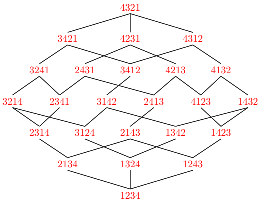
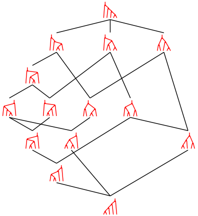
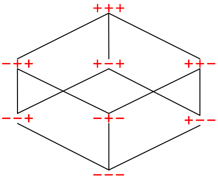
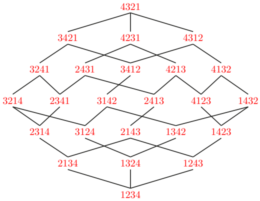
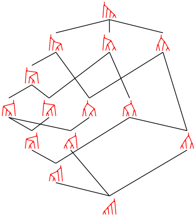
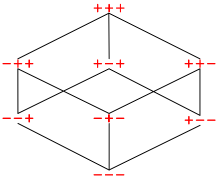
Lattice and Hopf algebra of integer relations

Viviane Pons
with Grégory Châtel and Vincent Pilaud

Université Paris-Sud

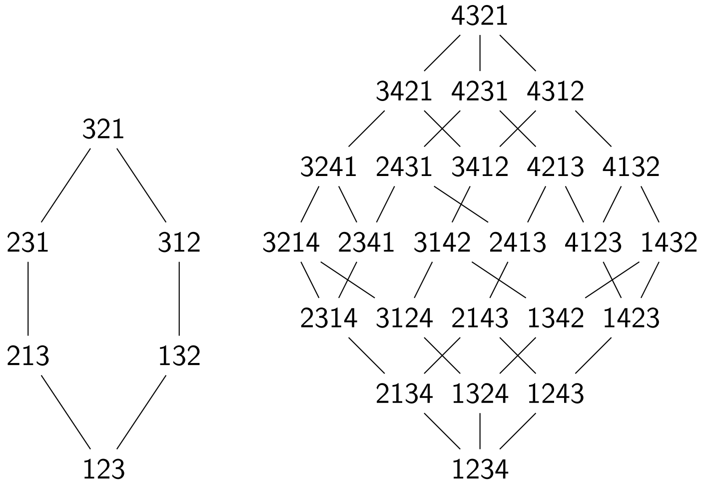
CARMA 30/06/2017

	permutations	binary trees	binary sequences
Combinatorics			
Algebra	Malvenuto-Reutenauer algebra	Loday-Ronco algebra	Descent Hopf algebra

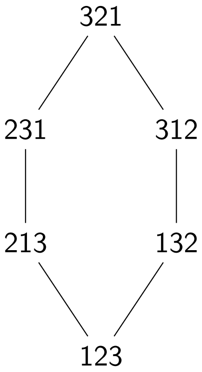
	permutations	binary trees	binary sequences
Combinatorics			
Algebra	Malvenuto-Reutenauer algebra	Loday-Ronco algebra	Descent Hopf algebra

→ All these objects (and more) can be interpreted in terms of *integer posets*.

Weak order


321
231
312
213
132
123
4321
3421 4231 4312
3241 2431 3412 4213 4132
3214 2341 3142 2413 4123 1432
2314 3124 2143 1342 1423
2134 1324 1243
1234

