From binary relations to Tamari lattice

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with Grégory Châtel and Vincent Pilaud

Université Paris-Sud

Discrete Structure Days

What is an order?

What is an order?

Natural order on integers: $1 < 2 < 3 < \ldots$

What is an order?

Natural order on integers: $1 < 2 < 3 < \dots$

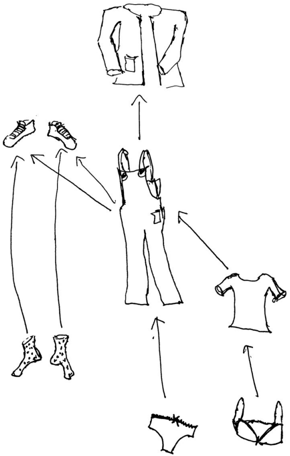
Partial orders?

What is an order?

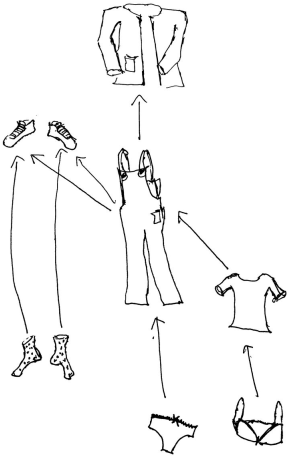
Natural order on integers: $1 < 2 < 3 < \dots$

Partial orders?

The closet-poset!

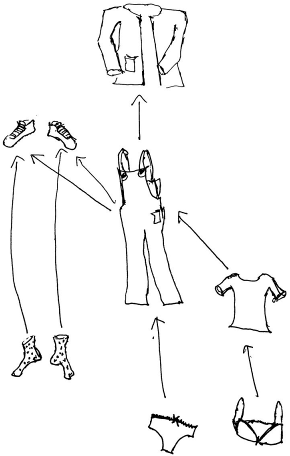


Properties:



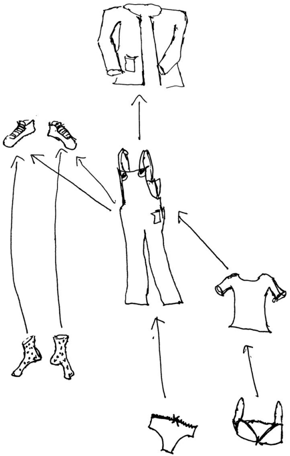
Properties:

- Antisymmetric



Properties:

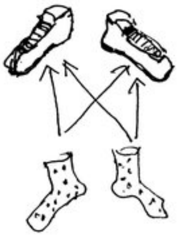
- Antisymmetric
- Transitive



Properties:

- Antisymmetric
- Transitive
- Lattice?

Non lattice example:



Bubble sort on permutations

251436

Bubble sort on permutations

251436

Bubble sort on permutations

251436

215436

Bubble sort on permutations

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Bubble sort on permutations

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Bubble sort on permutations

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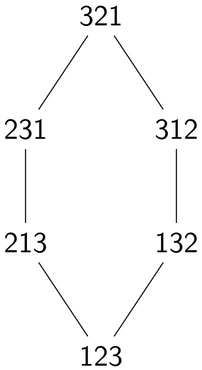
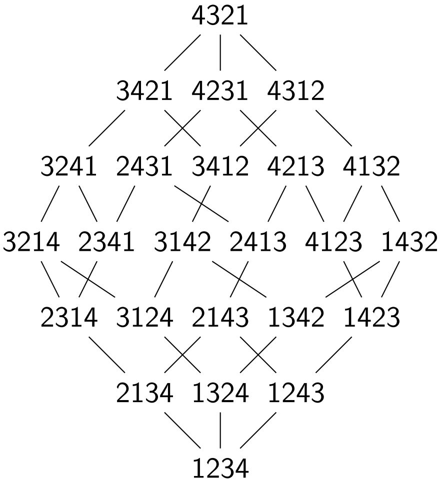
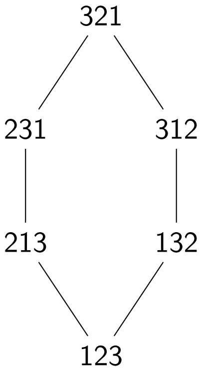
Weak Order


321
231
312
213
132
123


4321
3421 4231 4312
3241 2431 3412 4213 4132
3214 2341 3142 2413 4123 1432
2314 3124 2143 1342 1423
2134 1324 1243
1234

