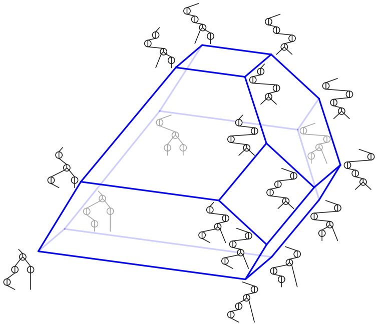
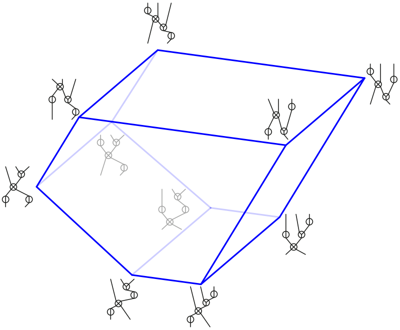
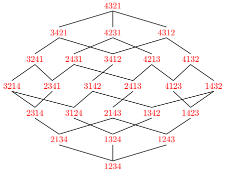
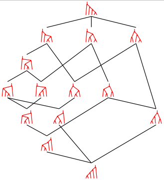
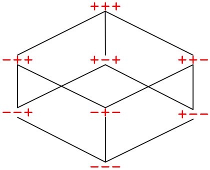
Permutrees

Vincent Pilaud – **Viviane Pons**
CNRS & Ecole Polytechnique – LRI, Univ. Paris-Sud





<table>
<tr><th></th><th>permutations</th><th>binary trees</th><th>binary sequences</th></tr>
<tr><td>Combinatorics</td><td>
</td><td></td><td>
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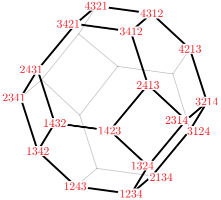
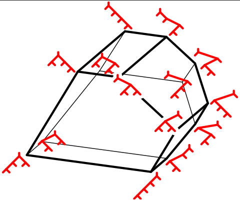
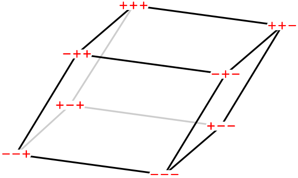
<tr><td>Geometry</td><td>
</td><td></td><td>
</td></tr>
<tr><td>Algebra</td><td>Malvenuto-Reutenauer algebra
$\mathrm{FQSym} = \mathrm{vect}\langle \mathbb{F}_\tau \mid \tau \in \mathfrak{S} \rangle$
$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\shuffle}\, \tau'} \mathbb{F}_\sigma$
$\triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$</td><td>Loday-Ronco algebra
$\mathrm{PBT} = \mathrm{vect}\langle \mathbb{P}_{\mathrm{T}} \mid \mathrm{T} \in \mathcal{BT} \rangle$
$\mathbb{P}_T \cdot \mathbb{P}_{T'} = \sum_{T^{\nearrow} T' \le T'' \le T_{\nwarrow} T'} \mathbb{P}_{T''}$
$\triangle \mathbb{F}_\gamma = \sum_{\gamma \text{ cut}} B(T,\gamma) \otimes A(T,\gamma)$</td><td>Solomon algebra
$\mathrm{Rec} = \mathrm{vect}\langle \mathbb{X}_\eta \mid \eta \in \pm^* \rangle$
$\mathbb{X}_\eta \cdot \mathbb{X}_{\eta'} = \mathbb{X}_{\eta+\eta'} + \mathbb{X}_{\eta-\eta'}$
$\triangle \mathbb{X}_\eta = \sum_{\gamma \text{ cut}} B(\eta,\gamma) \otimes A(\eta,\gamma)$</td></tr>
</table>

Some background

- Reading, *Cambiran Lattices* (2006).
- Chatel-Pilaud, *Cambiran Hopf algebra* (2014).
- **Pilaud-P., Permutrees (2016).**

The Permutree Recipe

- Take a word in {⦶, ☮, ⍦, ⊗}^{n}
- Take a permutation
- Do the insertion: get a Leveled Permutree (bijection)
- Remove the levels: get a Permutree (surjection)

The Permutree Recipe

- Take a word in $\{⦶, ☮, Ⓨ, \otimes\}^n$
- Take a permutation
- Do the insertion: get a <u>Leveled Permutree</u> (bijection)
- Remove the levels: get a <u>Permutree</u> (surjection)

