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THEORETICAL BASIS OF BLOCK-ORIENTED FRACTAL IMAGE ENCODING

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Theoretical Basis of Block-Oriented Fractal Image Coding

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Abstract

This paper details the theoretical basis of a block-oriented fractal image encoding scheme. Encoding a fractal image entails seeking a set of transforms W such that the attractor resembles a close approximation of the target image. Fisher's PIFS scheme enables one to create a set of transforms that obeys the general contractivity requirements and yet need not have every transform being attractive.

Keywords: domains, ranges, contractive, Z-contractive, eventually contractive.

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1 Introduction

Following the pioneering work of Williams [WILL71] and Hutchinson [HUTCH81] in the synthesis of deterministic fractals, Barnsley [BARN88], [BADE85], proposed the use of what he termed IFS Iterated Function Systems for the encoding of arbitrary monochrome images. It is important to note that in IFS encoding, a set of marked pixels is described in terms of a finite set of contractive maps of the form (w_i: i = 0, . . . ,n) where

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix} \quad \text{where } |a_i d_i - b_i c_i| < 1 \qquad (1)$$

Subsequently, Barnsley and co-workers [BASL88] further proposed that fractal encoding should be applicable, and effective for the compressive encoding of gray-scale and high quality colour images [BARN88]. Barnsley's demonstration of the encoding of high quality images from the 1987 National Geographic involved 'hand-encoding' as described in the appendix of [BARN88]. However, a solution to the problem of implementing fractal image coding is now known. Firstly, there has been a patent [BARN90b] granted to Michael Barnsley for the fractal encoding of images. The patent is vaguely worded and rather general in scope. Secondly, demonstration of a practical scheme for the block-oriented fractal encoding of an arbitrary gray-scale image has been given by Jacquin [JACQ92]. Following Jacquin, other workers, notably Fisher [FIJB89], have implemented practical schemes of block encoding. Jacquin's approach, which we follow in broad outline though not in detail, involves image blocking and transform for each domain block. The authors have published a preliminary report on the implementation of block-oriented fractal image encoding [CONG93]. This paper seeks to describe the theoretical basis for block-oriented fractal encoding.

Fractal image compression necessarily requires that on encoding an image, as discussed in [CONG93], to determine $\boldsymbol{W}$, a collection of transforms w_i , and that this transform $\boldsymbol{W}$ is able to store the image as a fixed point on the space $\mathbf{F} \rightarrow \mathbf{F}$ from a complete metric space to itself. Our image model f(x,y) represents a gray scale image with gray levels at point (x,y) of the image.

IFS theory requires that for a fixed point [$\boldsymbol{W}$] to exist, every transform w_i: is contractive. The collage theorem provides the motivation to find a close approximation. The contractive mapping fixed point theorem assures us that such a fixed point will exists for any initial image. PIFS Partitioned Iterated Function Systems [FIJB89],[FISH92] are a form of RIFS Recurrent Iterated Function Systems [BAJA88] which partitioned the image in blocks. PIFS however has a less stringent requirement by insisting that the transformation $W^{\circ m}$ can be contractive rather than W.

2 Encoding Images

Our image space $\mathbf{F}$ is a space which consists of all graphs of a real function $z = f(x,y)$ with $(x, y, f(x,y) \in I^3$. To construct the map W, such that w_i is applied to a part of the image called *domains* D_i and mapped to a copy called *ranges* R_i requires some mapping $v_i : \mathbf{I}^3 \mapsto \mathbf{I}^3$. As mentioned earlier, the transforms w_i is restricted to certain parts of the image defined as

$$w_i = v_i|_{D_i \times I}$$

These maps w_i will tile the image I^2 if

$$W(f) = \bigcup_{i=1}^{n} w(f) \quad \forall f \in F$$

D_i therefore defines a part of the image $f \cap (D_i \times I)$ to which w_i is restricted. The application of w_i to $f \cap (D_i \times I)$ must be a graph of a function over R_i with $I^2 = \cup_{i=1}^{n} R_i$. Since R_i must be disjoint, the union $\cup_{i=1}^{n} w_i(f)$ must yield a graph of a function over I^2. The union of w_i defines a map of the form

$$W = \bigcup_{i=1}^{n} w_i$$

To seek an approximation $\hat{f} = |W|$ requires that the distance metric $d(f, \hat{f})$ be at its minimal. Finding this $|W| = W(|W|) = \bigcup_{i=1}^{n} w_i(|W|)$ would require in PIFS to seek domains D_i and corresponding w_i to form an image f defined as

$$f \approx W(f) = \bigcup_{i=1}^{n} w_i(f) \tag{2}$$

Equation 2 is an expression which says that the union of these transforms w_i defines a map W. This map W is created by f covering with parts D_i of itself. How these parts would cover f is determined by w_i. The approximated image f should be close to the original image if $d(f, |W|)$ is at minimal. The collage theorem provides such a motivation.

The encoding process is therefore first to partition the image I^2 in a certain manner by the ranges R_i. Domains D_i are created by partitioning I^2 also. The $w_i : D_i \times I^3 \mapsto I^3$ and domains D_i corresponding to the range R_i is sought such that $w_i(f)$ and $f \cap (R_i \times I)$ has the closest distance, that is

$$d(f \cap (R_i \times I), w_i(f)) \tag{3}$$

is at its minimal. Maps w_i which specifies W must be chosen such that W or $W^{\circ m}$ is contractive.