Weak order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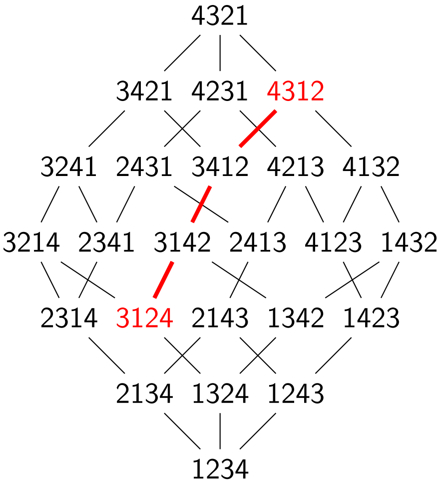
4321
3421 4231 4312
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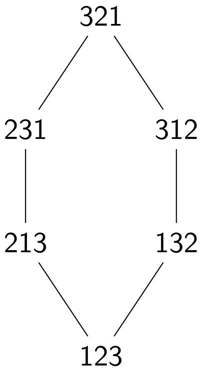
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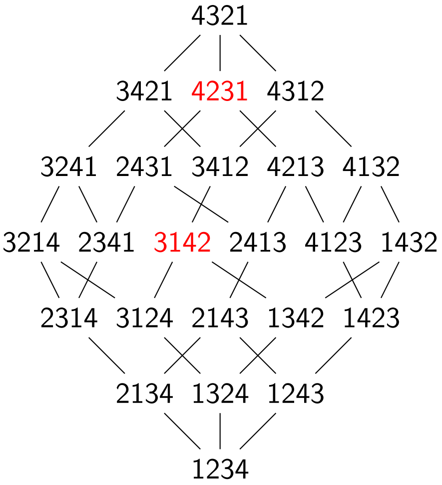
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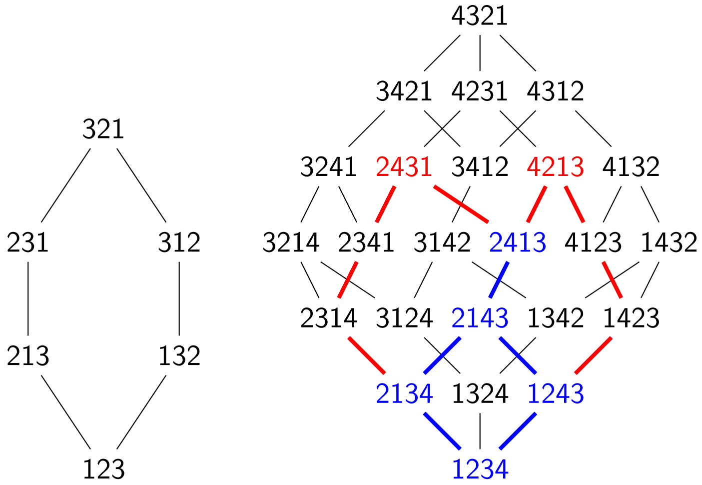
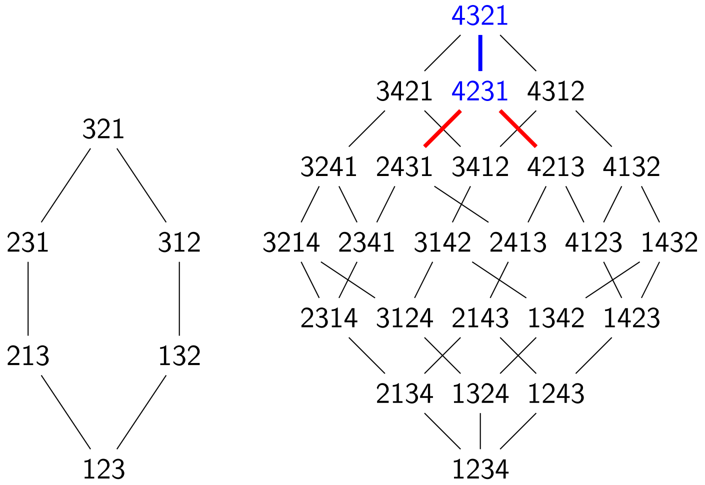


$$2413 \wedge 4213 = 2413$$

Weak order



$2413 \wedge 4213 = 2413$ $2413 \vee 4213 = 4231$

Permutation poset

4312

Permutation poset

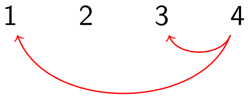
4312

Permutation poset

4312

Permutation poset

4312



Permutation poset

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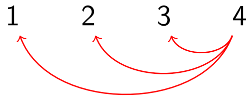