Weak Order


321
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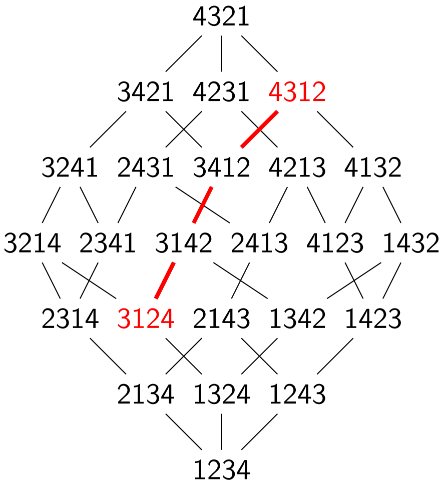
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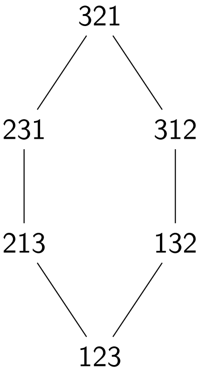
Weak Order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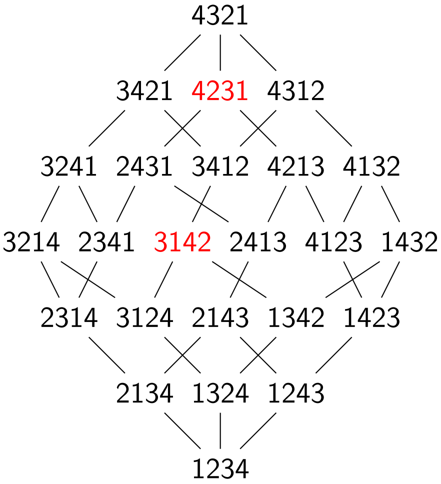
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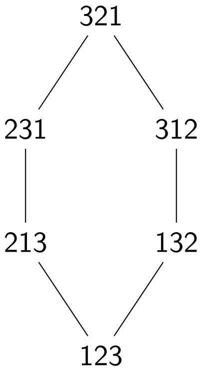
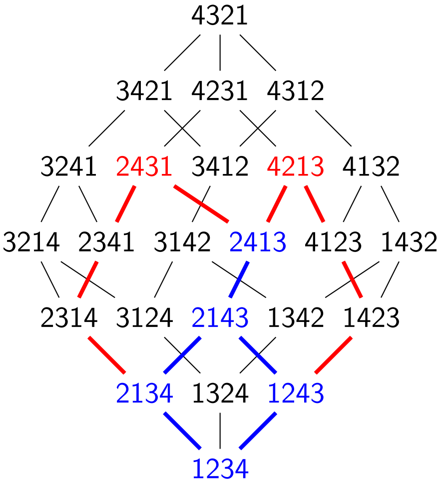
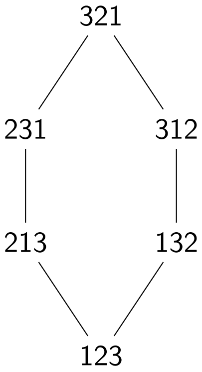
Weak Order

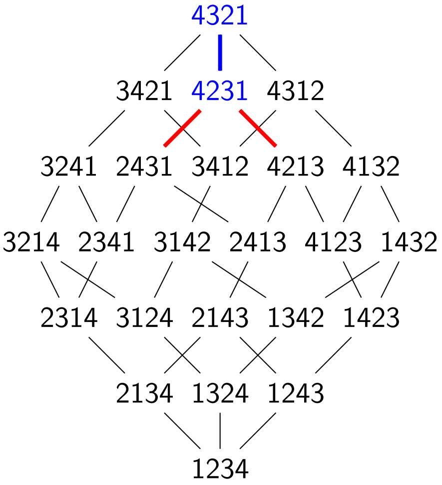




$$2413 \wedge 4213 = 2413$$

Weak Order

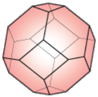


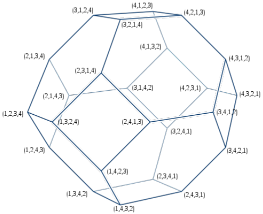


$2413 \wedge 4213 = 2413$ $\qquad$ $2413 \vee 4213 = 4231$

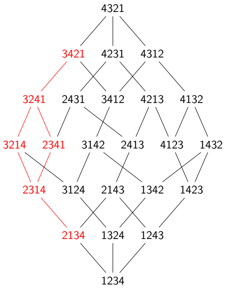
Triple interpretation

Geometry





Combinatorics



Algebra

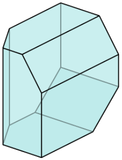
$$\mathbf{F}_{21} \, . \, \mathbf{F}_{12} = \mathbf{F}_{21 ⧢ 12}$$

