Basis of totally primitive elements of **WQSym**

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Packed words

Definition

A word over the alphabet $\mathbb{N}_{>0}$ is packed if all the letters from 1 to its maximum m appears at least once.

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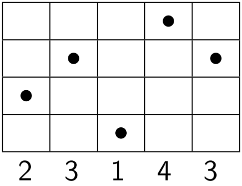
One representation : $\#$rows $\leq$ $\#$columns



Hopf algebra

- unitary associative product $\cdot$
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- Hopf relation $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$

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Example

WQSym

- $\mathbb{R}_w$ with $w \in \mathbf{PW}$
- $\mathbb{R}_{12}\mathbb{R}_{11} = \mathbb{R}_{1233} + \mathbb{R}_{1323} + \mathbb{R}_{1332} + \mathbb{R}_{3123} + \mathbb{R}_{3132} + \mathbb{R}_{3312}$
- $\Delta(\mathbb{R}_{24231}) = 1 \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21} + \mathbb{R}_{1312} \otimes \mathbb{R}_{1} + \mathbb{R}_{24231} \otimes 1$

Shuffle product on packed words

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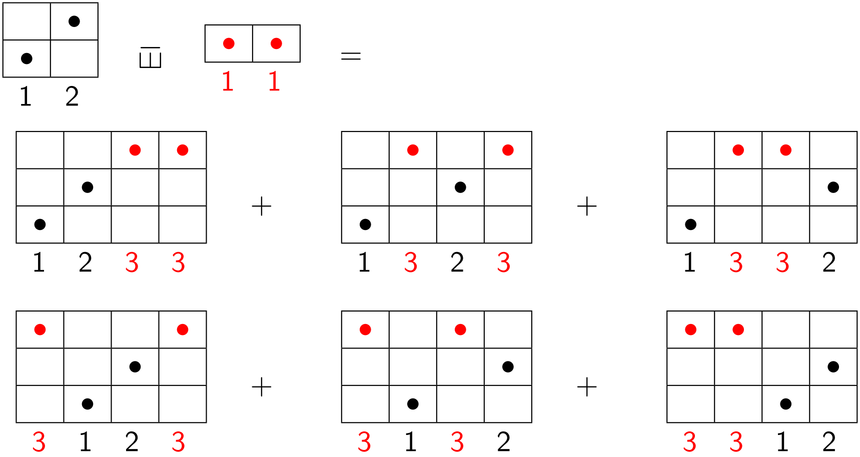
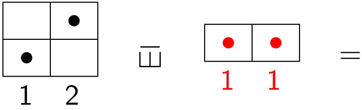


Shuffle product on packed words



$\mathbb{R}$

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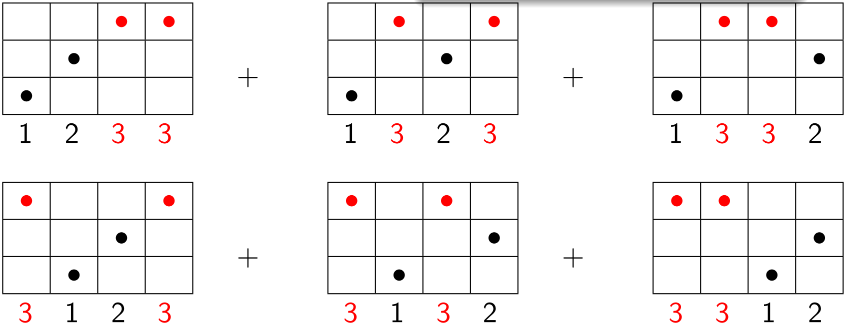
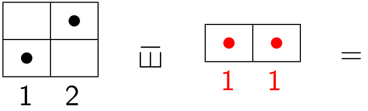
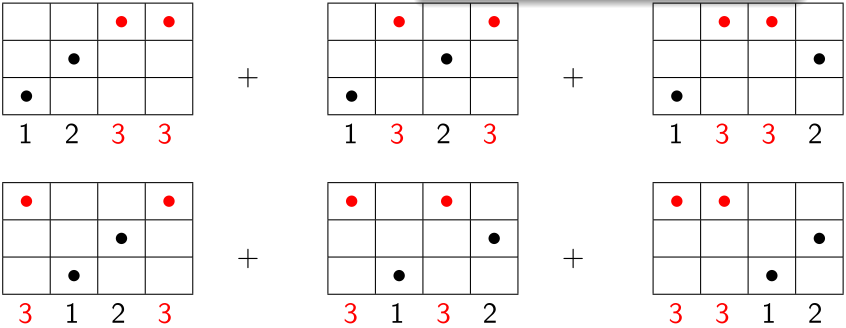


