

The principle of a finite density of information

Pablo Arrighi* and Gilles Dowek†

Abstract

The possibility to describe the laws of the Universe in a computational way seems to be correlated to a principle that the density of information is bounded. This principle, that is dual to that of a finite velocity of information, has already been investigated in Physics, and is correlated to the old idea that there is no way to know a magnitude with an infinite precision. It takes different forms in classical Physics and in quantum Physics.

1 Why cellular automata?

Stephen Wolfram has advocated in [Wol02] the idea that cellular automata might be a relevant formalism to study the laws of the Universe. This thesis can be seen as a consequence of three more fundamental principles that are implicit in the definition of the notion of a cellular automaton.

To define a cellular automaton, we must discretize space by partitioning it into an infinite set of identical *cells*, for instance cubes. We must also discretize time and observe the Universe at regular time steps. Then come three assumptions

1. that the state space of each cell is finite,
2. that the state space of a cell at a given time step depends only on the state of a finite number of neighbours at the previous time step,

*École normale supérieure de Lyon, LIP, 46 allée d'Italie, 69008 Lyon, France, and Université de Grenoble, LIG, 220 rue de la chimie, 38400 SMH, France, parrighi@imag.fr.

†INRIA, 23 avenue d'Italie, CS 81321, 75214 Paris Cedex 13, France, gilles.dowek@inria.fr.

3. that the state space is the same for all cells and the evolution acts the same everywhere and everywhen.

Assumption (3.) is a reformulation of a well-known principle of Physics: the homogeneity of space and time. So is assumption (2.), which is a reformulation of the bounded velocity of information. Assumption (1.), in contrast, seems to express the new idea, that the density of information also is bounded. Such a principle can be formulated as the fact that the amount of information that can be stored in a bounded region of space is bounded. It can also be formulated as the fact that the cardinality of the state space of a bounded region of space is bounded, since the amount of information storage is the logarithm of this cardinality.

There are other assumptions in the definition of a cellular automaton, such as the fact that space-time is absolute and passive: the neighbourhood relation on cells does not evolve as their states do. Weakening these assumptions is possible, leading for instance to *causal networks* [Wol02] or to *causal graph dynamics* [AD12].

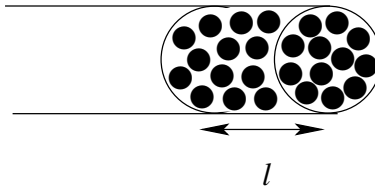
There is a historical discrepancy between the idea of a finite velocity of information and that of a finite density of information. The first has had a clear status as a principle of Physics, since special relativity, and we even know the bound on the velocity. The second seems to be less established. Yet, it is not completely new, as all three principles (1.), (2.), and (3.), have been stated by Robin Gandy in [Gan80], and the principle of a bounded density of information has also been stated by Jacob Bekenstein—although the Bekenstein bound is not on the amount of information but on the quotient of the amount of information and the energy contained in a sphere [Bek81].

2 Two dual principles

The principles of a bounded velocity and of a bounded density of information play a dual role.

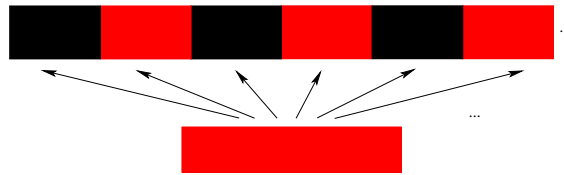
For instance, the amount I of information that can be output by a wire of section S in a given time t is bounded by the amount of information contained in the cylinder of section S and length l , where l is the distance that information can travel in time t .

If information has a bounded velocity, then l is bounded and if moreover information has a finite density, the amount of information I is bounded. Thus we need both principles to prove that the information flow of a wire is

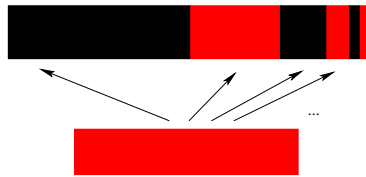


bounded. If one of these principles were invalid, it would be easy to build a wire with unbounded flow.

A more evolved example is the impossibility to build an infinitely parallel computing machine, i.e. a machine formed with an infinite number of processors communicating with a common server. If information has a bounded density, then the size of each machine has a lower bound and some machines must be arbitrarily far from the server and if moreover information has a finite velocity, some machines are too far to communicate with the server in due time. Again, if one of these principles were invalidated it would be easy to build an infinitely parallel machine, either with processors of the same size communicating infinitely fast with the server, or with arbitrarily small

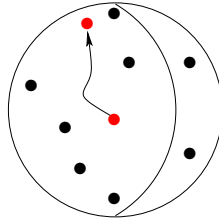


machines, of size $1, 1/2, 1/4, 1/8, 1/16, \dots$ that would lie at a finite distance of the server.



It seems that these two principles could even be summarized in one, as the fact that in a given time, a piece of information can only travel in a space which is itself populated by a finite amount of information.