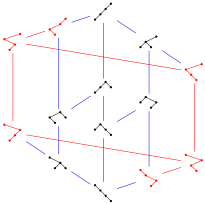
$$= \mathbf{F}_{2134} + \mathbf{F}_{2314} + \mathbf{F}_{2341} + \mathbf{F}_{3214} + \mathbf{F}_{3241} + \mathbf{F}_{3421}$$

Triple interpretation

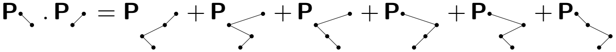
Combinatorics

Geometry





Algebra



The graph of a permutation

4312

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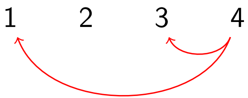

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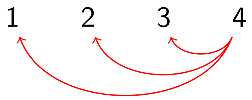

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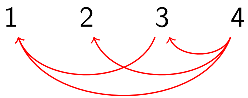

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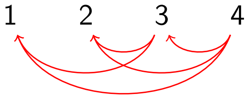

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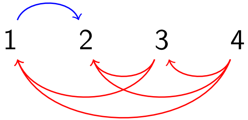

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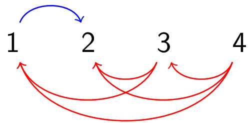

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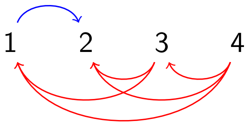

3412

The graph of a permutation

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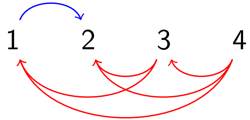
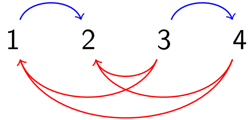


3412



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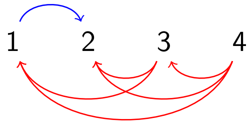

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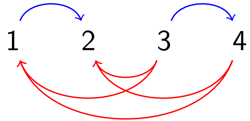
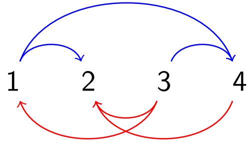


3142



Binary Relations on integers

Let R be a relation of size n.

$$1 \qquad 2 \qquad \cdots \qquad i \qquad \cdots \qquad j \qquad \cdots \qquad k \qquad \cdots \qquad n$$

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$i \,\mathrm{R}\, j$ $\qquad\qquad$ $k \,\mathrm{R}\, i$

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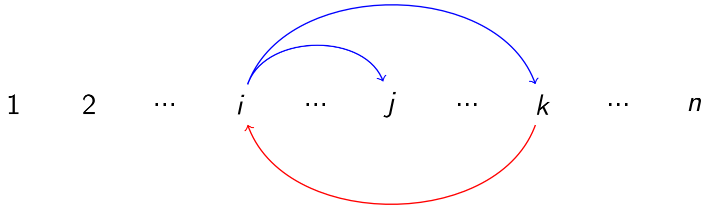

$i \mathrel{\mathrm{R}} j$ $\quad$ $k \mathrel{\mathrm{R}} i$ $\quad$ $i \mathrel{\mathrm{R}} k$

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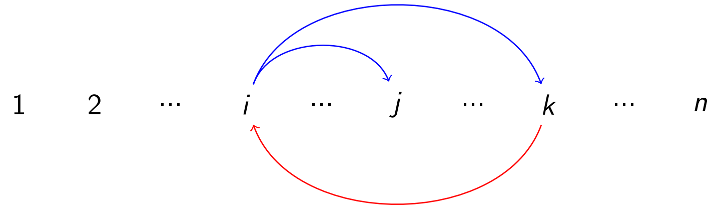

$i \mathrel{\mathrm{R}} j$ $\qquad$ $k \mathrel{\mathrm{R}} i$ $\qquad$ $i \mathrel{\mathrm{R}} k$

For size n: $2^{n(n-1)}$ possible binary relations.

Weak order on binary relations

Let R be a binary relation

$$\mathrm{R}^{\mathsf{Inc}} = \{i \,\mathrm{R}\, j, i < j\}$$
$$\mathrm{R}^{\mathsf{Dec}} = \{j \,\mathrm{R}\, i, i < j\}$$

Weak order on binary relations

Let R be a binary relation

$$\mathrm{R}^{\mathsf{Inc}} = \{i \,\mathrm{R}\, j, i < j\}$$
$$\mathrm{R}^{\mathsf{Dec}} = \{j \,\mathrm{R}\, i, i < j\}$$

Let R and S be two binary relations.

