Permutrees

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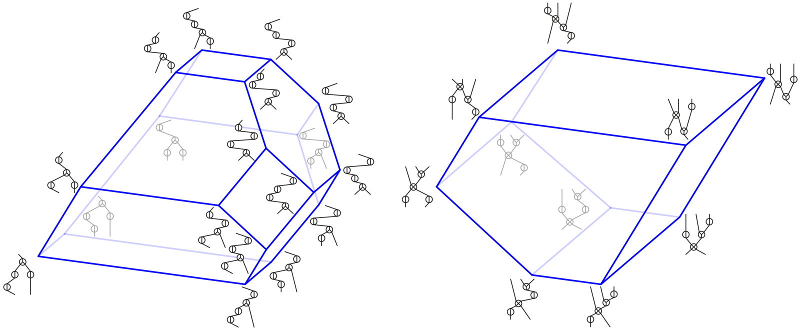
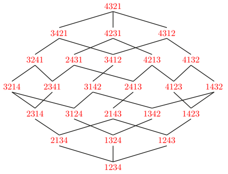
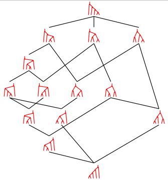
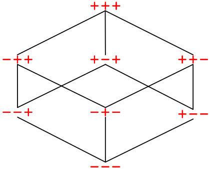
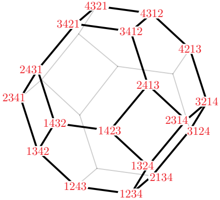
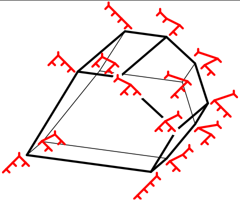
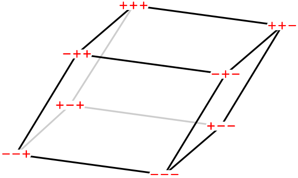
CNRS & Ecole Polytechnique – LRI, Univ. Paris-Sud



	permutations	binary trees	binary sequences
Combinatorics			
Geometry			
Algebra	Malvenuto-Reutenauer algebra FQSym = vect $\langle \mathbb{F}_\tau \mid \tau \in \mathfrak{S} \rangle$ $\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\sqcup}\, \tau'} \mathbb{F}_\sigma$ $\triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$	Loday-Ronco algebra PBT = vect $\langle \mathbb{P}_\mathrm{T} \mid \mathrm{T} \in \mathcal{BT} \rangle$ $\mathbb{P}_T \cdot \mathbb{P}_{T'} = \sum_{T \nearrow^{T'} \le T'' \le {}^{T}\nwarrow T'} \mathbb{P}_{T''}$ $\triangle \mathbb{F}_\gamma = \sum_{\gamma \text{ cut}} B(T,\gamma) \otimes A(T,\gamma)$	Solomon algebra Rec = vect $\langle \mathbb{X}_\eta \mid \eta \in \pm^* \rangle$ $\mathbb{X}_\eta \cdot \mathbb{X}_{\eta'} = \mathbb{X}_{\eta+\eta'} + \mathbb{X}_{\eta-\eta'}$ $\triangle \mathbb{X}_\eta = \sum_{\gamma \text{ cut}} B(\eta,\gamma) \otimes A(\eta,\gamma)$

The Permutree Recipe

- Take a word in $\{$⦶, ☮, Ⓨ, $\otimes\}^n$
- Take a permutation
- Do the insertion: get a Leveled Permutree (bijection)
- Remove the levels: get a Permutree (surjection)

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Example

$⦶^n$	$\longleftrightarrow$	permutations of $[n]$
$☮^n$	$\longleftrightarrow$	standard binary search trees
$\{☮, Ⓨ\}^n$	$\longleftrightarrow$	Cambrian trees
$\otimes^n$	$\longleftrightarrow$	binary sequences

The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

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Decorated permutation: $\overline{2}\overline{7}5\underline{1}3\overline{\underline{4}}\underline{6}$

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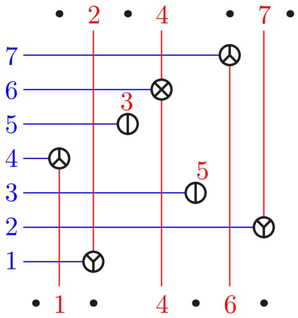