Finally, the two principles are comparable, and both differ from the third: homogeneity of space and time. Indeed, the latter is pre-supposed by the very notion of a ‘Physics law’. For instance, suppose that some law would be valid



in Paris but not in Boston, then this would not be much of a law. The word ‘law’ itself carries a notion of universality.

But what about the other two? It turns out that a it is possible to do very respectable Physics without these principles. Yet, it is still the case a notion of applicability of a Physics laws underpins both the finite velocity of information and the bounded density of information principles. If a law broke the finite velocity of information principle, this would entail that, in order to determine the new state of a physical system after one second has past, one may potentially need to look at an unbounded number of neighbours. This is no doubt unpractical, unless far-away contributors can be neglected.

In the same way, if a law broke the finite density of information principle, in order to determine the new state of a physical system, one would need to use a description of the state of the neighbours that might potentially contain an unbounded amount of information. Again this law would hardly be applicable, unless this information can be rounded up in some way.

3 Newtonian Physics

It is well-know that Newtonian Physics contradicts the principle of a finite velocity of information. Whenever a bee flies half a centimeter, it instantaneously modifies the gravitational field on a distant galaxy. It is only with General Relativity that we understood how to control the consequences of the flight of a bee.

Newtonian Physics not only contradicts the finite velocity principle, it also contradicts the principle of a finite density of information as the distance between the jaws of a caliper for instance is measured by a real number. So if we imagine an infinite amount of information expressed, for instance, as an infinite sequence of 0 and 1, we can see this infinite sequence as the digits of a real number and place the jaws of a caliper at this exact distance, recording this way the infinite amount of information in a finite volume.

Of course, we have all been taught that this idea is nonsense and that the distance between the jaws of a caliper is defined with a finite precision: three or four digits. It is surprising that this thesis that a physical magnitude is always defined with a finite precision has never been given the status of a Physics principle as have the homogeneity of space and time. Yet, this thesis may be a very early occurrence of this idea of a finite density of information.

4 The finite-density versus the superposition principle

The principle of a finite density of information seems to be challenged by quantum theory and its superposition principle. Indeed, regardless of whether a region of space is bounded or not, whenever its state space contains two states \mathbf{u} and \mathbf{v} , then it must also contain all the linear combinations $\lambda\mathbf{u} + \mu\mathbf{v}$ up to a renormalization factor. Hence it is infinite, and the finite density principle cannot be formulated as the fact that the set of states of a bounded region of space is finite.

These complex amplitudes λ and μ can be compared with probabilities. But this comparison has its limits. Indeed, since probabilities are positive real numbers, they can only add up constructively to make an event more likely. In contrast amplitudes are complex numbers, then can also be subtracted to one another, i.e. add up destructively, making an event either more likely or less likely; or sometimes even impossible.

This is why when two different scenarios happen in a superposition, we cannot quite think of them as two non-interacting branches of what may happen in the Universe: these amplitudes cannot be ignored. This makes the superposition principle a difficulty when extending the finite-density principle to quantum theory.

Yet, the infinity of the state space does not mean that the amount of possible outcomes, when measuring the state of this region, is itself infinite, because in quantum theory, we cannot measure the state of the system, but only one of its observables. Thus, an alternative formulation of the bounded density of information principle is that each projective measurement of a bounded region, at any given point in time, may only yield a finite number of possible outcomes. This requirement amounts to the fact that the state space of a bounded region of space is a finite-dimensional vector space.

This constitutes a good quantum alternative to the formulation of the finite density of information principle, one which does not demand anything to be actually measured in any way.

5 The finite-density principle versus correlations

Moreover, yet another problem arises in the quantum setting, namely that of ‘correlations’; which in this context are also referred to as ‘entanglement’. To understand what entanglement is, one must apply the superposition principle again, but this time to pairs of systems. For instance if systems A and B may be in state \mathbf{uu} , meaning that both are in state \mathbf{u} , or in state \mathbf{vv} , meaning that both are in state \mathbf{v} , then the state $\lambda(\mathbf{uu}) + \mu(\mathbf{vv})$ is also allowed. This entangled state corresponds to a superposition of two global, and everywhere different scenarios. It is possible to get an intuition of the meaning of this entangled state by appealing to our understanding about usual probabilities. In probability theory, it is a commonplace that knowing the marginal probabilities over the individual systems does not entail knowing the joint probability distributions. For instance Alice and Bob may have a half-half chance of having a blue or red ball in each of their boxes, but it could be that each of their boxes have been prepared in such a way that both balls are of the same colour. The state $\lambda(\mathbf{uu}) + \mu(\mathbf{vv})$ tells the same story not for probabilities, but for amplitudes.

This is an issue, because the bounded velocity of information principle seems very weak in that respect: after all, all it says is that the state of each individual system is determined by that of the neighbours, but it does not say how the entanglement between the individual states is determined. If we retake the probability analogy; the bounded velocity of information principle tells you that the probabilities of Alice finding a red or a blue ball in her box are determined locally; but what about the probability that this ball is of the same color as Bob’s?