$$\mathrm{R} \preccurlyeq \mathrm{S} \Leftrightarrow \qquad \mathrm{R}^{\mathsf{Inc}} \supseteq \mathrm{S}^{\mathsf{Inc}} \text{ and } \mathrm{R}^{\mathsf{Dec}} \subseteq \mathrm{S}^{\mathsf{Dec}}$$

Binary relations of size 2

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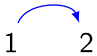
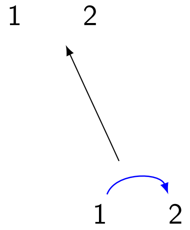
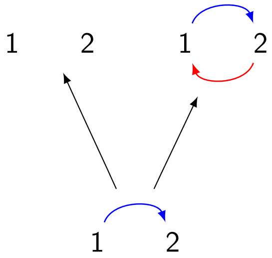


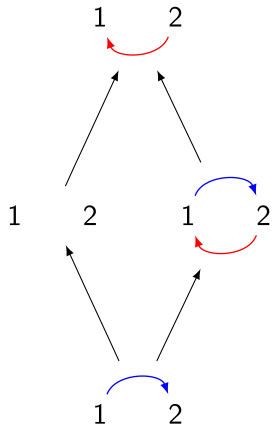
Binary relations of size 2

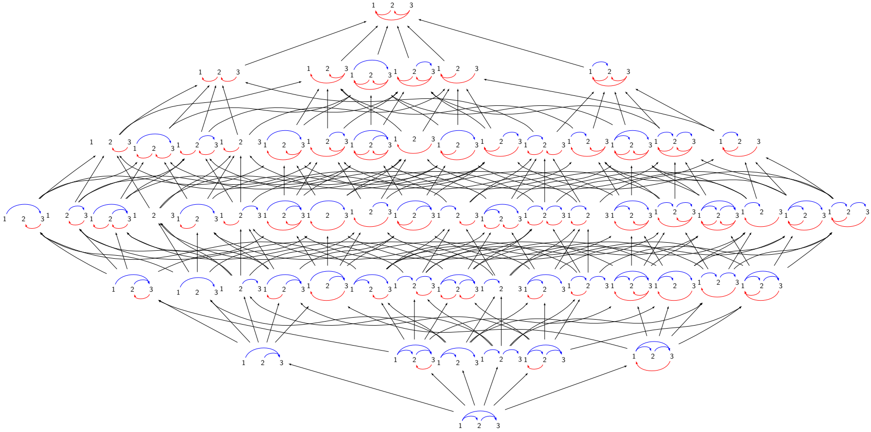


Binary relations of size 2

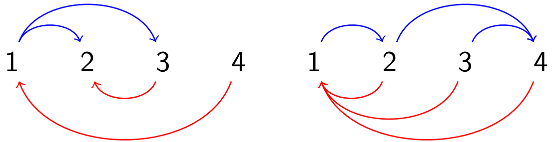


Binary relations of size 2





Meet and join



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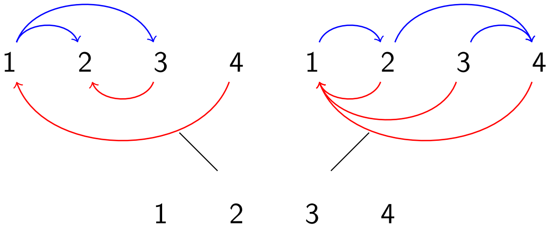

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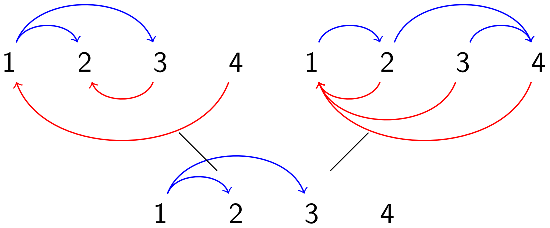

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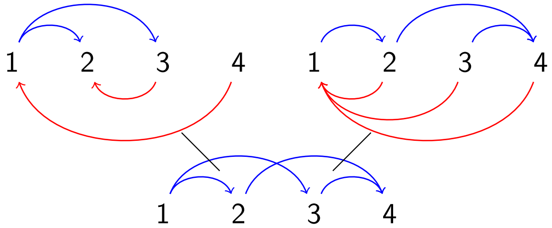

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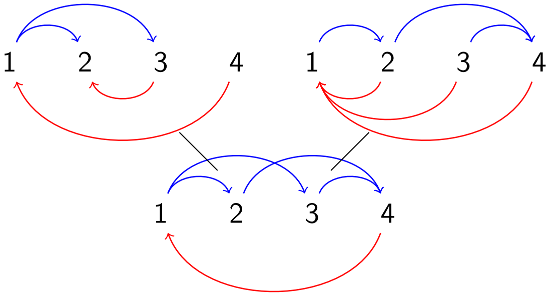