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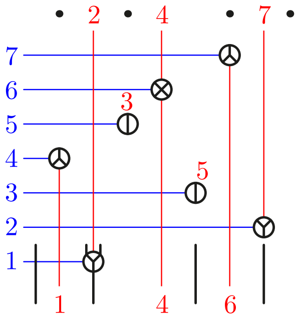

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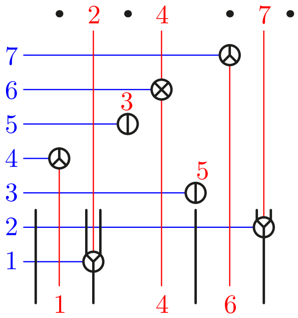

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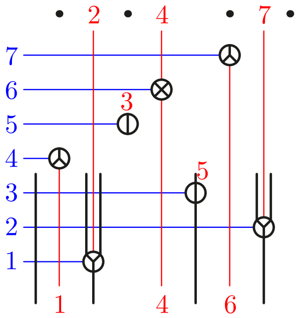

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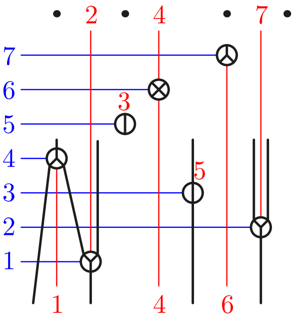

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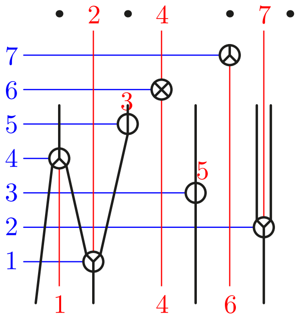

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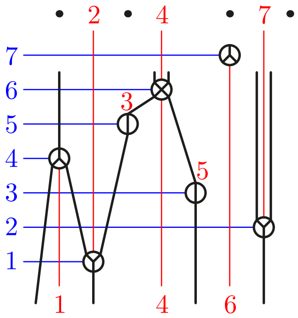

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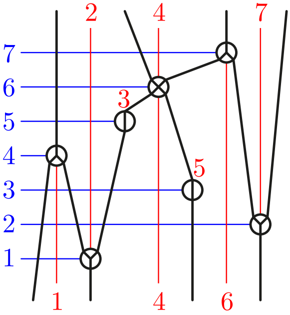

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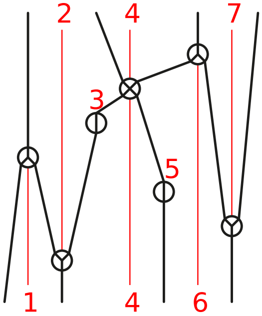

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Leveled permutrees:

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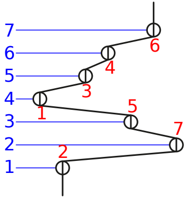

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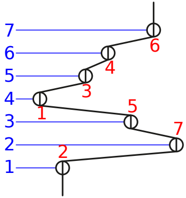

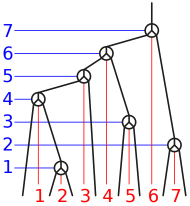

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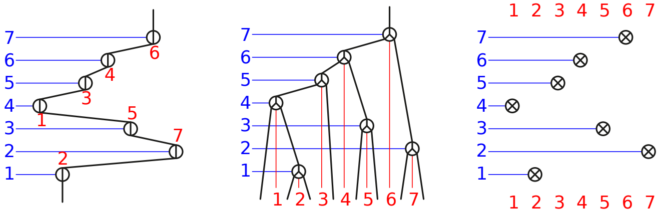

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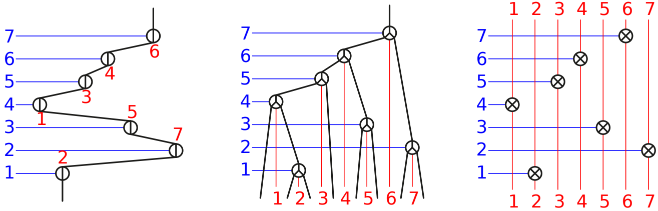

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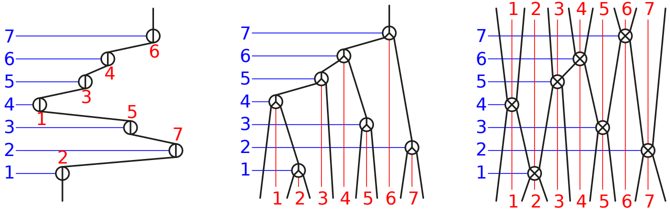