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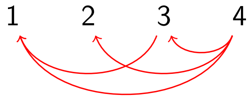

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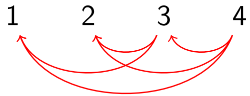

Permutation poset

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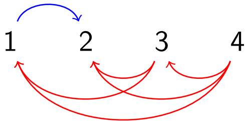

Permutation poset

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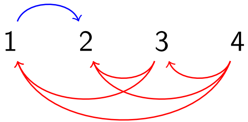

Permutation poset

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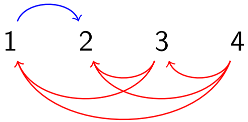

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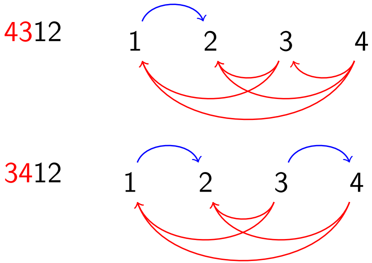
Permutation poset



Permutation poset

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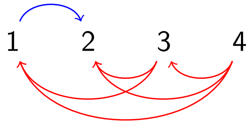

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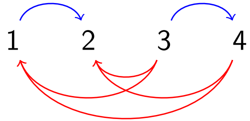
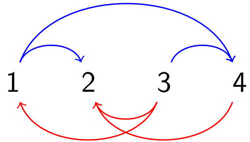


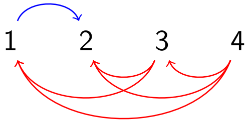
3142



Relation associated to a permutation

$\dots x \dots y \dots \Rightarrow x \mathrm{R}\, y$

4312



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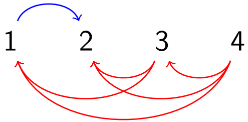
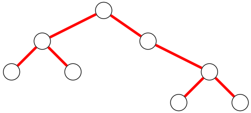


Weak order

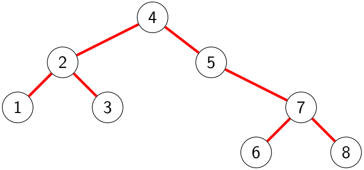
$$\mathrm{R}^{\mathsf{Inc}} = \{ i \,\mathrm{R}\, j, i < j \}$$
$$\mathrm{R}^{\mathsf{Dec}} = \{ j \,\mathrm{R}\, i, i < j \}$$

$$\mathrm{R} \preccurlyeq \mathrm{S} \Leftrightarrow \qquad \mathrm{R}^{\mathsf{Inc}} \supseteq \mathrm{S}^{\mathsf{Inc}} \text{ and } \mathrm{R}^{\mathsf{Dec}} \subseteq \mathrm{S}^{\mathsf{Dec}}$$

Binary tree poset



Binary tree poset



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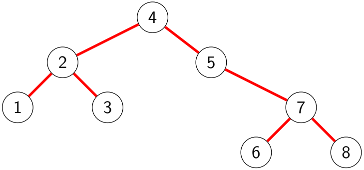

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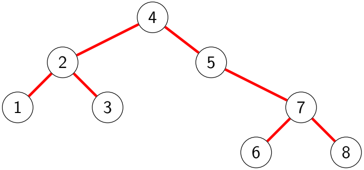

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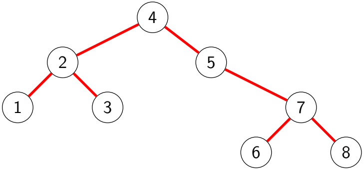

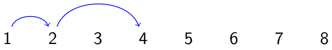

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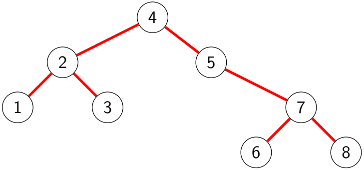

