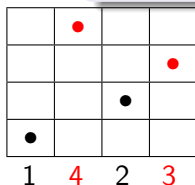


=

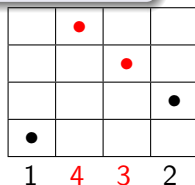
$$\mathbb{F} \mathbb{F}_{12} \mathbb{F}_{21} = \mathbb{F}_{1243} + \mathbb{F}_{1423} + \mathbb{F}_{1432} + \mathbb{F}_{4123} + \mathbb{F}_{4132} + \mathbb{F}_{4312}$$



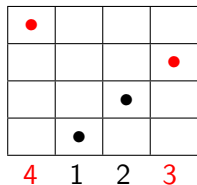
+



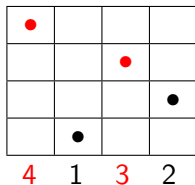
+



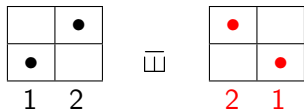
+



+

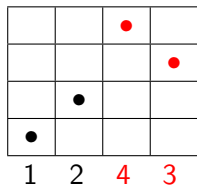


Shuffle product

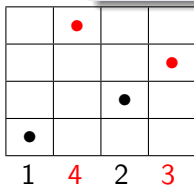


=

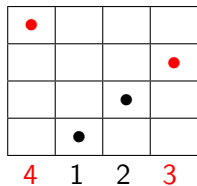
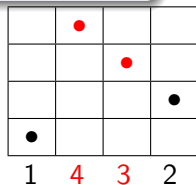
$$\mathbb{F} \text{ } \mathbb{F}_\sigma \mathbb{F}_\mu := \sum_{\nu \in \sigma \bar{\sqcup} \mu} \mathbb{F}_\nu$$



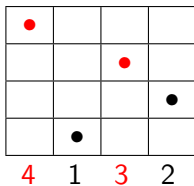
+



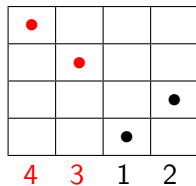
+



+



+



Shuffle product on values

$$\begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \\ \hline \end{array} \quad \underline{\underline{\sqcup}} \quad \begin{array}{|c|c|} \hline \bullet & \\ \hline & \bullet \\ \hline \end{array} =$$

1 2
2 1

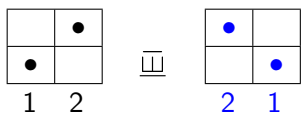
$$\begin{array}{|c|c|c|c|} \hline & & \bullet & \\ \hline & & & \bullet \\ \hline & \bullet & & \\ \hline \bullet & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & \bullet & \\ \hline & \bullet & & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & \bullet & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline \end{array}$$

1 2 4 3
1 3 4 2
1 4 3 2

$$\begin{array}{|c|c|c|c|} \hline & & \bullet & \\ \hline & \bullet & & \\ \hline \bullet & & & \\ \hline & & & \bullet \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & \bullet & \\ \hline \bullet & & & \\ \hline & & & \bullet \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline \bullet & & & \\ \hline & & \bullet & \\ \hline & & & \bullet \\ \hline \end{array}$$

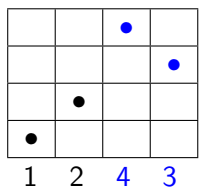
2 3 4 1
2 4 3 1
3 4 2 1

Shuffle product on values

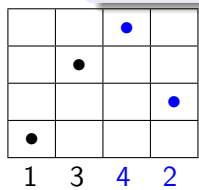


$$\mathbb{G}$$

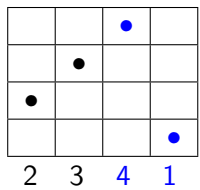
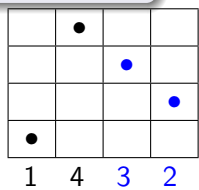
$$\mathbb{G}_{12}\mathbb{G}_{21} = \mathbb{G}_{1243} + \mathbb{G}_{1342} + \mathbb{G}_{1432} + \mathbb{G}_{2341} + \mathbb{G}_{2431} + \mathbb{G}_{3421}$$



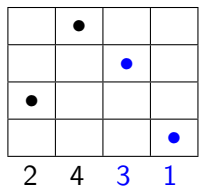
+



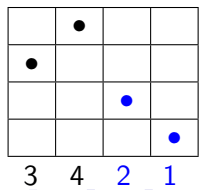
+



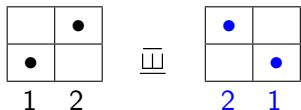
+



+

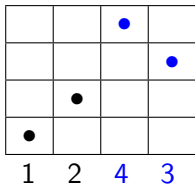


Shuffle product on values

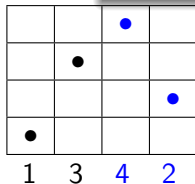


=

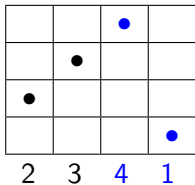
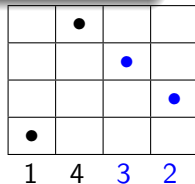
$$\begin{array}{c}
 \mathbb{G} \\
 \mathbb{G}_\sigma \mathbb{G}_\mu := \sum_{\substack{\nu=uv, \\ \text{std}(u)=\sigma, \\ \text{std}(v)=\mu}} \mathbb{G}_\nu
 \end{array}$$



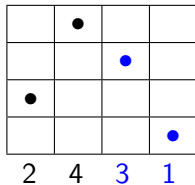
+



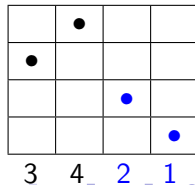
+



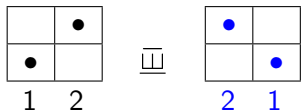
+



+

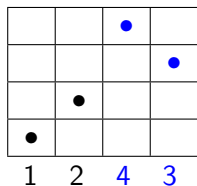


Shuffle product on values

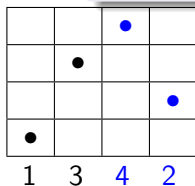


=

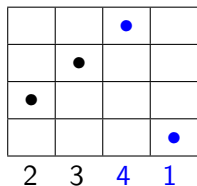
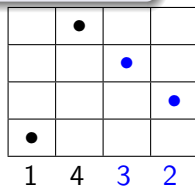
$$\mathbb{G} \sigma \mathbb{G} \mu := \sum_{\nu \in \sigma \underline{\underline{\mu}}} \mathbb{G} \nu$$