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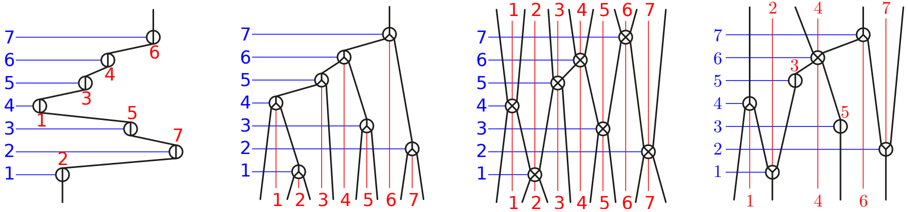

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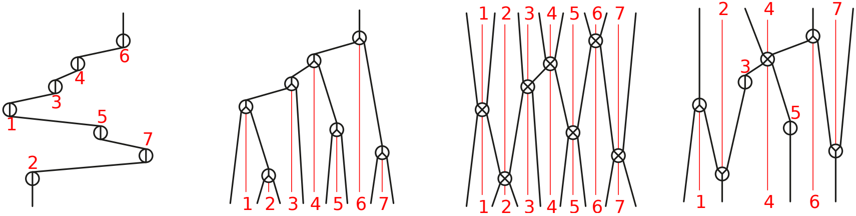
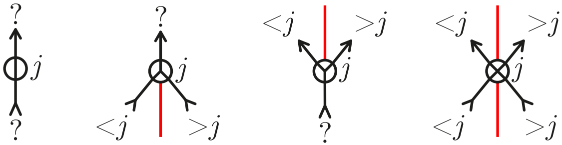
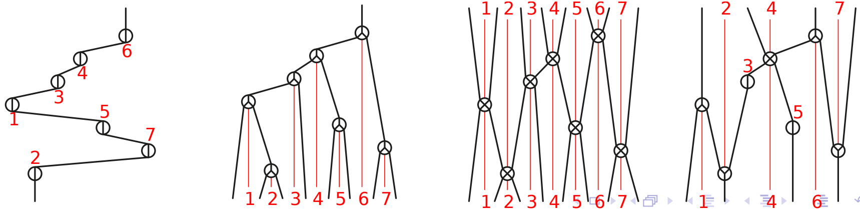


Definition of a permutree

directed (bottom to top) and labeled (bijectively by $[n]$) tree such that





Insertion

$$\mathfrak{S}_n \times \{⦶, ☮, Ⓨ, \otimes\}^n \longrightarrow \text{Permutrees}$$

$$(\sigma, \delta) \longrightarrow \mathbf{P}_\delta(\sigma)$$

Insertion

$$\mathfrak{S}_n \times \{\text{⦶}, \text{☮}, \text{Ⓨ}, \otimes\}^n \longrightarrow \text{Permutrees}$$
$$(\sigma, \delta) \longrightarrow \mathbf{P}_\delta(\sigma)$$

Congruence

$$\dots ac \dots \underline{b} \dots \equiv_\delta \dots ca \dots \underline{b} \dots \qquad (\delta_b \in \{\text{☮}, \otimes\})$$
$$\dots \overline{b} \dots ac \dots \equiv_\delta \dots \overline{b} \dots ca \dots \qquad (\delta_b \in \{\text{Ⓨ}, \otimes\})$$

Insertion

$$\begin{aligned} \mathfrak{S}_n \times \{⦶, ☮, Ⓨ, \otimes\}^n &\longrightarrow \text{Permutrees} \\ (\sigma, \delta) &\longrightarrow \mathbf{P}_\delta(\sigma) \end{aligned}$$

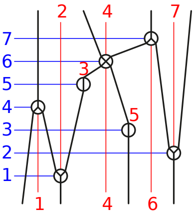
Congruence

$$\dots ac \dots \underline{b} \dots \equiv_\delta \dots ca \dots \underline{b} \dots \qquad (\delta_b \in \{☮, \otimes\})$$

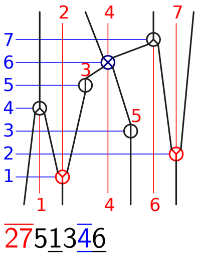
$$\dots \overline{b} \dots ac \dots \equiv_\delta \dots \overline{b} \dots ca \dots \qquad (\delta_b \in \{Ⓨ, \otimes\})$$