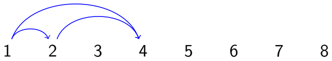

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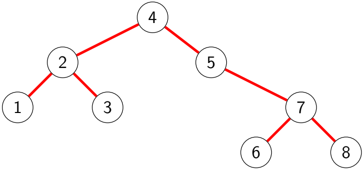

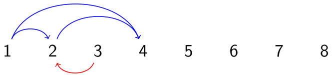

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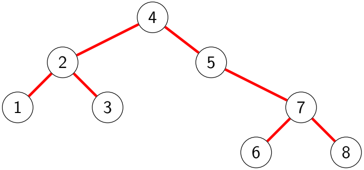

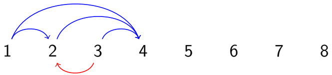

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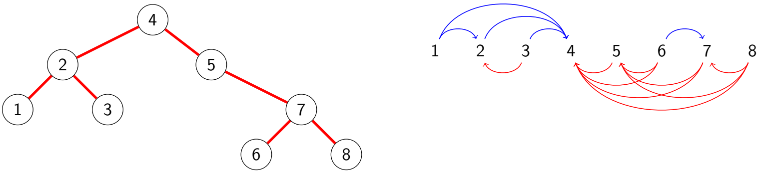

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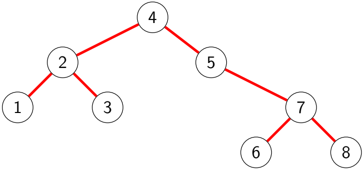
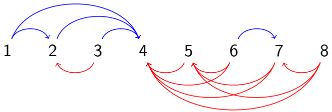




$\rightarrow$ Tamari lattice

Binary sequence poset

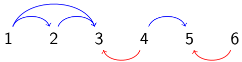
$- - + - +$ 1 2 3 4 5 6

Binary sequence poset

$- - + - +$

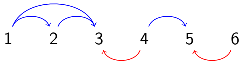
Binary sequence poset

$- - + - +$



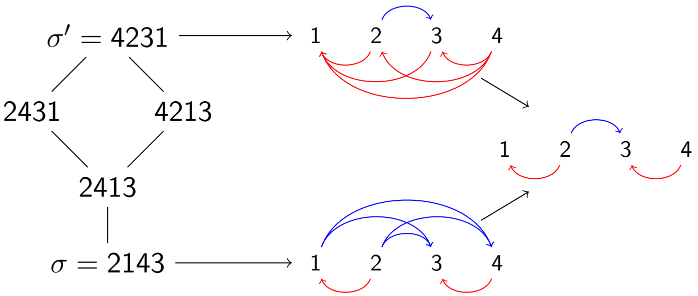
Binary sequence poset

$- - + - +$



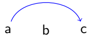
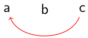
$\rightarrow$ Boolean lattice

Permutation intervals



Interval-poset characterization

Posets (anti-symmetric, transitive) and

	intervals of permutations (weak order)	intervals of binary trees (Tamari lattice)	intervals of binary sequence (boolean lattice)
 ⇒			
 ⇒			

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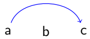
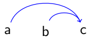
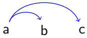
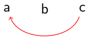
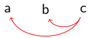
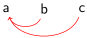
 ⇒	 or 		
 ⇒	 or 		

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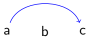
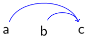
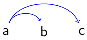
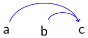
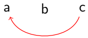
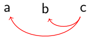
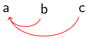
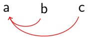
 ⇒	 or 		
 ⇒	 or 		