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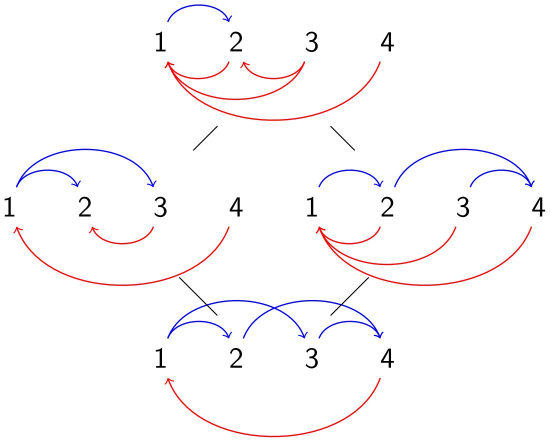
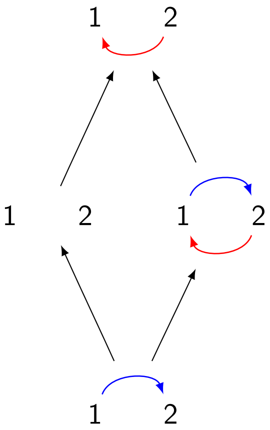


We want to keep binary relations which are both

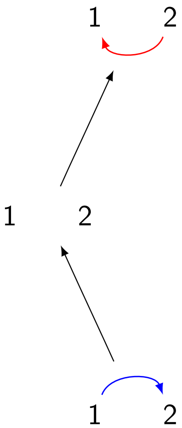
- antisymmetric
- transitive

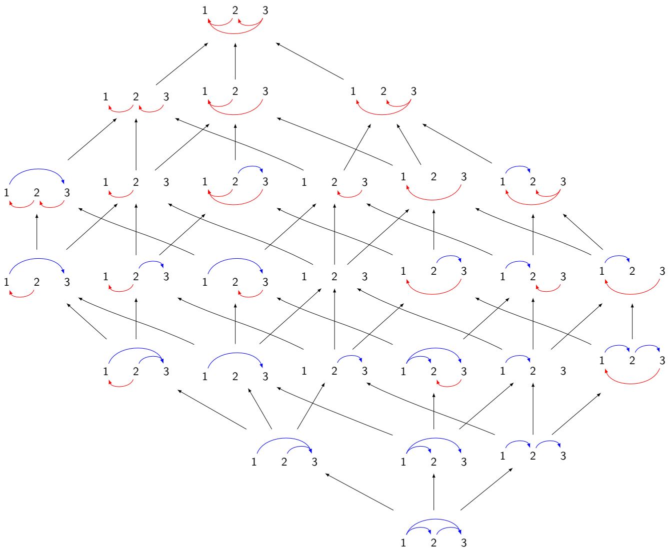
(posets)

Antisymmetry



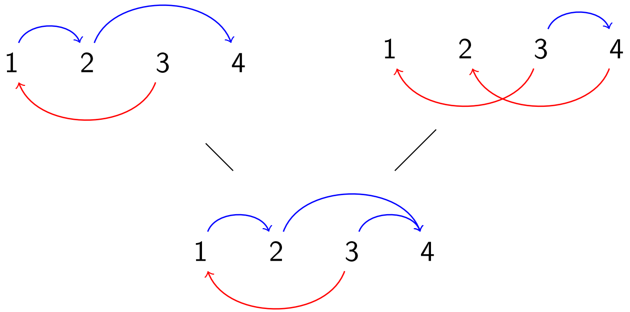
Antisymmetry





Sublattice ?

Let R and S be antisymmetric, is $\mathrm{R} \wedge \mathrm{S}$ also antisymmetric ?



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$$\cdots \quad i \quad \cdots \quad j \quad \cdots \qquad\qquad \cdots \quad i \quad \cdots \quad j \quad \cdots$$

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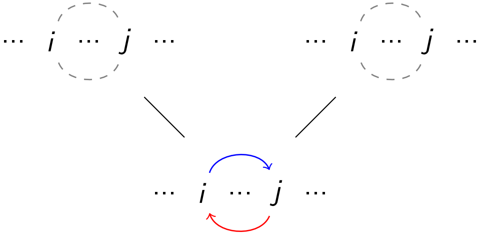

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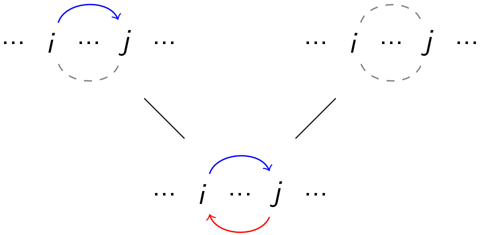