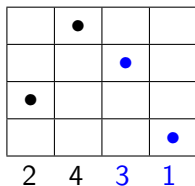
+



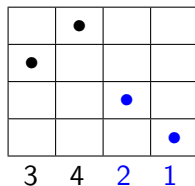
+



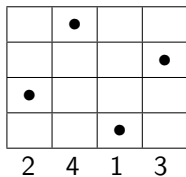
+



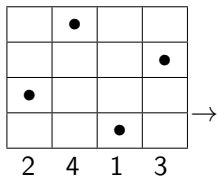
+



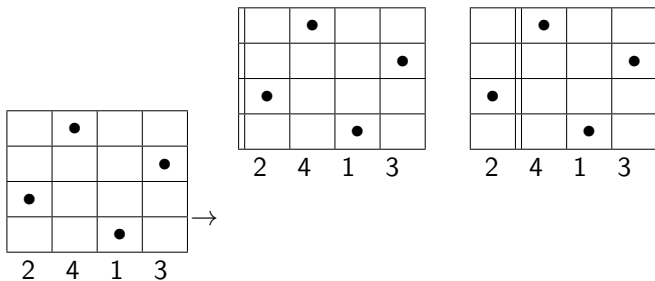
Vertical disassembly



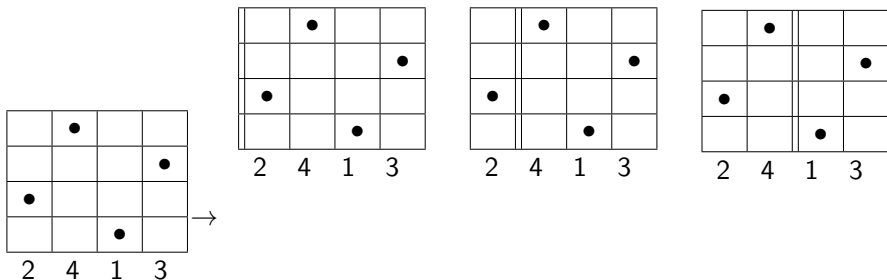
Vertical disassembly



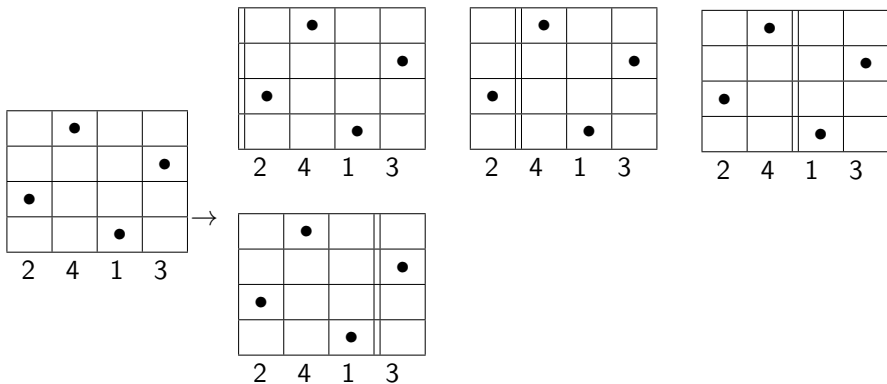
Vertical disassembly



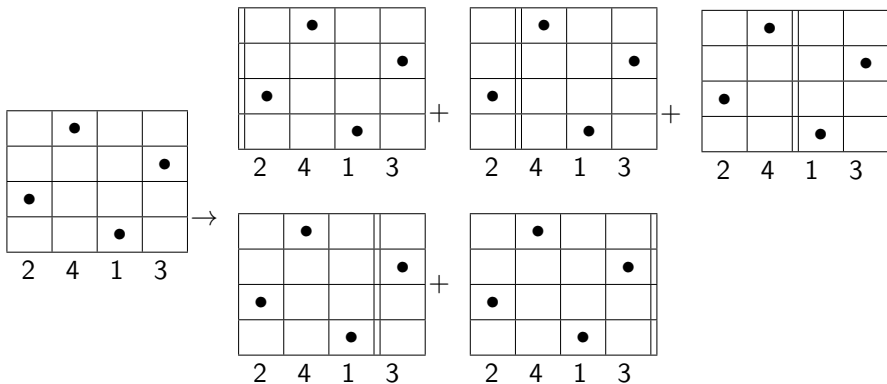
Vertical disassembly



Vertical disassembly

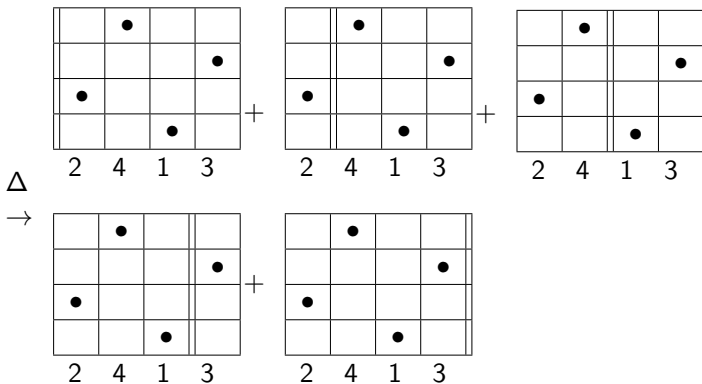


Vertical disassembly

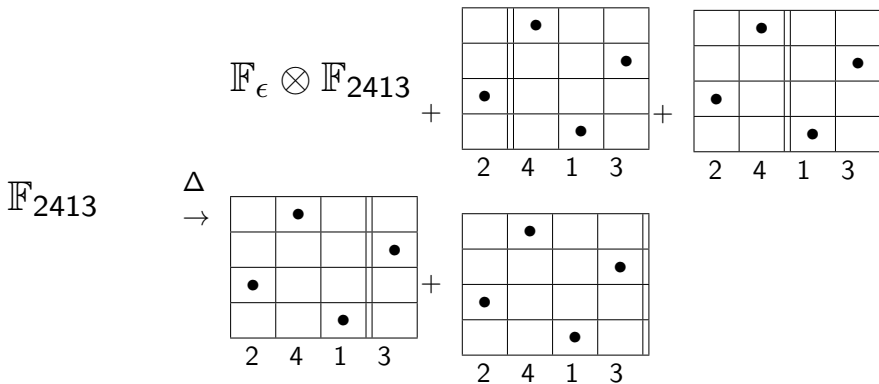


Vertical disassembly

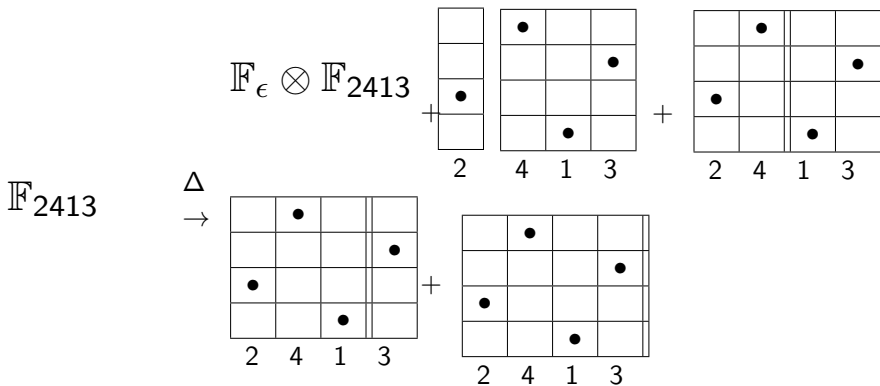
\mathbb{F}_{2413}



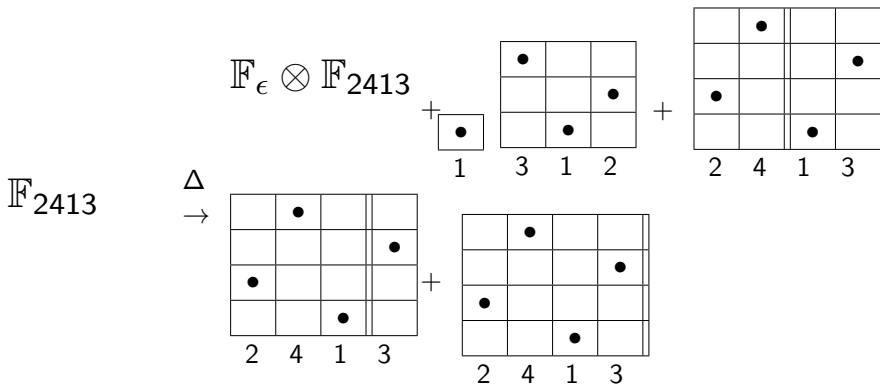
Vertical disassembly



Vertical disassembly



Vertical disassembly



Vertical disassembly

$$\mathbb{F}_{2413} \xrightarrow{\Delta} \mathbb{F}_{\epsilon} \otimes \mathbb{F}_{2413} + \mathbb{F}_1 \otimes \mathbb{F}_{312} + \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline & & \bullet & \\ \hline 2 & 4 & 1 & 3 \\ \hline \end{array}$$