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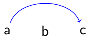
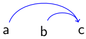
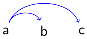
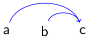
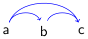
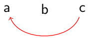
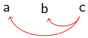
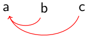
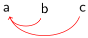
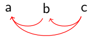
	intervals of permutations (weak order)	intervals of binary trees (Tamari lattice)	intervals of binary sequence (boolean lattice)
 ⇒	 or 		
 ⇒	 or 		

Binary relations on integer

R is a binary relation of size n.

$$1 \quad 2 \quad \cdots \quad i \quad \cdots \quad j \quad \cdots \quad k \quad \cdots \quad n$$

Binary relations on integer

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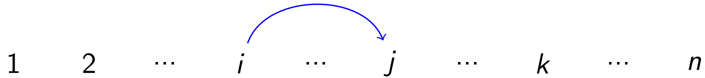

$i \, \mathrm{R} \, j$

Binary relations on integer

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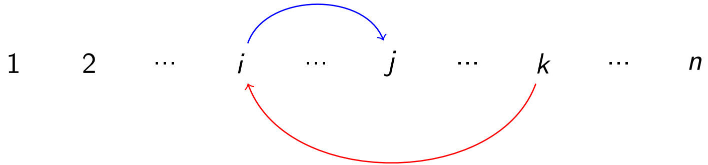
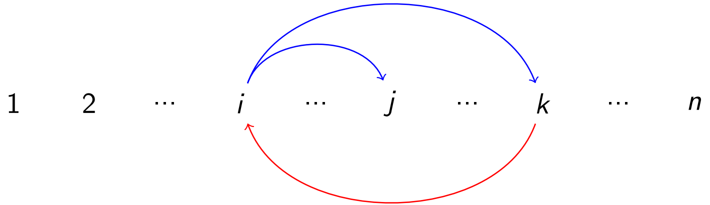


$i \mathrm{R} j$ $\quad$ $k \mathrm{R} i$

Binary relations on integer

R is a binary relation of size n.



$i \mathrel{\mathrm{R}} j$ $\quad$ $k \mathrel{\mathrm{R}} i$ $\quad$ $i \mathrel{\mathrm{R}} k$

Binary relations on integer

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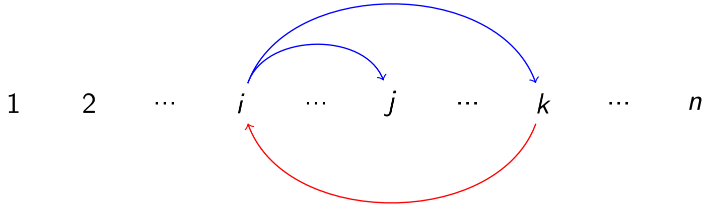


$i \,\mathrm{R}\, j$ $\quad$ $k \,\mathrm{R}\, i$ $\quad$ $i \,\mathrm{R}\, k$

Family of size $2^{n(n-1)}$.

Weak order

Let R an integer binary relation

$$\mathrm{R}^{\mathsf{Inc}} = \{i \,\mathrm{R}\, j, i < j\}$$
$$\mathrm{R}^{\mathsf{Dec}} = \{j \,\mathrm{R}\, i, i < j\}$$

Weak order

Let R an integer binary relation

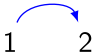
$$\mathrm{R}^{\mathsf{Inc}} = \{ i \,\mathrm{R}\, j, i < j \}$$
$$\mathrm{R}^{\mathsf{Dec}} = \{ j \,\mathrm{R}\, i, i < j \}$$

Let R and S be two integer binary relations

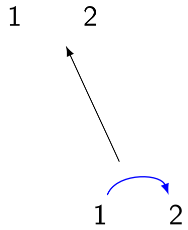
$$\mathrm{R} \preccurlyeq \mathrm{S} \Leftrightarrow \qquad \mathrm{R}^{\mathsf{Inc}} \supseteq \mathrm{S}^{\mathsf{Inc}} \text{ et } \mathrm{R}^{\mathsf{Dec}} \subseteq \mathrm{S}^{\mathsf{Dec}}$$