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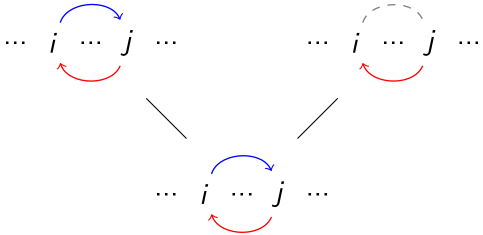
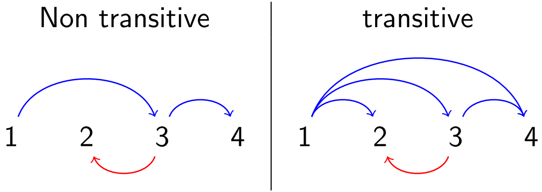
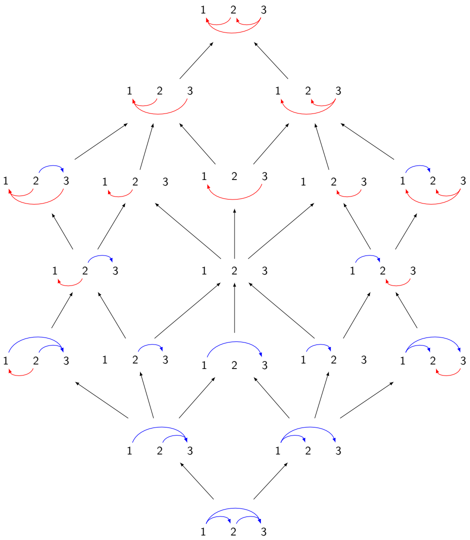


Transitivity

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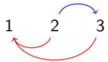
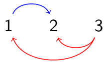
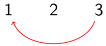
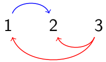
Sublattice ?

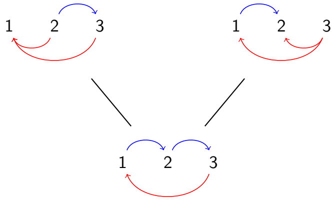


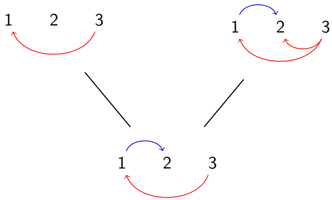




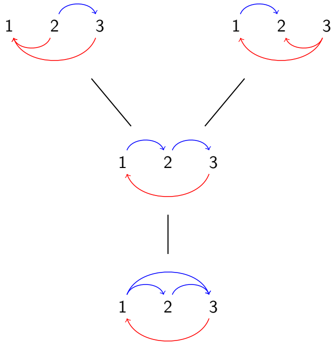


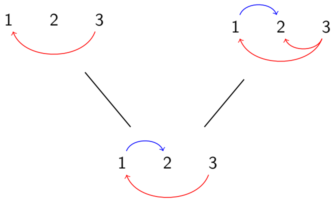
Sublattice ? No!



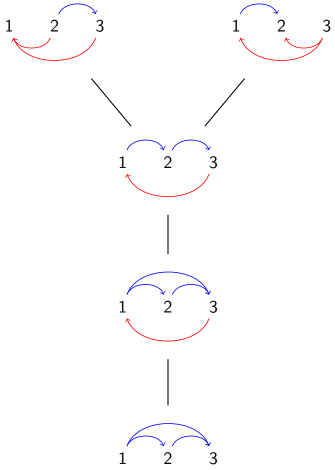


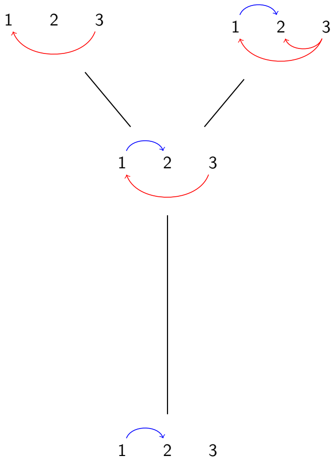
Sublattice ? No!