To reconstruct the image given W, use an arbitrary image and iterate n times computing $W(f_0), W(W(f_0)), \ldots$ until the attractor $|W|$ for the image appears [1].

2.1 Z-Contractive Maps

The theory here provides a method to map three-dimensional objects onto a two-dimensional space and yet obey the general contractivity requirements.

Let $\pi_z : \mathbf{I}^3 \mapsto \mathbf{I}^2$ be the projection operation defined as $\pi_z(x, y, z) = (x, y)$.
Then, a map $w : \mathbf{R}^3 \mapsto \mathbf{R}^3$ is said to be *z-contractive* if there is a positive real number $s < 1$ such that for all $x, y, z_1, z_2 \in \mathbf{R}$

$$|w_i(x, y, z_1) - w_i(x, y, z_2)| < s_i \cdot |z_1 - z_2|$$

Now $\pi_z \circ w(x, y, z)$ is independent of z.

Define $(D_i : i = 1, \ldots, n)$ as subsets of $\mathbf{I}^2$ and some mappings $(v_i : i = 1, \ldots, n) : \mathbf{I}^3 \mapsto \mathbf{I}^3$. Define w_i to be some mappings with restrictions as such

$$w_i = v_i|_{D_i \times I}$$

Maps w_i would now tile $\mathbf{I}^2$ if for all images $f \in \mathbf{F}$, and the union of such mappings also belong to $\mathbf{F}$. This implied that for an image $f \in \mathbf{F}$, each D_i defines a part of the image given as

$$d(f \cap (R_i \times I), w_i(f))$$

into which w_i is restricted. When w_i is applied to this part, the result must be a graph of a function $z = f(x, y)$ over R_i which is the result of $\pi_z \circ w_i(f)$.

2.2 Eventually Contractive Maps

PIFS attractors are formed by the same iteration process as IFS with its limit set defined as

$$|W| \equiv |W^{\circ m}|f_\infty = \lim_{n \to \infty} W^{\circ n}(f_0)$$

but with less stringent contractivity requirements. It has been found [FIJB89] that the attractor in PIFS is bounded as long as W is *eventually contractive*. A map $W : \mathbf{F} \mapsto \mathbf{F}$ is eventually contractive if there exists a positive integer m called the *exponent of eventual contractivity* such that $W^{\circ m}$ is contractive [FIJB89]. If every transform w_i obeys the contractivity condition, as in IFS then the transform is eventually contractive also, but this cannot be said of a map that is eventually contractive must necessarily compose of all transforms that are contractive . When these maps w_i are choosen through a suitably metric to yield z-contractivity the following results

$$|w_i(x, y, z_1) - w_i(x, y, z_2)| < s_i \cdot |z_1 - z_2|$$

then W will be d_{sup}-contractive if and only if every w_i is d_{sup}-contractive. This happens when each $s_i < 1$. If at least one $s_i \geq 1$ then W will not be d_{sup}-contractive. Recall that

[1] examples given in [CONG93]

the map W is defined as

$$W(\cdot) = \bigcup_{i=1}^{n} w_i(\cdot)$$

which is a union of maps w_i, but such maps are mapping onto disjoint parts of the image. The iterate transform $W^{\circ m}$ is then composed of the union of compositions of the form

$$w_{i1} \circ w_{i2} \circ \ldots \circ w_{im}$$

The product of these composition must be that it bounds the contractivity of the composition if there is sufficient 'mixing' with those contractive w_i eventually dominating the non-contractive ones.

Generalized Collage Theorem([FIJB91])
For $f \in \mathbf{F}$ and $W : \mathbf{F} \mapsto \mathbf{F}$ *eventually contractive with minimum exponent of eventual-contractivity* m *and eventual-contractivity* $\sigma < 1$,

$$d(|W|, f) \leq \frac{1}{1-\sigma} \frac{1 - s_{max}^m}{1 - s_{max}} d(W(f), f).$$

This theorem relates the metric difference between the attractor $|W|$ of the mapping W and the target image f, to the like difference between the target image f and the mapping of f by the mapping, viz $W(f)$. This is of course a significant generalization of the well-known collage theorem for IFS encoding of Barnsley and Sloan. Note that $s_{max} = max_{i=1,\ldots,n}\{\text{s : s = z-contractivity of } w_i\}$ This theorem as in the earlier IFS Collage theorem only serves as a motivation and has been found to not provide useful bounds [FIJB91].

3 Conclusion

In this paper, we have detailed the theoretical basis of block-oriented fractal compression scheme. Our discussion has been directed towards fractal compressing using linear mappings, as used by us [CONG93],in Jacquin's system and PIFS encoding; it would appear trivial to extend the discussion to fractal encoding using bilinear mappings such as the recently described Bath Fractal Transform [WINM94]. In the above discussion, we have not dealt with the issue of convergence which has vital significance for practical decoding: a specific study of this aspect has been dealt with by Hürtgen [JUHA94].

Images are essentially three-dimensional quantities. Fisher's Z-contractive maps defines a way to map such a three-dimensional object into two-dimensional object. In this framework eventually contractive maps are shown to be effective.

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