Binary relations of size 2

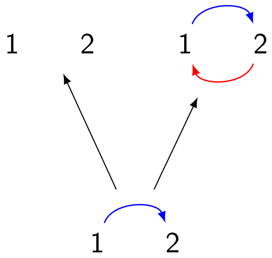
Binary relations of size 2



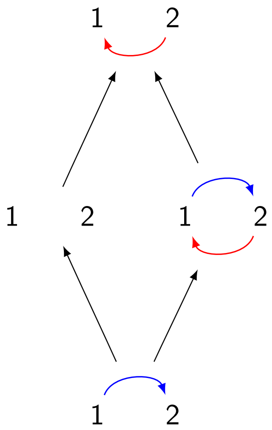
Binary relations of size 2

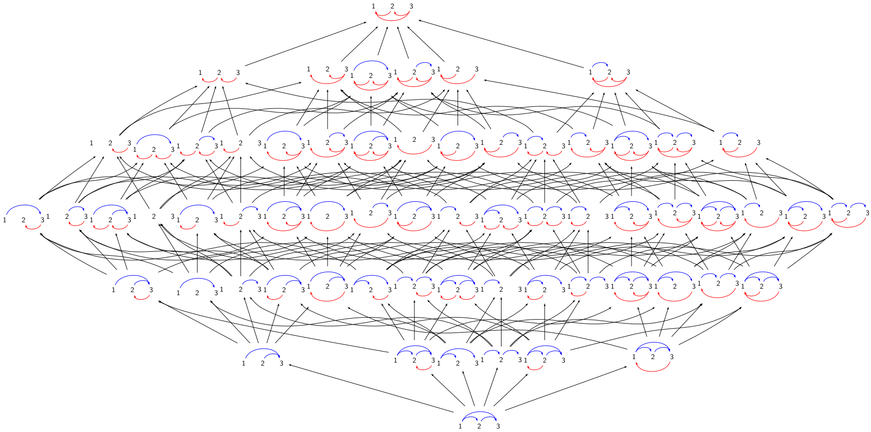


Binary relations of size 2

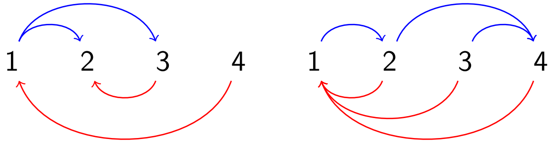


Binary relations of size 2

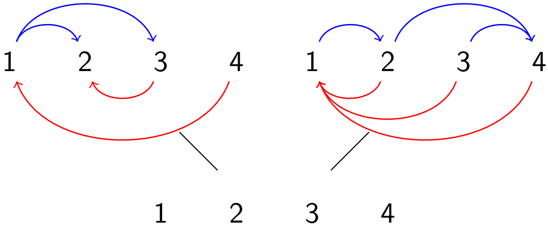




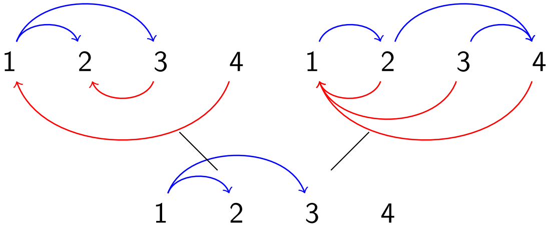
Meet and join



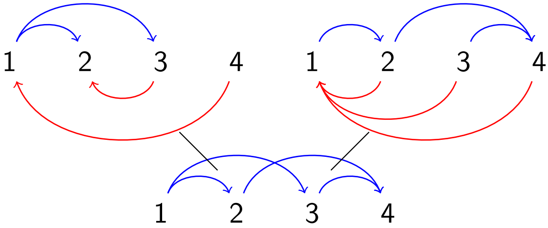
Meet and join



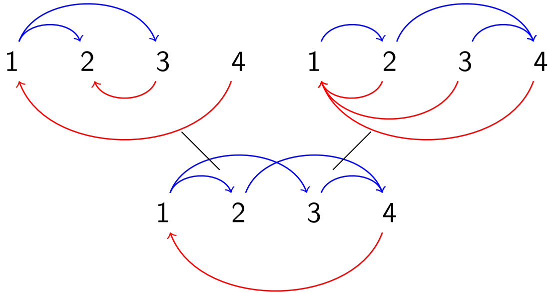
Meet and join



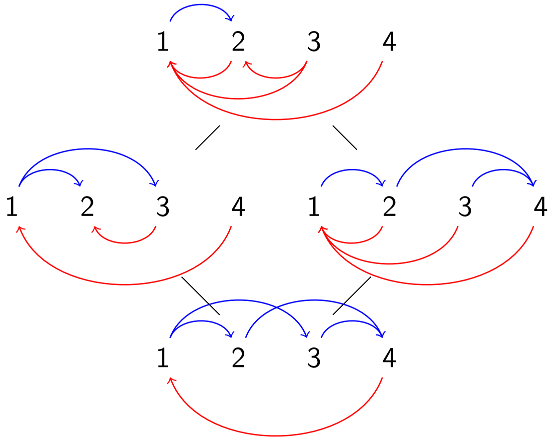
Meet and join



Meet and join