\mathbb{F}_{2413} is represented by a 4x4 grid with a vertical line between columns 3 and 4. The grid contains dots at (row, col) positions: (1,2), (2,4), (3,1), and (4,3). Below the grid are the numbers 2, 4, 1, 3.

$\mathbb{F}_{\epsilon} \otimes \mathbb{F}_{2413}$ is represented by a 4x4 grid with a vertical line between columns 3 and 4. The grid contains dots at (1,2), (2,4), and (4,3). Below the grid are the numbers 2, 4, 1, 3.

$\mathbb{F}_1 \otimes \mathbb{F}_{312}$ is represented by a 4x4 grid with a vertical line between columns 3 and 4. The grid contains dots at (3,1) and (2,4). Below the grid are the numbers 2, 4, 1, 3.

The final term is a 4x4 grid with a vertical line between columns 3 and 4. The grid contains dots at (1,2), (2,4), (3,1), and (4,3). Below the grid are the numbers 2, 4, 1, 3.

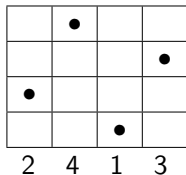
Vertical disassembly

$$\mathbb{F}_\epsilon \otimes \mathbb{F}_{2413} \quad + \quad \mathbb{F}_1 \otimes \mathbb{F}_{312} \quad + \quad \mathbb{F}_{12} \otimes \mathbb{F}_{12}$$

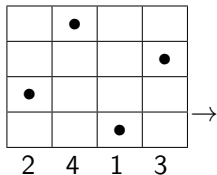
 \mathbb{F}_{2413}
 Δ
 \rightarrow

$$\mathbb{F}_{231} \otimes \mathbb{F}_1 \quad + \quad \mathbb{F}_{2413} \otimes \mathbb{F}_\epsilon$$