Shuffle product on packed words

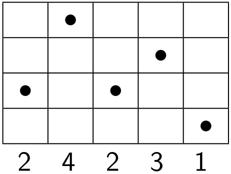


ℝ

$$\mathbb{R}_\sigma \mathbb{R}_\mu := \sum_{\nu \in \sigma \overline{\sqcup\!\sqcup} \mu} \mathbb{R}_\nu$$



Deconcatenation



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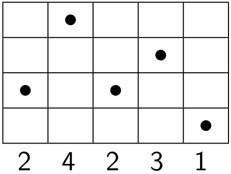

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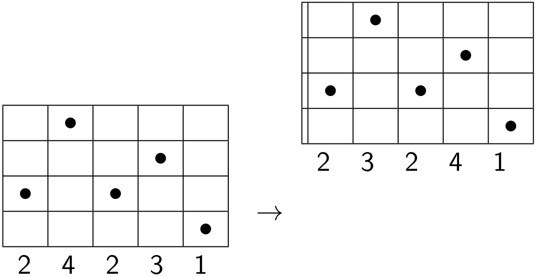

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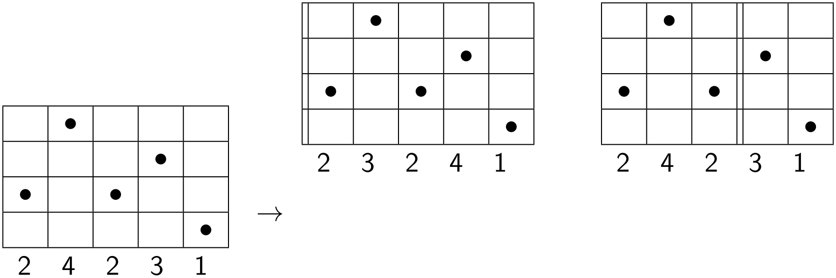

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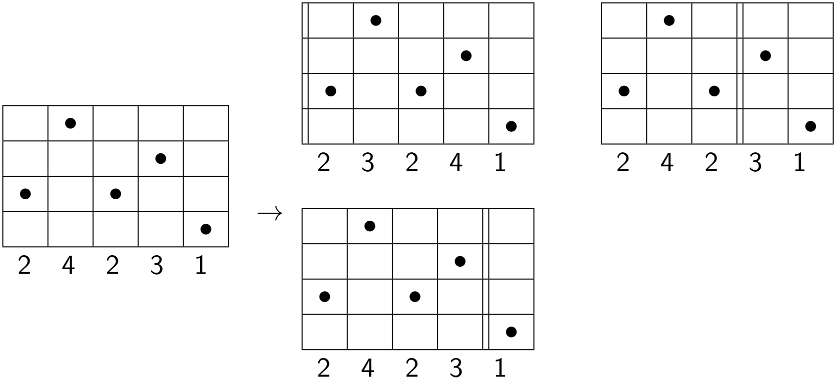

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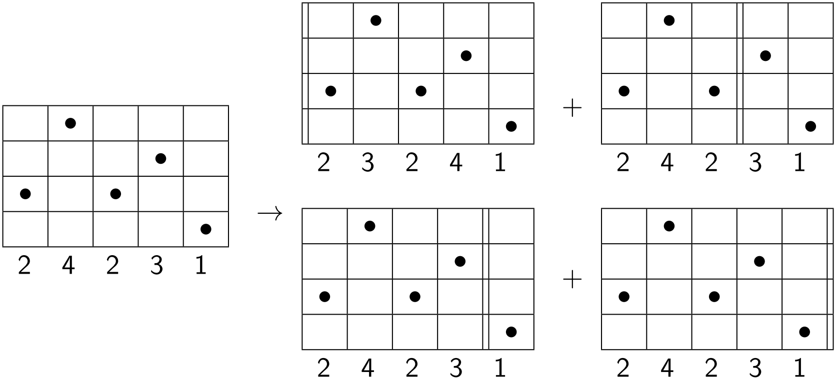
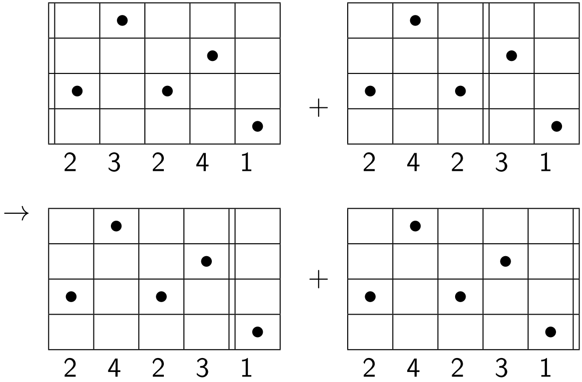