Property

$$\sigma \equiv_\delta \tau \Leftrightarrow \mathbf{P}_\delta(\sigma) = \mathbf{P}_\delta(\tau)$$



$\overline{27}51\underline{1}3\underline{\overline{4}6}$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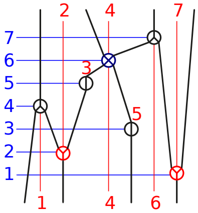
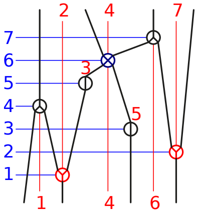


$\overline{7}\overline{2}5\underline{1}3\overline{\underline{4}}\underline{6} \equiv$



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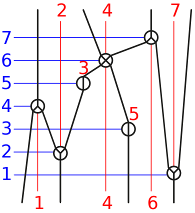
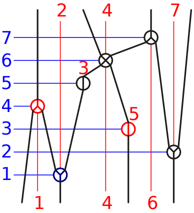


$\overline{72}51\underline{}3\overline{\underline{46}}$ $\equiv$



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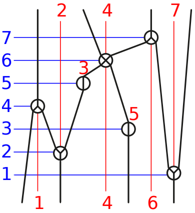

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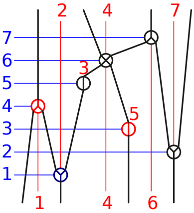

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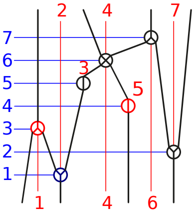
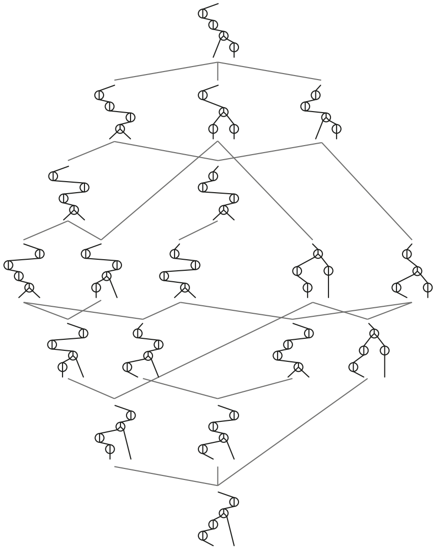
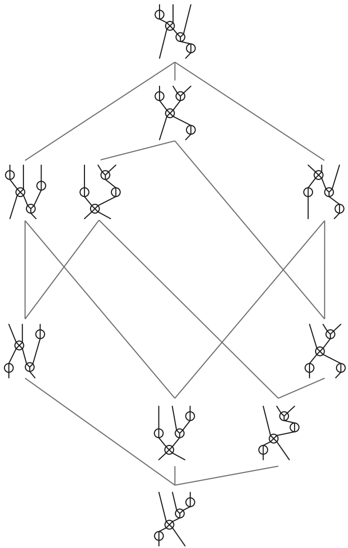


$\equiv \overline{27}\underline{1}53\underline{\overline{4}6}$

Lattice definition





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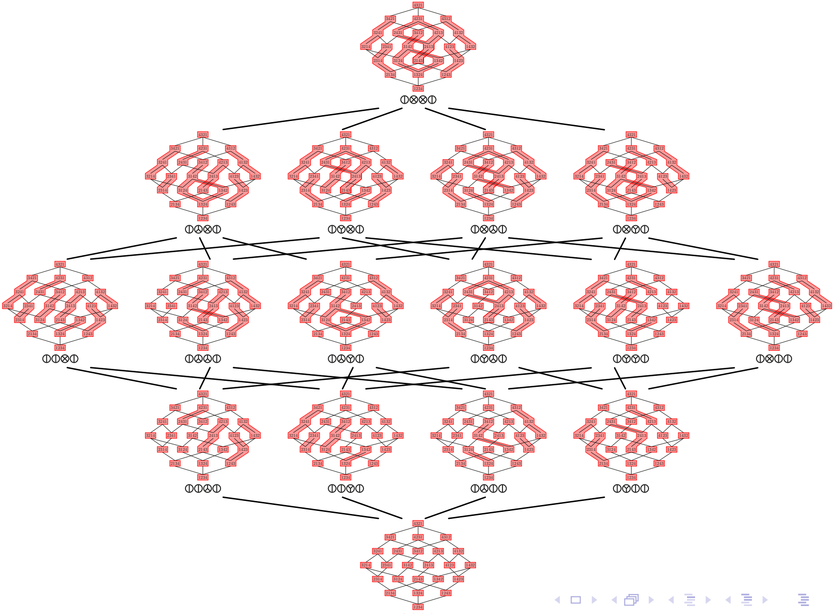
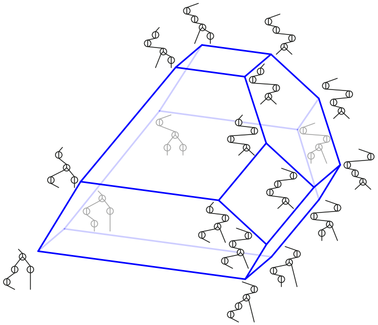