Sublattice ? No! But still a lattice





Back to permutations

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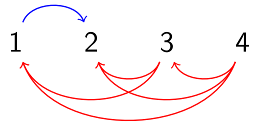
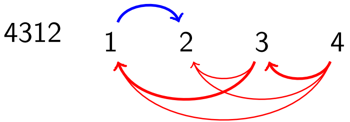


We have $i \mathrm{R} j$ iif the number i appears before j in the permutations. The relation is then

- antisymmetric
- transitive
- **and a total order**

Back to permutations



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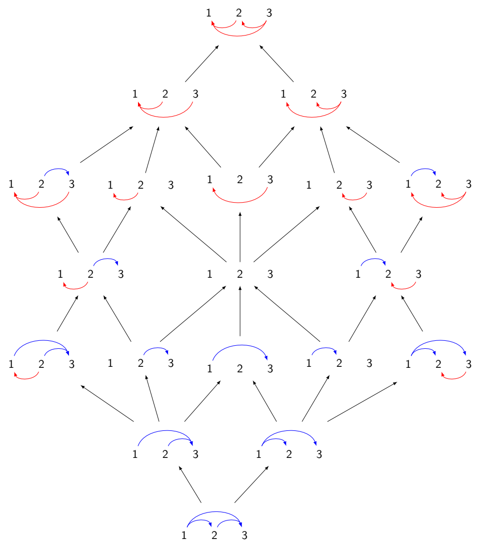

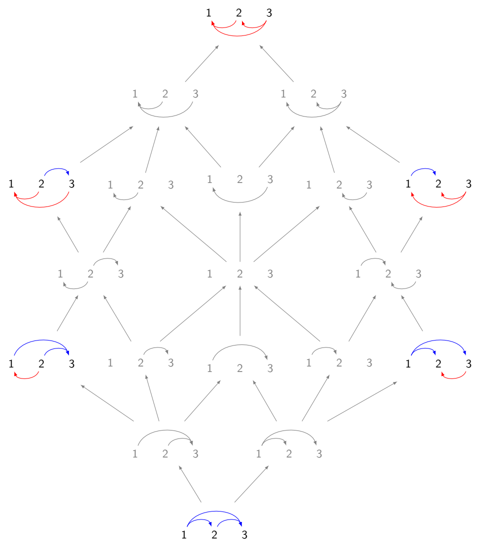

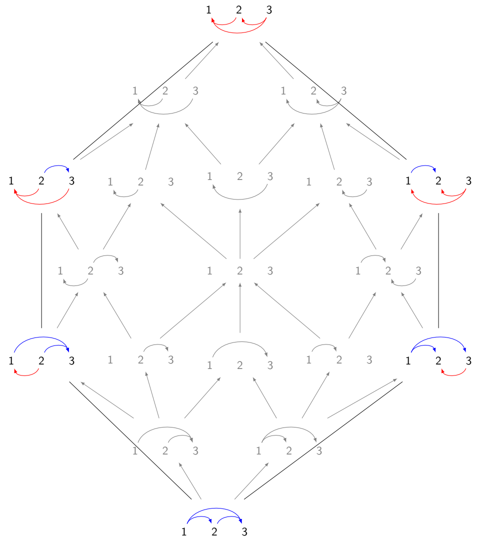
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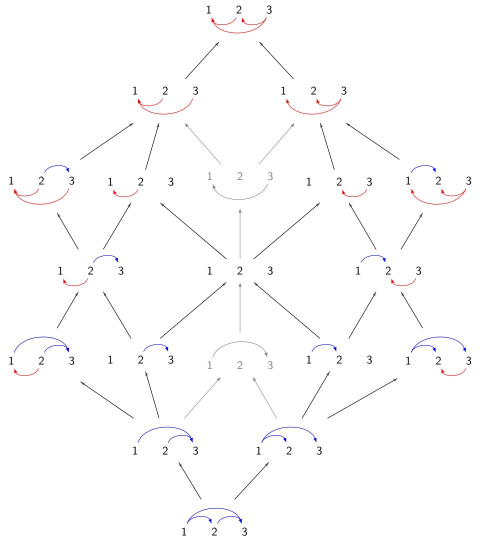