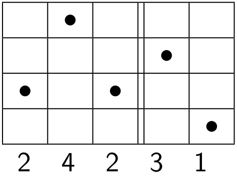
Deconcatenation

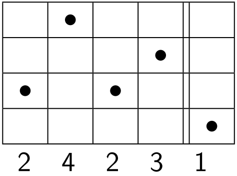
$\mathbb{R}_{24231}$



Deconcatenation

$$\mathbb{R}_{24231} \xrightarrow{\Delta} \mathbb{R}_{\epsilon} \otimes \mathbb{R}_{24231} +$$





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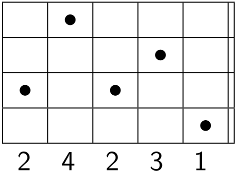

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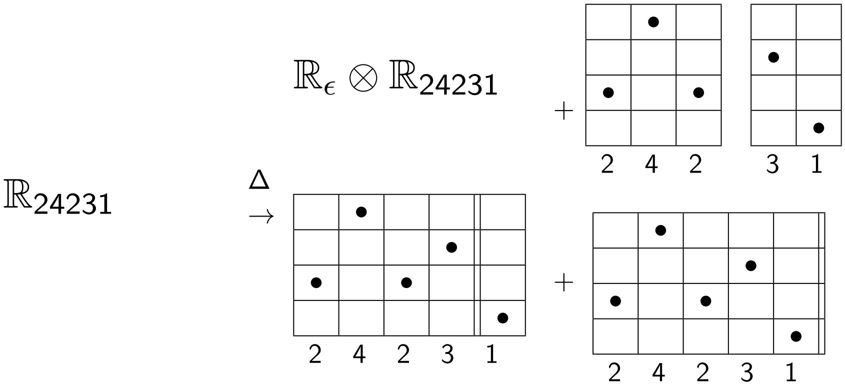
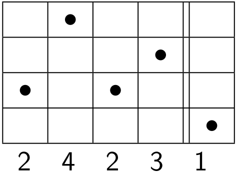


Deconcatenation

$$\mathbb{R}_{24231} \overset{\Delta}{\rightarrow} \mathbb{R}_{\epsilon} \otimes \mathbb{R}_{24231}$$



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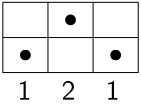

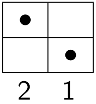

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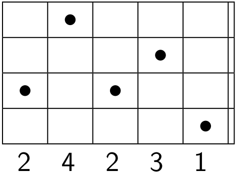
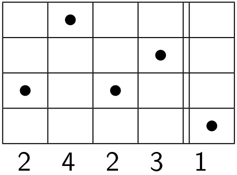


Deconcatenation

$$\mathbb{R}_{24231} \xrightarrow{\Delta} \mathbb{R}_{\epsilon} \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21}$$



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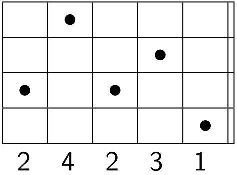

Deconcatenation

$$\mathbb{R}_{24231} \xrightarrow{\Delta} \mathbb{R}_{\epsilon} \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21} + \mathbb{R}_{1312} \otimes \mathbb{R}_{1} + \mathbb{R}_{24231} \otimes \mathbb{R}_{\epsilon}$$

Half products

Recursive definition of Shuffle product

- $\epsilon \sqcup\!\sqcup w = w \sqcup\!\sqcup \epsilon = w$
- $ua \sqcup\!\sqcup vb = (u \sqcup\!\sqcup vb)a + (ua \sqcup\!\sqcup v)b$

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Example of left and right products

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- $\mathbb{R}_{12} \prec \mathbb{R}_{11} = \mathbb{R}_{1332} + \mathbb{R}_{3132} + \mathbb{R}_{3312}$
- $\mathbb{R}_{12} \succ \mathbb{R}_{11} = \mathbb{R}_{1233} + \mathbb{R}_{1323} + \mathbb{R}_{3123}$