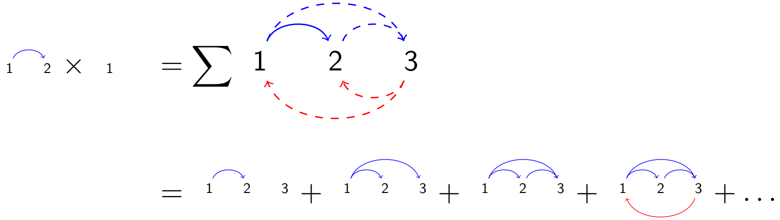


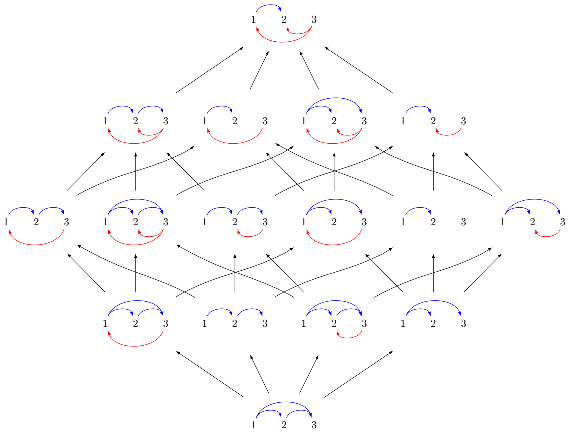
Meet and join



Hopf algebra

$$\mathrm{R} \times \mathrm{S} = \sum \mathrm{R} \qquad \mathrm{S}$$

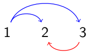

1 2 × 1
= Σ 1 2 3
= 1 2 3 + 1 2 3 + 1 2 3 + 1 2 3 + . . .

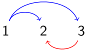


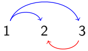
Coproduct

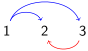
$$\triangle \mathrm{R} = \sum \mathrm{P} \otimes \mathrm{Q}$$

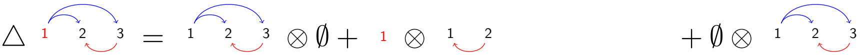
where $\mathrm{R} = \mathrm{P} \cup \mathrm{Q} \cup (\mathrm{P} \rightarrow \mathrm{Q})$