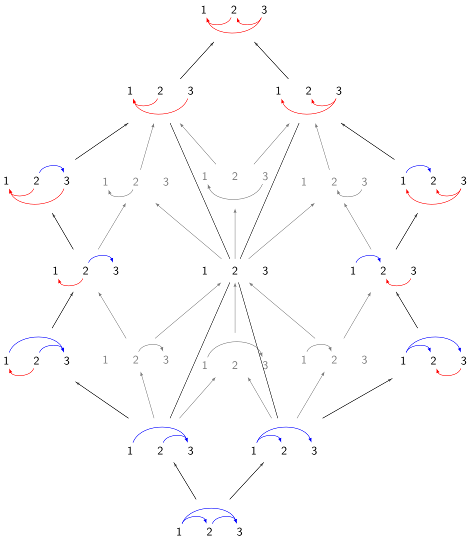

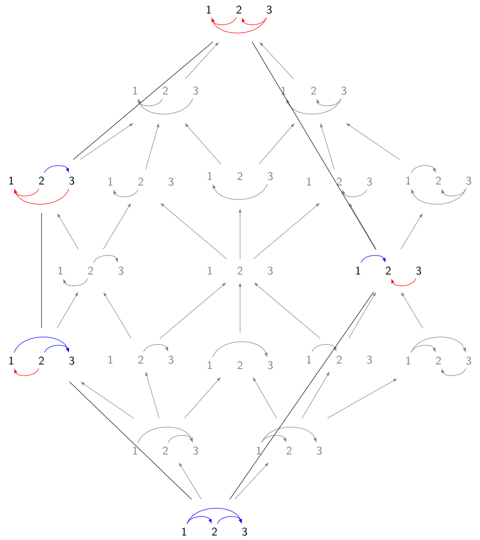
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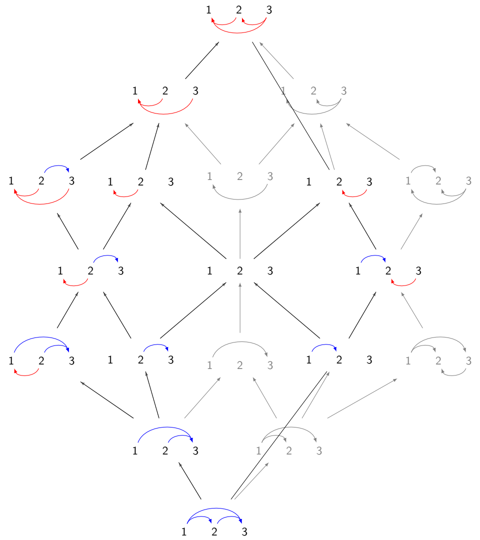
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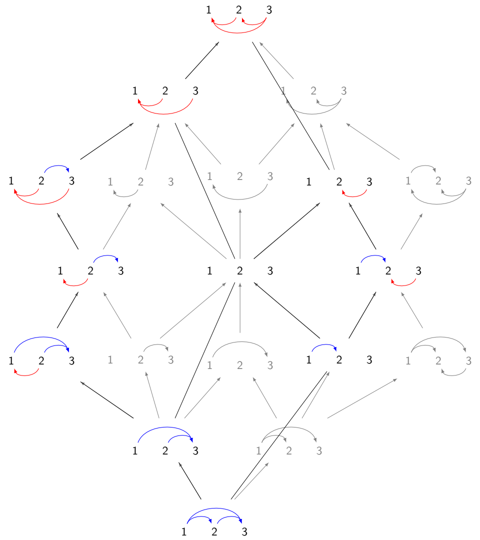

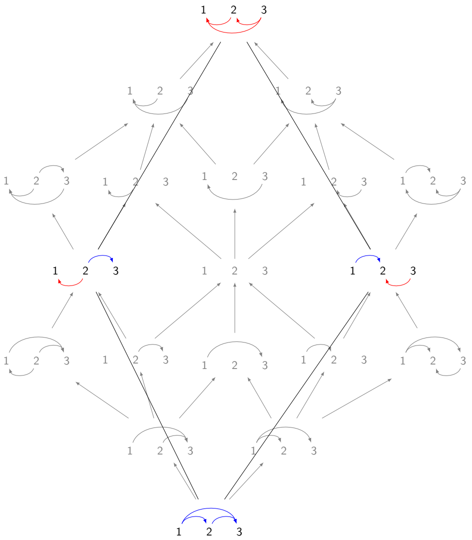

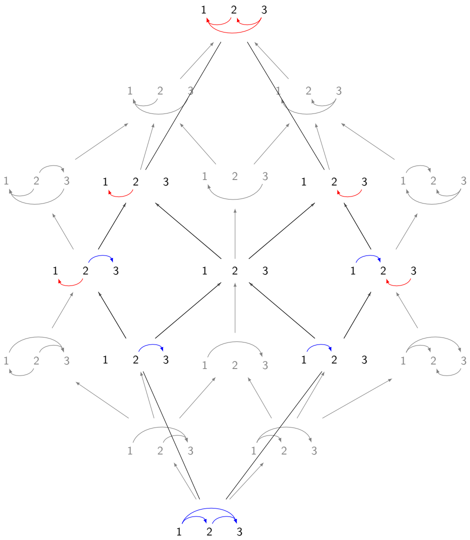
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References

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- Pilaud, P. *Permutrees* arXiv:1606.09643
- Pilaud, P. *The Hopf algebra of integer posets* (work in progress)