Half coproducts

Example of left and right coproducts

- $\tilde{\Delta}(\mathbb{R}_{2425531}) = \mathbb{R}_{121} \otimes \mathbb{R}_{3321} + \mathbb{R}_{12133} \otimes \mathbb{R}_{21} + \mathbb{R}_{131442} \otimes \mathbb{R}_{1}$

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Half coproducts

Definitions

- $\Delta_{\prec}(\mathbb{R}_u) := \sum_{\substack{i=k \\ \{u_1,\dots,u_i\}\cap\{u_{i+1},\dots,u_n\}=\emptyset \\ u_k=\max(u)}}^{n-1} \mathbb{R}_{pack(u_1\cdots u_i)} \otimes \mathbb{R}_{pack(u_{i+1}\cdots u_n)}$,
- $\Delta_{\succ}(\mathbb{R}_u) := \sum_{\substack{i=1 \\ \{u_1,\dots,u_i\}\cap\{u_{i+1},\dots,u_n\}=\emptyset \\ u_k=\max(u)}}^{k-1} \mathbb{R}_{pack(u_1\cdots u_i)} \otimes \mathbb{R}_{pack(u_{i+1}\cdots u_n)}$

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Goal

- Automorphism,
- Subspace that generate the algebra with (half) products.

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Answer

Totally primitive elements

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Primitive elements

P is a primitive element $\iff \tilde{\Delta}(P) = 0$

Ex : $\mathbb{R}_{1213} - \mathbb{R}_{2321}$

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Totally primitive Element

P is a totally primitive element $\iff \Delta_{\prec}(P) = \Delta_{\succ}(P) = 0$

Ex : $\mathbb{R}_{12443} - \mathbb{R}_{21443} - \mathbb{R}_{23441} + \mathbb{R}_{32441}$

Notations

Let A be a bidendriforme bialgebra

$$\begin{aligned}
\mathrm{Prim}(A) &= \mathrm{Ker}(\tilde{\Delta}),\\
\mathrm{TPrim}(A) &= \mathrm{Ker}(\Delta_{\prec}) \cap \mathrm{Ker}(\Delta_{\succ}),\\
\mathcal{A}(z) &= \sum_{n=1}^{+\inf} \dim(A_n) z^n,\\
\mathcal{P}(z) &= \sum_{n=1}^{+\inf} \dim(\mathrm{Prim}(A_n)) z^n,\\
\mathcal{T}(z) &= \sum_{n=1}^{+\inf} \dim(\mathrm{TPrim}(A_n)) z^n.
\end{aligned}$$

Theorems

Bidendriforme version of Carier Milnor-Moore [Foissy]

Let A a bidendriforme bialgebra. Then A is freely generated as a dendriform algebra by $\mathrm{TPrim}(A)$. Moreover :

$$\mathcal{P}(z) = \frac{\mathcal{A}(z)}{\mathcal{A}(z)+1}, \qquad \mathcal{T}(z) = \frac{\mathcal{A}(z)}{(\mathcal{A}(z)+1)^2}.$$

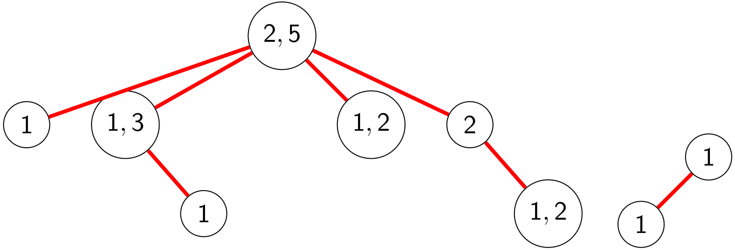
or equivalently

$$\mathcal{A}(z) = \frac{\mathcal{P}(z)}{1-\mathcal{P}(z)}, \qquad \mathcal{P}(z) = \mathcal{T}(z)(1+\mathcal{A}(z)).$$

n	1	2	3	4	5	6	7	OEIS
a_n	1	3	13	75	541	4 683	47 293	A000670
p_n	1	2	8	48	368	3 376	35 824	A095989
t_n	1	1	4	28	240	2 384	26 832	

Decorated forests

An exemple of decorated biplan forest:



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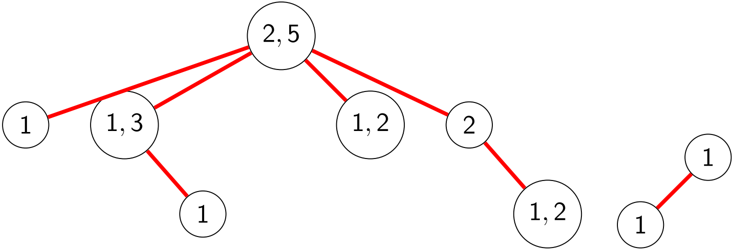

On board: 8767595394312.