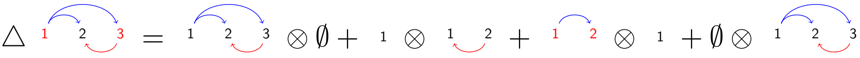
Δ  $=$

Δ 

$=$ 

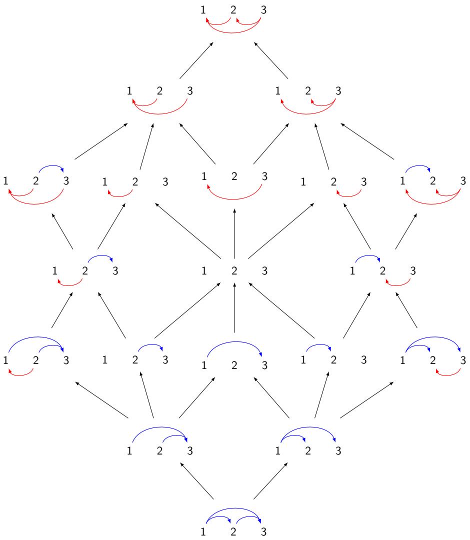
$\otimes \emptyset + \emptyset \otimes$ 


△
1 2 3
=
1 2 3
⊗ ∅ +
1
⊗
1 2
+ ∅ ⊗
1 2 3



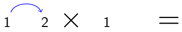
Integer Posets

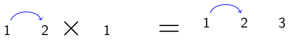
We *restrict* ourselves to transitive antisymmetric relations.

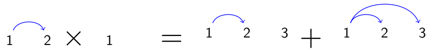


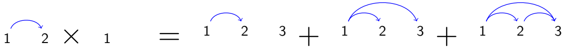
Hopf algebra of integer poset

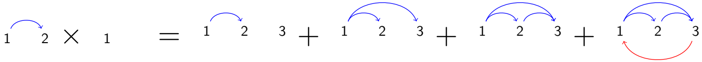
Quotient of the integer relations Hopf algebra
$\mathrm{R} \equiv 0$ if R is not a poset.

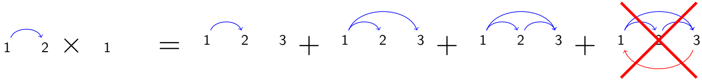

1
2
×
1
=

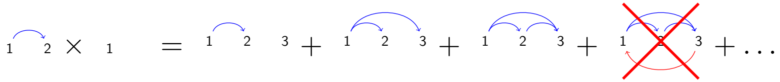

1
2
×
1
=
1
2
3

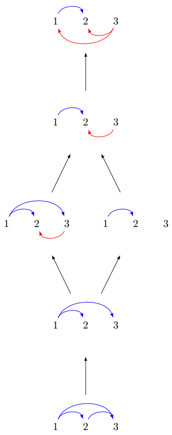

1 2 × 1 = 1 2 3 + 1 2 3


1 2 × 1 = 1 2 3 + 1 2 3 + 1 2 3




1 2 × 1 = 1 2 3 + 1 2 3 + 1 2 3 + 1 2 3


1 2 × 1 = 1 2 3 + 1 2 3 + 1 2 3 + 1 2 3 + ...


1 2 3
1 2 3
1 2 3
1 2 3
1 2 3
1 2 3

To check: is it well defined?

$\mathrm{P} \times 0 = 0$

$\triangle 0 = 0$

Question: can we restrict to intervals of permutations?

Question: can we restrict to intervals of permutations?

Yes!

Question: can we restrict to intervals of binary trees (and binary sequences)?

Question: can we restrict to intervals of binary trees (and binary sequences)?

Not exactly... But

Question: can we restrict to intervals of binary trees (and binary sequences)?

Not exactly... But

Lattice: ok
Hopf algebra: as a sub algebra of the Hopf algebra on permutations intervals

References

- Châtel, Pilaud, P. *The weak order on integer posets* arXiv:1701.07995
- Pilaud, P. *Permutrees* arXiv:1606.09643
- Pilaud, P. *The Hopf algebra of integer posets* (work in progress)