Fortunately on this issue of whether the correlations can be determined locally, quantum theory is a bit friendlier than probability theory. Indeed, it has been shown that whereas the bounded velocity of information principle is way too weak in the context of probabilistic cellular automata [AFNT11], surprisingly it turns out to be sufficient in the context of quantum cellular

automata [ANW11] for the evolution to be described as a composition of local functions mapping neighbourhoods to neighbourhoods, that is functions from a finite-dimensional vector space to itself. Another way to phrase this is that in the quantum setting, any evolution that respects the bounded velocity of information principle necessarily takes the form of a circuit of gates being applied on systems and their neighbours. Thus correlations can be ‘tamed’ and the global evolution broken into functions from finite-dimensional vector spaces to themselves.

6 A computable quantum Universe

Yet, moving from a finite set to a vector space, even a finite-dimensional one, is a big leap and many advantages of describing the Universe as a cellular automaton might be lost by doing so. Indeed, the main advantage of having a finite state space S for cells was that the local evolution function of the cellular automaton, that is a function from S^n to S , where n is the number of neighbours of a cell, was a function from a finite set to a finite set. Such a function can be described by a table giving the image of each element of the domain and it is always computable. This point was the key to the computability of cellular automata, and thus to the idea that the Universe is computable if it can be described as a cellular automaton. This idea of a computable Universe—the physical Church-Turing thesis—is one of the main goals of both Robin Gandy and Stephen Wolfram.

Fortunately, another feature of the quantum theory comes to repair the damages created by the superposition principle. According to the quantum theory, these local functions of a quantum cellular automata cannot be any function from neighbourhoods to neighbourhoods, but must be linear. There are much less linear functions than functions from a finite-dimensional vector space to another. In particular any such linear function can be described by a matrix, and matrix multiplication boils down to additions and multiplications that are computable operations. Thus if scalars are chosen in an adequate manner as discussed below, all such functions are computable. Thus, infinity is tamed by finite-dimension and linearity and this formulation of the principle of a finite density of information is sufficient to prove the Universe computable.

7 Scalars

When saying that addition and multiplication of scalars are computable, we need to be a little careful, as scalars are complex numbers and computability is mostly defined for natural numbers.

Computability can be extended from natural numbers to real and complex numbers [Wei00], but then the picture is a somewhat different. Not only functions from complex numbers to complex numbers can be computable or not, but complex numbers themselves can be computable or not. Thus, although multiplication is computable, multiplication by a fixed scalar may be non-computable if the fixed scalar is non-computable.

Michael Nielsen has noticed that having a process building a superposed state $\lambda\mathbf{u} + \mu\mathbf{v}$, where λ and μ are non computable numbers, can *per se* lead to non computability, as this state can be tomographed by repeated measurements and λ be extracted with increased precision from the state.

Thus, a physical system has two ways to encode an infinite amount of information: either by having an infinite dimensional state space, or by encoding an infinite amount of information in a scalar, exactly in the same way as an infinite amount of information can be encoded in a distance in Newtonian Physics.

To express the finite-density principle, it seems that it not sufficient to restrict to finite-dimensional vector spaces for the state spaces of cells, but we must also restrict to a smaller set of scalars, yielding an interesting problem for physicists about the fields that can be used for the scalars in quantum theory.

With this precise formulation of the finite density of information, we can formally prove that the physical Church-Turing thesis is a consequence of the three hypotheses we have mentioned: homogeneity of space and time, bounded velocity of information and bounded density of information [AD11].

Some scientists, such as David Deutsch [Deu85] propose to take the physical Church-Turing thesis as a principle of Physics. Some others, such as Michael Nielsen [Nie97], propose to deduce this thesis from more fundamental principles of Physics. Like Robin Gandy, we have chosen to start from these three fundamental principles. An advantage of this approach is that these three principles are falsifiable provided explicit bounds are given. Thus, from a logical point of view, they are purely universal formulas, while the physical Church-Turing thesis itself has a much more complex logical form $\forall\exists\forall$, namely for all physical system m , there exists a computable function f

such that for all initial state x , the evolution of m leads from x to $f(x)$.

8 Conclusion: cellular automata and quantum cellular automata

It is one thing to say that the Universe is computable and thus can be simulated by any model of computation, e.g Cellular Automata. It is another to seek to closely model physical systems in a space-preserving manner. Pursuing this second, more demanding aim leads us to investigate how to model the state space of a bounded region of space. Supposing that this state space is finite is adequate in the classical setting, but must be refined in the quantum setting to a finite-dimensional vector space over a reasonable field. Indeed it seems that simulating a quantum systems by classical cellular automata not only leads to an exponential slowdown, but also fails to respect the geometry of the system. This justifies Feynman's proposal to model quantum systems with Quantum Cellular Automata. This quantum extension of the Cellular Automata model shows its liveliness and robustness.

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