The Permutreehedron

Thm (Pilaud, P.) for every $\delta \in \{⦶, ☮, Ⓨ, \otimes\}^n$, there is an explicit construction of a

- a complete simplicial fan, the δ-permutree fan $\mathcal{F}(\delta)$;
- a polytope, the Permutreehedron $\mathbb{PT}(\delta)$, whose normal fan is $\mathcal{F}(\delta)$.



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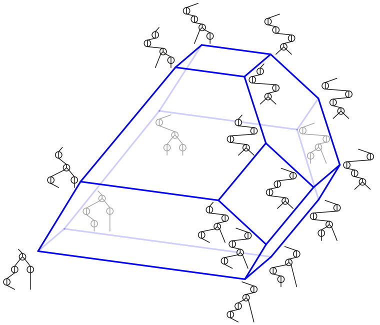

The Permutreehedron can be constructed by convex hull or hyperplane intersection.

The Permutreehedron

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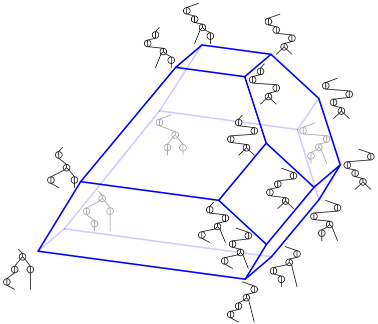

The vertices of $\mathbb{PT}(\delta)$ are the δ-permutrees.

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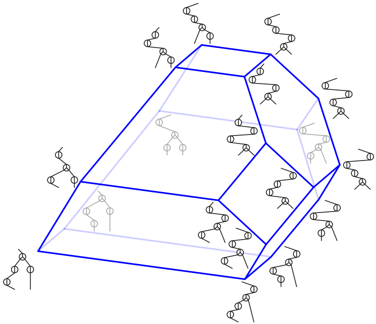

The oriented graph of $\mathbb{PT}(\delta)$ is the Hasse diagram of the δ-permutree lattice.

The Permutreehedron

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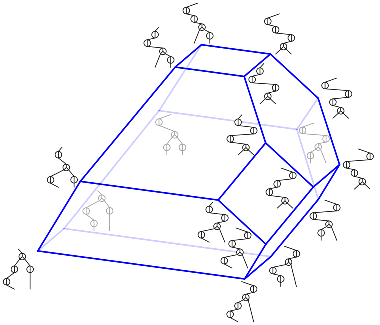
- a complete simplicial fan, the δ-permutree fan $\mathcal{F}(\delta)$;
- a polytope, the Permutreehedron $\mathbb{PT}(\delta)$, whose normal fan is $\mathcal{F}(\delta)$.

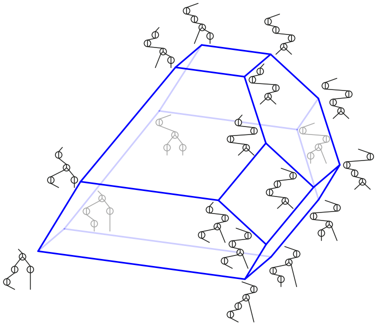


Combinatorics of the faces: Schröder permutrees.

The Permutreehedron

Thm (Pilaud, P.) for every $\delta \in \{⦶, ☮, ⊗\}^n$, there is an explicit construction of a

- a complete simplicial fan, the δ-permutree fan $\mathcal{F}(\delta)$;
- a polytope, the Permutreehedron $\mathbb{PT}(\delta)$, whose normal fan is $\mathcal{F}(\delta)$.



Show the 3D-surprise!

Matriochka Permutreehedra

refinement $\delta \preccurlyeq \delta'$ $\Longrightarrow$ inclusion $\mathbb{PT}(\delta) \subset \mathbb{PT}(\delta')$