Algorithm on 8767595394312

- global descente factorization

- 87675953943 12

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- remove maximums

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- $87675 \cdot 53 \cdot 4$

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- 87675953943 12
- $87675 \cdot 53 \cdot 4$
- 8 767 $5 \cdot 5$ $3 \cdot 4$

Algorithm on 8767595394312

- global descente factorization
- remove maximums
- global descente factorization
- left and right groups

- $87675953943 \quad 12$
- $87675 \cdot 53 \cdot 4$
- $8 \quad 767 \quad 5 \cdot 5 \quad 3 \cdot 4$
- $8 \quad 767 \mid 5 \cdot 5 \quad 3 \cdot 4$

Algorithm on 8767595394312

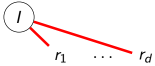
- global descente factorization
- remove maximums
- global descente factorization
- left and right groups
- positions of maximums
- recursively

- $87675953943 \quad 12$
- $87675 \cdot 53 \cdot 4$
- $8 \quad 767 \quad 5 \cdot 5 \quad 3 \cdot 4$
- $8 \quad 767 \mid 5 \cdot 5 \quad 3 \cdot 4$
- $\cdot \quad \cdots \mid 123 \quad 456$

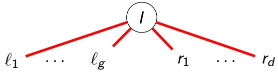
New basis $\mathbb{P}$

$$\mathbb{P}_{\circ} := \mathbb{R}_1,$$

$$\mathbb{P}_{t_1,\dots,t_k} := (\dots(\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1},$$

$\mathbb{P}$ 

$$:= \Phi_l(\mathbb{P}_{r_1,\dots,r_d}),$$

$\mathbb{P}$ 

$$:= \langle \mathbb{P}_{l_1}, \mathbb{P}_{l_2}, \dots, \mathbb{P}_{l_k}; \Phi_l(\mathbb{P}_{r_1,\dots,r_d})\rangle.$$

Theorem

- $(\mathbb{P}_f)_{f\in\mathfrak{F}_n}$ is a basis of $\mathbf{WQSym}_n$,
- $(\mathbb{P}_t)_{t\in\mathfrak{T}_n}$ is a basis of Prim_n,
- $(\mathbb{P}_t)_{t\in\mathfrak{P}_n}$ is a basis of TPrim_n.

Théorèmes

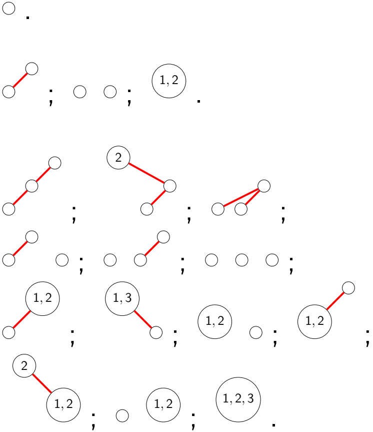
Corolaire du Théorème Carier Milnor-Moore version dendriforme

Soit A une algèbre de Hopf bidendriforme. Prim(A) est généré librement par TPrim(A) en tant que algèbre de brace avec l'opération n-multilinéaire suivante :

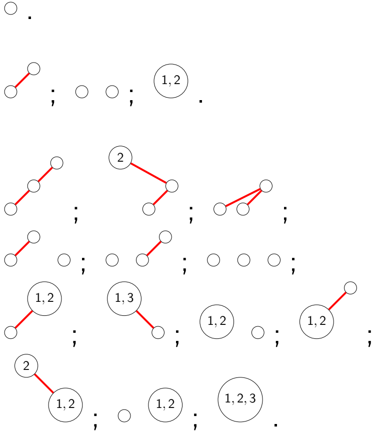
$$\langle p_1, \ldots, p_{n-1}; p_n \rangle = \sum_{i=0}^{n-1} (-1)^{n-1-i}$$

$$(p_1 \prec (p_2 \prec (\ldots \prec p_i)\ldots) \succ p_n \prec (\ldots (p_{i+1} \succ p_{i+2}) \succ \ldots) \succ p_{n-1}).$$