Example

$⦶^n$	$\longleftrightarrow$	permutations of $[n]$
$☮^n$	$\longleftrightarrow$	standard binary search trees
$\{☮, Ⓨ\}^n$	$\longleftrightarrow$	Cambrian trees
$\otimes^n$	$\longleftrightarrow$	binary sequences

The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

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Decorated permutation: $\overline{2}\overline{7}5\underline{1}3\underline{\overline{4}}\underline{6}$

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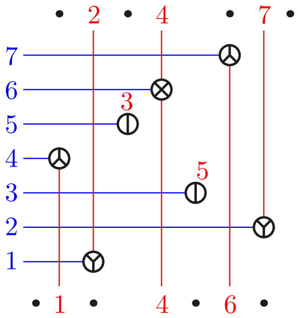

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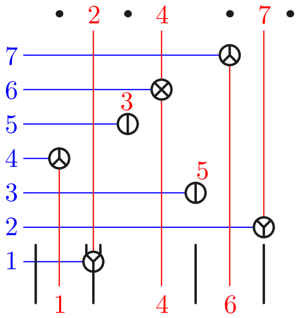

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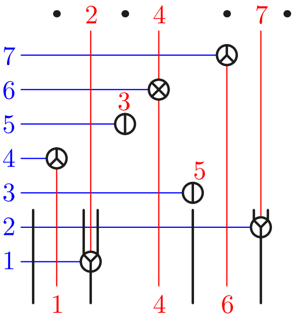

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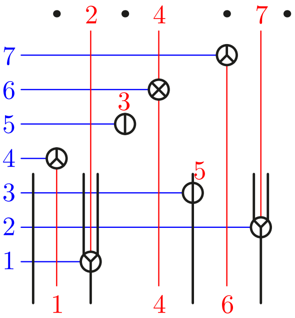

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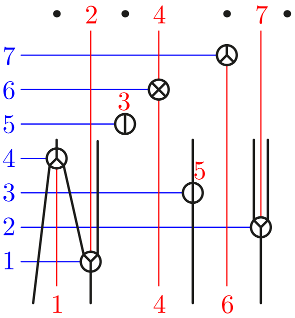

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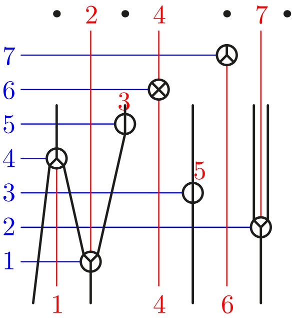

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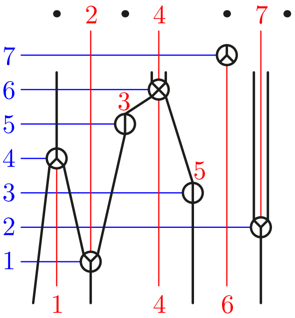

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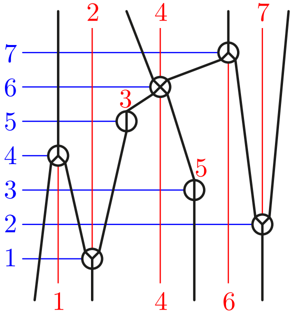

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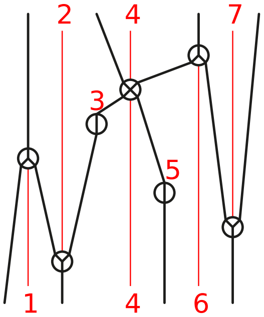

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⦶⦶⦶⦶⦶⦶⦶ ☮☮☮☮☮☮☮ ⊗⊗⊗⊗⊗⊗⊗ ☮Ⓨ⦶⊗⦶☮Ⓨ

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Leveled permutrees:

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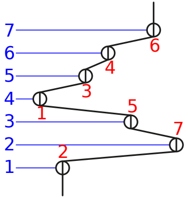

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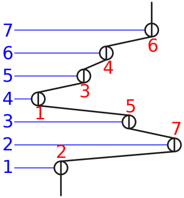

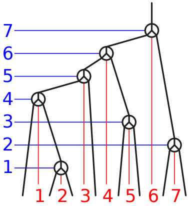

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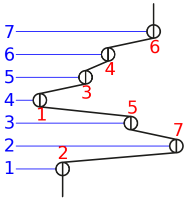