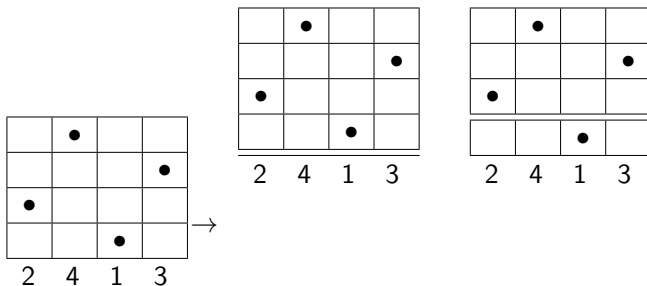
Horizontal disassembly



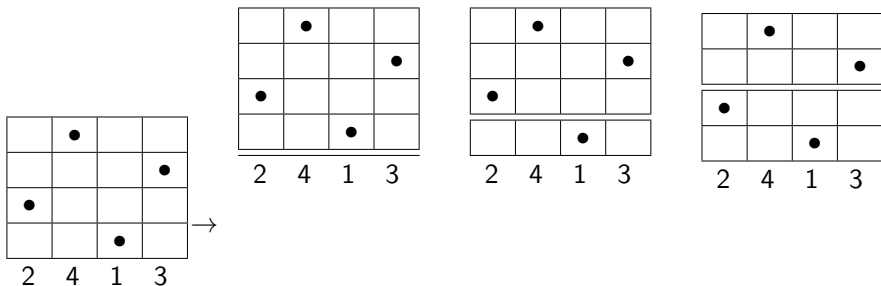
Horizontal disassembly



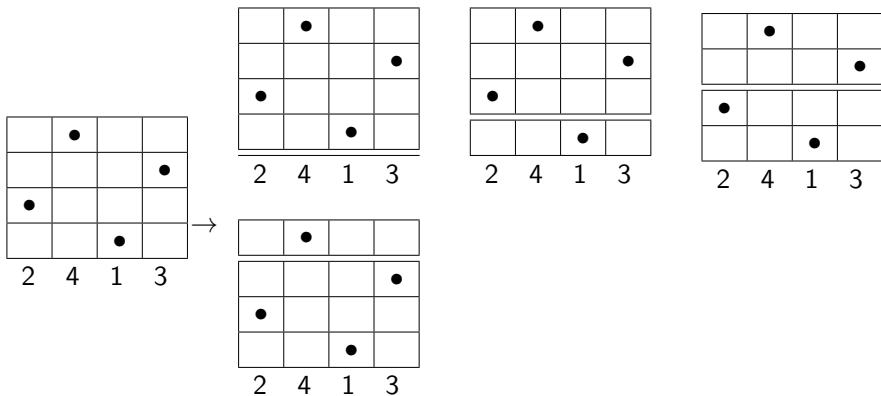
Horizontal disassembly



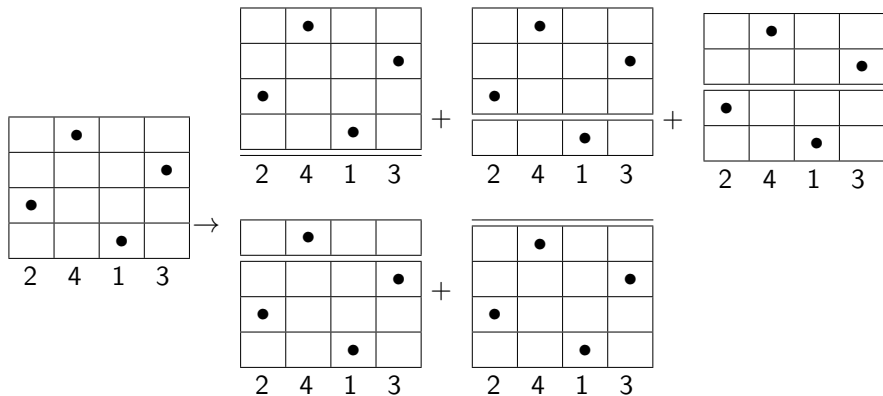
Horizontal disassembly



Horizontal disassembly

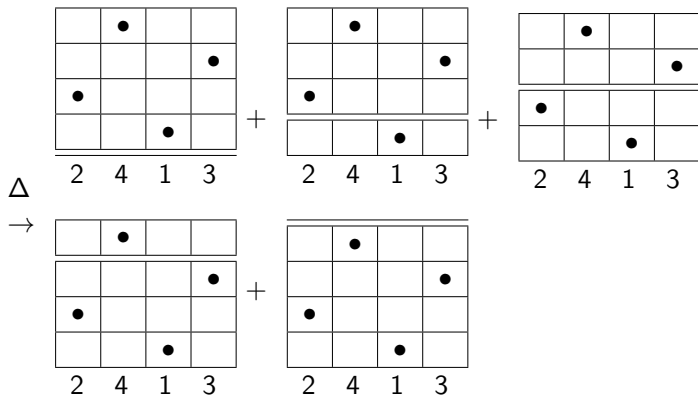


Horizontal disassembly



Horizontal disassembly

\mathbb{G}_{2413}

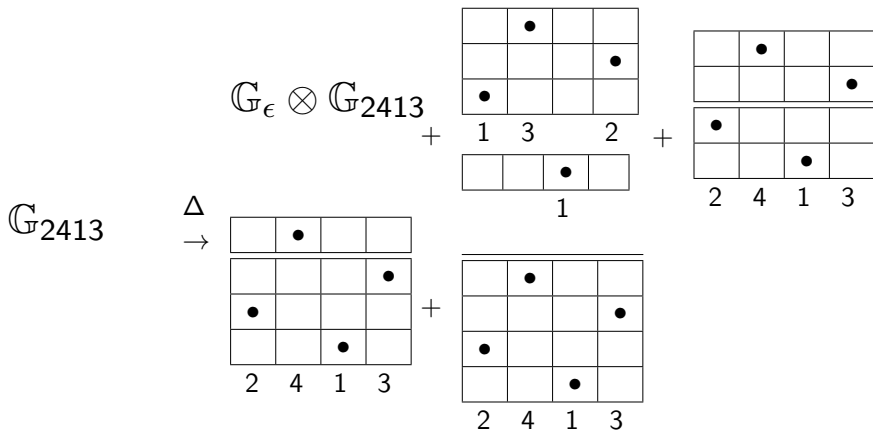


Horizontal disassembly

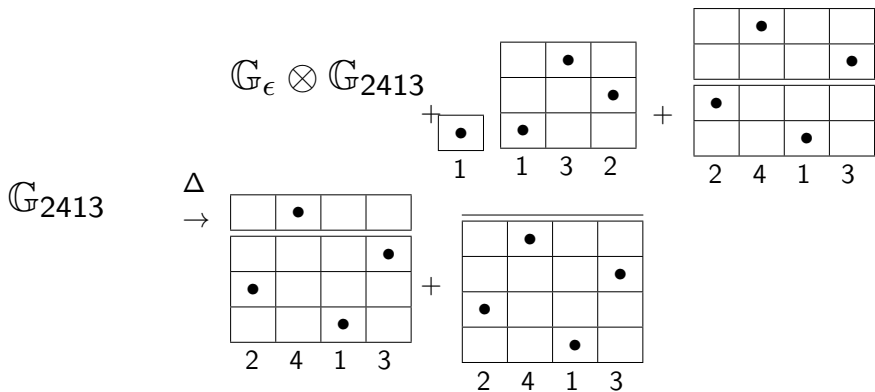
$$\mathbb{G}_{2413} \xrightarrow{\Delta} \mathbb{G}_{\epsilon} \otimes \mathbb{G}_{2413} + \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & \bullet & & \\ \hline \bullet & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \bullet & & \bullet & \\ \hline \end{array}$$

2 4 1 3
2 4 1 3
2 4 1 3
2 4 1 3
2 4 1 3
2 4 1 3
2 4 1 3

Horizontal disassembly



Horizontal disassembly



Horizontal disassembly

$$\mathbb{G}_{2413} \xrightarrow{\Delta} \mathbb{G}_{\epsilon} \otimes \mathbb{G}_{2413} + \mathbb{G}_1 \otimes \mathbb{G}_{132} + \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline & & \bullet & \\ \hline 2 & 4 & 1 & 3 \\ \hline \end{array}$$

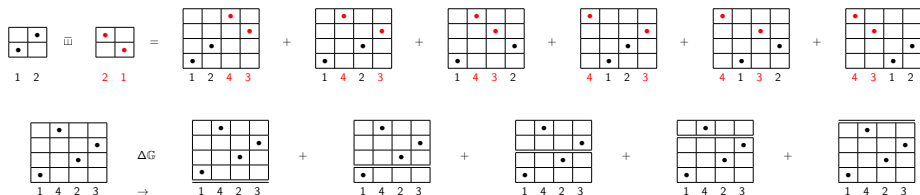
The diagram shows the horizontal disassembly of the permutation \mathbb{G}_{2413} . On the left, \mathbb{G}_{2413} is represented by a 4x4 grid with dots at (1,2), (2,4), (3,1), and (4,3), with columns labeled 2, 4, 1, 3 below. An arrow labeled Δ points to the right. On the right, the disassembly is shown as a sum of three terms:

- $\mathbb{G}_{\epsilon} \otimes \mathbb{G}_{2413}$: A 4x4 grid with dots at (1,2), (2,4), (3,1), and (4,3), with columns labeled 2, 4, 1, 3 below.