Les premiers arbres



Les premiers arbres



1

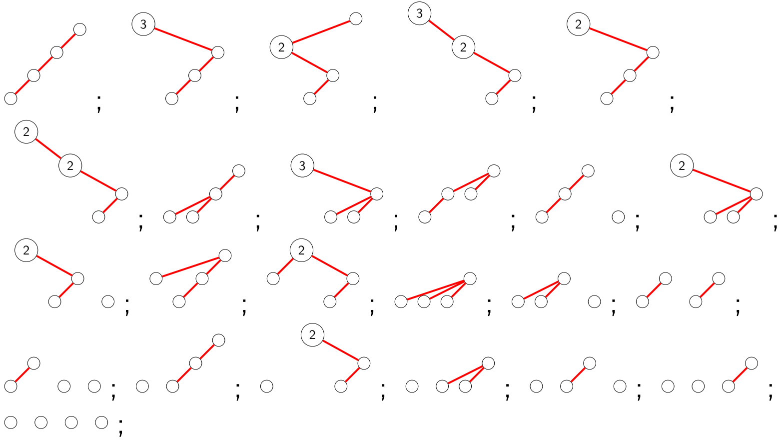
12; 21; 11

123; 132; 213;

231; 312; 321;

122; 212; 221; 112;

121; 211; 111.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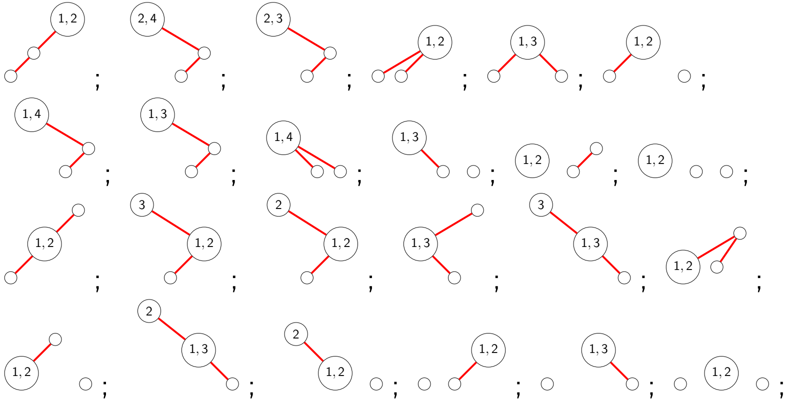
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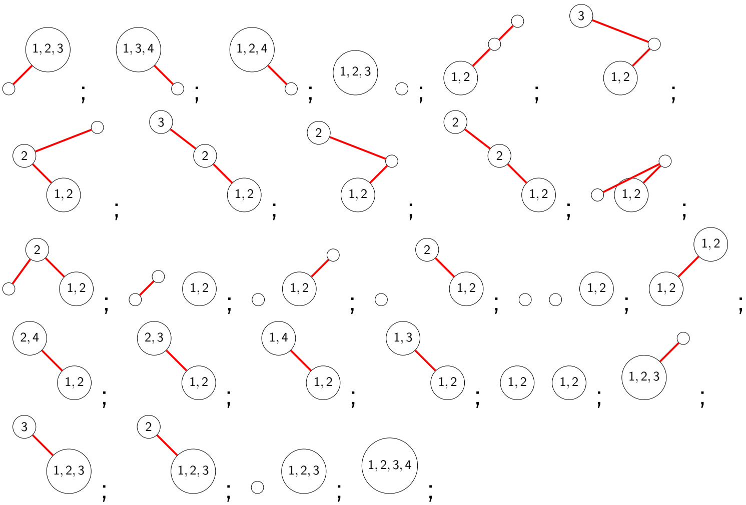
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2 1, 2 ; 3 2 1, 2 ; 2 1, 2 ; 2 2 1, 2 ; 1, 2 ;
2 1, 2 ; 1, 2 ; 1, 2 ; 2 1, 2 ; 1, 2 ; 1, 2 1, 2 ;
2, 4 1, 2 ; 2, 3 1, 2 ; 1, 4 1, 2 ; 1, 3 1, 2 ; 1, 2 1, 2 ; 1, 2, 3 ;
3 1, 2, 3 ; 2 1, 2, 3 ; 1, 2, 3 ; 1, 2, 3, 4 ;