- $\mathbb{G}_1 \otimes \mathbb{G}_{132}$: A 4x4 grid with dots at (1,2), (2,4), (3,1), and (4,3), with columns labeled 2, 4, 1, 3 below.
- A 4x4 grid with dots at (1,2), (2,4), (3,1), and (4,3), with columns labeled 2, 4, 1, 3 below.

Horizontal disassembly

$$\begin{array}{c}
 \mathbb{G}_{2413} \xrightarrow{\Delta} \mathbb{G}_{\epsilon} \otimes \mathbb{G}_{2413} + \mathbb{G}_1 \otimes \mathbb{G}_{132} + \mathbb{G}_{21} \otimes \mathbb{G}_{21} \\
 \rightarrow \mathbb{G}_{213} \otimes \mathbb{G}_1 + \mathbb{G}_{2413} \otimes \mathbb{G}_{\epsilon}
 \end{array}$$

Duality of FQSym



Duality

If H is a Hopf algebra,

$$\langle \Delta(z), x \otimes y \rangle = \langle z, x \cdot y \rangle$$

$$\langle y \cdot z, x \rangle = \langle y \otimes z, \Delta(x) \rangle$$

$$\forall x, y \in H, z \in H^*,$$

$$\forall x \in H, y, z \in H^*$$

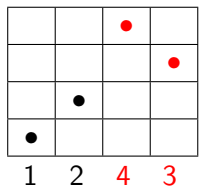
	●
●	

 $\bar{\cup}$

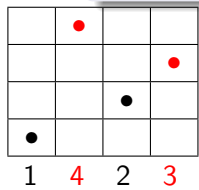
●	
	●

1 2 2 1

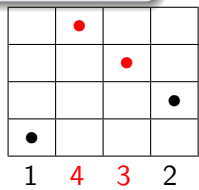
\mathbb{F}
 Explicite isomorphisme :
 $\mathbb{F}_\sigma \rightarrow \mathbb{G}_{\sigma^{-1}}$



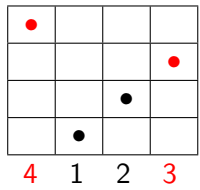
+



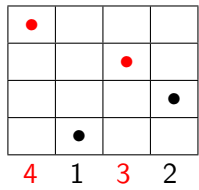
+



+



+



Packed words

Definition

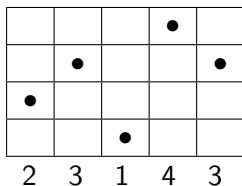
A word w with letters in $\{1, \dots, n\}$ is a packed word if for each number $k > 1$ appearing in w , the number $k - 1$ appears in w too.

Packed words

Definition

A word w with letters in $\{1, \dots, n\}$ is a packed word if for each number $k > 1$ appearing in w , the number $k - 1$ appears in w too.

With the same representation : $\#lines \leq \#columns$

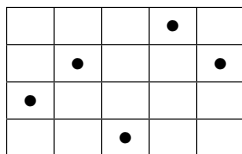


Packed words

Definition

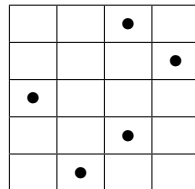
A word w with letters in $\{1, \dots, n\}$ is a packed word if for each number $k > 1$ appearing in w , the number $k - 1$ appears in w too.

With the same representation : $\#lines \leq \#columns$



2 3 1 4 3

→ transposition →

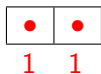
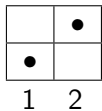


5

3 1 2 4

→ inversion →

Shuffle product on packed words



Shuffle product on packed words

$$\begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \\ \hline \end{array} \quad \boxplus \quad \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \end{array} =$$

1 2 1 1

$$\begin{array}{|c|c|c|c|} \hline & & \bullet & \bullet \\ \hline & & & \\ \hline \bullet & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & \bullet & & \bullet \\ \hline & & \bullet & \\ \hline \bullet & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & \bullet & \bullet & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline \end{array}$$

1 2 3 3 1 3 2 3 1 3 3 2

$$+ \begin{array}{|c|c|c|c|} \hline \bullet & & & \bullet \\ \hline & & \bullet & \\ \hline & \bullet & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \bullet & & \bullet & \\ \hline & & & \bullet \\ \hline & \bullet & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & & \\ \hline & & & \bullet \\ \hline & & \bullet & \\ \hline \end{array}$$

3 1 2 3 3 1 3 2 3 3 1 2

Shuffle product on packed words

$$\begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \\ \hline \end{array} \quad \bar{\sqcup} \quad \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \end{array} = \mathcal{S}$$

$\mathcal{S}_{12}\mathcal{S}_{11} = \mathcal{S}_{1233} + \mathcal{S}_{1323} + \mathcal{S}_{1332} + \mathcal{S}_{3123} + \mathcal{S}_{3132} + \mathcal{S}_{3312}$

$$\begin{array}{|c|c|c|c|} \hline & & \bullet & \bullet \\ \hline & & & \\ \hline \bullet & & & \\ \hline \end{array} \quad + \quad \begin{array}{|c|c|c|c|} \hline & \bullet & & \bullet \\ \hline & & \bullet & \\ \hline \bullet & & & \\ \hline \end{array} \quad + \quad \begin{array}{|c|c|c|c|} \hline & \bullet & \bullet & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline \end{array}$$

$\begin{array}{|c|c|c|c|} \hline \bullet & & & \bullet \\ \hline & & & \\ \hline & & \bullet & \\ \hline \end{array} \quad + \quad \begin{array}{|c|c|c|c|} \hline \bullet & & \bullet & \\ \hline & & & \bullet \\ \hline & \bullet & & \\ \hline \end{array} \quad + \quad \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & & \\ \hline & & & \bullet \\ \hline & & \bullet & \\ \hline \end{array}$

$\begin{array}{|c|c|c|c|} \hline \bullet & & & \bullet \\ \hline & & & \\ \hline & \bullet & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline \bullet & & \bullet & \\ \hline & & & \bullet \\ \hline & \bullet & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & & \\ \hline & & & \bullet \\ \hline & & \bullet & \\ \hline \end{array}$

$\begin{array}{|c|c|c|c|} \hline & & \bullet & \bullet \\ \hline & & & \\ \hline \bullet & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline & \bullet & & \bullet \\ \hline & & \bullet & \\ \hline \bullet & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline & \bullet & \bullet & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline \end{array}$

$\begin{array}{|c|c|c|c|} \hline \bullet & & & \bullet \\ \hline & & & \\ \hline & & \bullet & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline \bullet & & \bullet & \\ \hline & & & \bullet \\ \hline & \bullet & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & & \\ \hline & & & \bullet \\ \hline & & \bullet & \\ \hline \end{array}$

$\begin{array}{|c|c|c|c|} \hline & & \bullet & \bullet \\ \hline & & & \\ \hline \bullet & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline & \bullet & & \bullet \\ \hline & & \bullet & \\ \hline \bullet & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline & \bullet & \bullet & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline \end{array}$

$\begin{array}{|c|c|c|c|} \hline \bullet & & & \bullet \\ \hline & & & \\ \hline & \bullet & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline \bullet & & \bullet & \\ \hline & & & \bullet \\ \hline & \bullet & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & & \\ \hline & & & \bullet \\ \hline & & \bullet & \\ \hline \end{array}$

Shuffle product on packed words

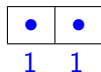
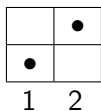
$$\begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \\ \hline \end{array} \quad \bar{\sqcup} \quad \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \end{array} = \sum_{\nu \in \sigma \bar{\sqcup} \mu} \mathbb{S}_\nu$$

$\mathbb{S}_\sigma \mathbb{S}_\mu := \sum_{\nu \in \sigma \bar{\sqcup} \mu} \mathbb{S}_\nu$

$$\begin{array}{|c|c|c|c|} \hline & & \bullet & \bullet \\ \hline & & & \\ \hline \bullet & & & \\ \hline \end{array} \quad + \quad \begin{array}{|c|c|c|c|} \hline & \bullet & & \bullet \\ \hline & & \bullet & \\ \hline \bullet & & & \\ \hline \end{array} \quad + \quad \begin{array}{|c|c|c|c|} \hline & \bullet & \bullet & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline \end{array}$$

$\begin{array}{|c|c|c|c|} \hline \bullet & & & \bullet \\ \hline & & \bullet & \\ \hline & \bullet & & \\ \hline \end{array} \quad + \quad \begin{array}{|c|c|c|c|} \hline \bullet & & \bullet & \\ \hline & & & \bullet \\ \hline & \bullet & & \\ \hline \end{array} \quad + \quad \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & & \\ \hline & & & \bullet \\ \hline & & \bullet & \\ \hline \end{array}$

Quasi shuffle product on values


 \mathbb{M}

$$\mathbb{M}_\sigma \mathbb{M}_\mu := \sum_{\substack{\nu=uv, \\ \text{pack}(u)=\sigma, \\ \text{pack}(v)=\mu}} \mathbb{M}_\nu$$

Quasi shuffle product on values

<table border="1" style="border-collapse: collapse; width: 40px; height: 40px; margin: 0 auto;"> <tr><td style="width: 20px; height: 20px;"></td><td style="width: 20px; height: 20px; text-align: center;">●</td></tr> <tr><td style="width: 20px; height: 20px; text-align: center;">●</td><td style="width: 20px; height: 20px;"></td></tr> </table> <div style="display: flex; justify-content: center; gap: 10px;"> 1 2 </div>		●	●		$\underline{\underline{=}}$	<table border="1" style="border-collapse: collapse; width: 60px; height: 40px; margin: 0 auto;"> <tr><td style="width: 30px; height: 20px; text-align: center;">●</td><td style="width: 30px; height: 20px; text-align: center;">●</td></tr> </table> <div style="display: flex; justify-content: center; gap: 10px;"> 1 1 </div>	●	●	=	<div style="border: 1px solid black; background-color: #e0e0e0; padding: 10px; margin: 0 auto; width: fit-content;"> <div style="background-color: #000080; color: white; padding: 2px 5px; font-weight: bold; margin-bottom: 5px;">M</div> $M_\sigma M_\mu := \sum_{\substack{\nu=uv, \\ \text{pack}(u)=\sigma, \\ \text{pack}(v)=\mu}} M_\nu$ </div>																														
	●																																							
●																																								
●	●																																							
<table border="1" style="border-collapse: collapse; width: 100px; height: 60px; margin: 0 auto;"> <tr><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px; text-align: center;">●</td><td style="width: 25px; height: 20px; text-align: center;">●</td></tr> <tr><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px; text-align: center;">●</td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px;"></td></tr> <tr><td style="width: 25px; height: 20px; text-align: center;">●</td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px;"></td></tr> </table> <div style="display: flex; justify-content: center; gap: 10px;"> 1 2 3 3 </div>			●	●		●			●				+	<table border="1" style="border-collapse: collapse; width: 100px; height: 60px; margin: 0 auto;"> <tr><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px; text-align: center;">●</td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px;"></td></tr> <tr><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px; text-align: center;">●</td><td style="width: 25px; height: 20px; text-align: center;">●</td></tr> <tr><td style="width: 25px; height: 20px; text-align: center;">●</td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px;"></td></tr> </table> <div style="display: flex; justify-content: center; gap: 10px;"> 1 3 2 2 </div>		●					●	●	●				+	<table border="1" style="border-collapse: collapse; width: 100px; height: 60px; margin: 0 auto;"> <tr><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px; text-align: center;">●</td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px;"></td></tr> <tr><td style="width: 25px; height: 20px; text-align: center;">●</td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px;"></td></tr> <tr><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px; text-align: center;">●</td><td style="width: 25px; height: 20px; text-align: center;">●</td></tr> </table> <div style="display: flex; justify-content: center; gap: 10px;"> 2 3 1 1 </div>		●			●						●	●
		●	●																																					
	●																																							
●																																								
	●																																							
		●	●																																					
●																																								
	●																																							
●																																								
		●	●																																					
<table border="1" style="border-collapse: collapse; width: 100px; height: 60px; margin: 0 auto;"> <tr><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px; text-align: center;">●</td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px;"></td></tr> <tr><td style="width: 25px; height: 20px; text-align: center;">●</td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px; text-align: center;">●</td><td style="width: 25px; height: 20px; text-align: center;">●</td></tr> </table> <div style="display: flex; justify-content: center; gap: 10px;"> 1 2 1 1 </div>		●			●		●	●	+	<table border="1" style="border-collapse: collapse; width: 100px; height: 60px; margin: 0 auto;"> <tr><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px; text-align: center;">●</td><td style="width: 25px; height: 20px; text-align: center;">●</td><td style="width: 25px; height: 20px; text-align: center;">●</td></tr> <tr><td style="width: 25px; height: 20px; text-align: center;">●</td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px;"></td><td style="width: 25px; height: 20px;"></td></tr> </table> <div style="display: flex; justify-content: center; gap: 10px;"> 1 2 2 2 </div>		●	●	●	●																									
	●																																							
●		●	●																																					
	●	●	●																																					
●																																								

Quasi shuffle product on values

$$\begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline & \\ \hline \bullet & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline & \\ \hline \end{array} \quad = \quad \begin{array}{|c|c|c|c|} \hline & & \bullet & \bullet \\ \hline & & & \\ \hline \bullet & & & \\ \hline \end{array} \quad + \quad \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & \bullet & \bullet \\ \hline \bullet & & & \\ \hline \end{array} \quad + \quad \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline \bullet & & & \\ \hline & & \bullet & \bullet \\ \hline \end{array}$$

$M_{12}M_{11} = M_{1233} + M_{1322} + M_{2311} + M_{1211} + M_{1222}$

$$\begin{array}{|c|c|c|c|} \hline & & \bullet & \bullet \\ \hline & & & \\ \hline \bullet & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & \bullet & \bullet \\ \hline \bullet & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline \bullet & & & \\ \hline & & \bullet & \bullet \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline \bullet & & \bullet & \bullet \\ \hline & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline & \bullet & \bullet & \bullet \\ \hline \bullet & & & \\ \hline & & & \\ \hline \end{array}$$

Shuffle product on values

$$\begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \\ \hline \end{array} \quad \underline{\underline{\sqcup}} \quad \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \end{array} = \mathbb{L}$$

1 2 1 1

$$\begin{array}{|c|c|c|c|} \hline & & \bullet & \bullet \\ \hline & & & \\ \hline \bullet & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & \bullet & \bullet \\ \hline \bullet & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & \bullet & \\ \hline \bullet & & & \\ \hline & & \bullet & \bullet \\ \hline \end{array}$$

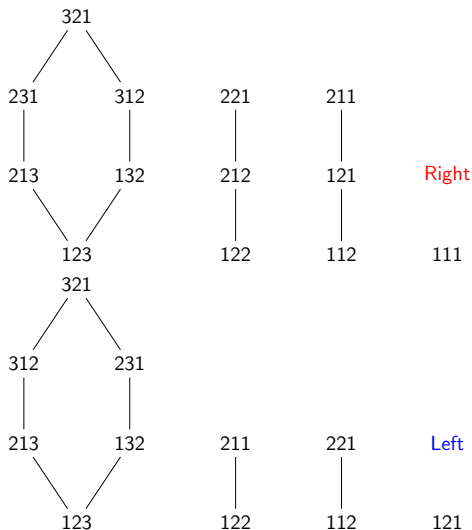
1 2 3 3 1 3 2 2 2 3 1 1

$$\mathbb{L}$$

$$\mathbb{L}_{12}\mathbb{L}_{11} = \mathbb{L}_{1233} + \mathbb{L}_{1322} + \mathbb{L}_{2311}$$

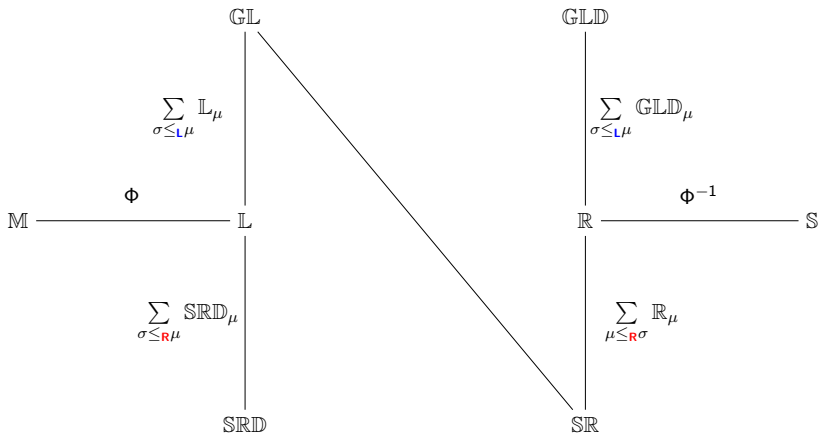
$$\mathbb{L}_{\sigma}\mathbb{L}_{\mu} := \sum_{\nu \in \sigma \underline{\underline{\sqcup}} \mu} \mathbb{L}_{\nu}$$

Poset



Poset :
 Reflexivity,
 Transitivity,
 Antisymmetrie.

From M to S in WQSym



Some matrices

	123	132	213	231	312	321	122	212	221	112	121	211	111
123	1
132	.	-1	1	1
213	.	1	-1	.	1
231	.	1
312	.	.	1
321	1
122	1	1	1	-1	.	.	.
212	1	1
221	1
112	-1	.	.	1	1	1	.
121	1	1	.	.
211	1	.	.	.
111	1

Figure: Transformation matrix from the basis \mathbb{L} to \mathbb{R} over packed words of size 3.

Some matrices

	123	132	213	231	312	321	122	212	221	112	121	211	111
123	0	0	0	0	0	1	0	0	1/2	0	0	1/2	1/6
132	0	0	0	1	0	0	0	0	1/2	0	1/2	0	1/6
213	0	0	0	0	1	0	0	1/2	0	0	0	1/2	1/6
231	0	1	0	1	-1	0	1/2	-1/2	1/2	0	1	-1/2	1/6
312	0	0	1	-1	1	0	0	1	-1/2	1/2	-1/2	1/2	1/6
321	1	0	0	0	0	0	1/2	0	0	1/2	0	0	1/6
122	0	0	0	1/2	0	1/2	0	0	3/2	0	1/4	1/4	2/3
212	0	0	1/2	-1/2	1	0	0	7/4	-5/4	1/4	-1/4	1/2	1/6
221	1/2	1/2	0	1/2	-1/2	0	3/2	-5/4	1/4	1/4	1/2	3/4	2/3
112	0	0	0	0	1/2	1/2	0	1/4	1/4	0	0	3/2	2/3
121	0	1/2	0	1	-1/2	0	1/4	-1/4	1/2	0	7/4	-5/4	1/6
211	1/2	0	1/2	-1/2	1/2	0	1/4	1/2	3/4	3/2	-5/4	1/4	2/3
111	1/6	1/6	1/6	1/6	1/6	1/6	2/3	1/6	2/3	2/3	1/6	2/3	13/6

Figure: Transformation matrix from the basis \mathbb{S} to \mathbb{M} over packed words of size 3.

Contributions

- Implementing packed words in Sage, *#25916 implement Packed Words*.

Contributions

- Implementing packed words in Sage, *#25916 implement Packed Words*.
- Implementing of WQSym in Sage with its 8 bases, *#25930 implementation of different basis of WQSym*.

Contributions

- Implementing packed words in Sage, *#25916 implement Packed Words*.
- Implementing of WQSym in Sage with its 8 bases, *#25930 implementation of different basis of WQSym*.
- Large scale tests.

Contributions

- Implementing packed words in Sage, *#25916 implement Packed Words*.
- Implementing of WQSym in Sage with its 8 bases, *#25930 implementation of different basis of WQSym*.
- Large scale tests.
- Study of the combinatorics of these basis changes thanks to the display of matrices and graphs obtained in Sage.

Contributions

- Implementing packed words in Sage, *#25916 implement Packed Words*.
- Implementing of WQSym in Sage with its 8 bases, *#25930 implementation of different basis of WQSym*.
- Large scale tests.
- Study of the combinatorics of these basis changes thanks to the display of matrices and graphs obtained in Sage.
- Some conjectures

Contributions

- Implementing packed words in Sage, *#25916 implement Packed Words*.
- Implementing of WQSym in Sage with its 8 bases, *#25930 implementation of different basis of WQSym*.
- Large scale tests.
- Study of the combinatorics of these basis changes thanks to the display of matrices and graphs obtained in Sage.
- Some conjectures
 - This describe an infinity of automorphisme of **WQSym**.
 - Generalization to **PQSym** (on parking functions).