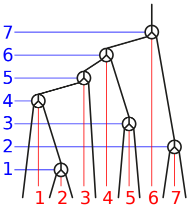

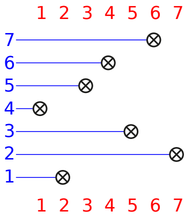

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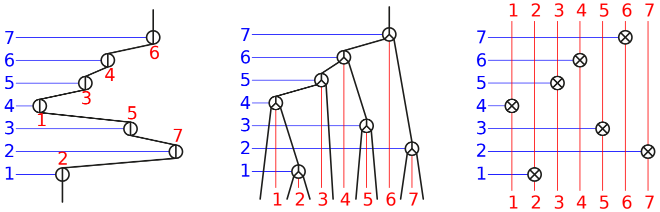

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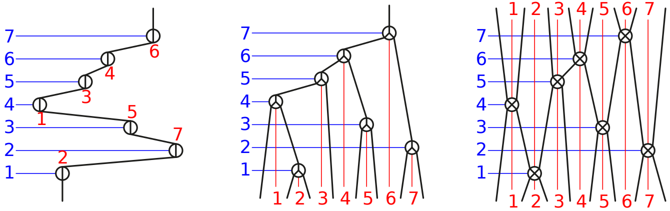

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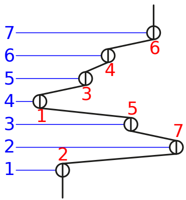
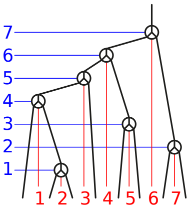
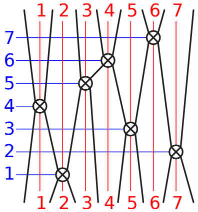
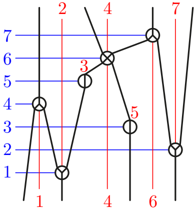

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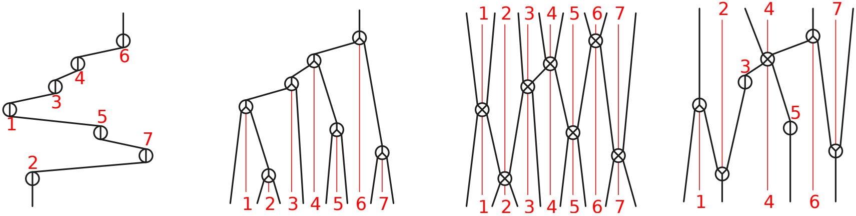
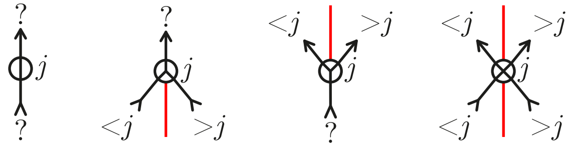
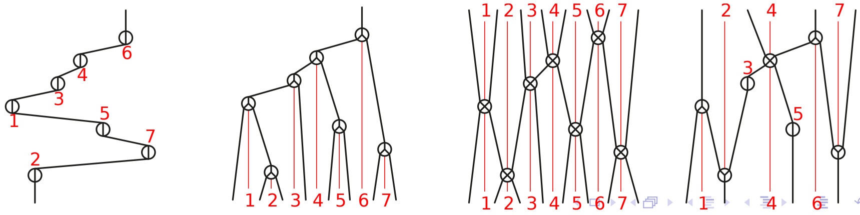


Definition of a permutree

directed (bottom to top) and labeled (bijectively by $[n]$) tree such that





Insertion

$$\mathfrak{S}_n \times \{⦶, ☮, ⏀, \otimes\}^n \longrightarrow \text{Permutrees}$$
$$(\sigma, \delta) \longrightarrow \mathbf{P}_\delta(\sigma)$$

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Congruence

$$\dots ac \dots \underline{b} \dots \equiv_\delta \dots ca \dots \underline{b} \dots \qquad (\delta_b \in \{☮, \otimes\})$$
$$\dots \overline{b} \dots ac \dots \equiv_\delta \dots \overline{b} \dots ca \dots \qquad (\delta_b \in \{Ⓨ, \otimes\})$$

Insertion

$$\mathfrak{S}_n \times \{⦶, ⦸, ⦹, \otimes\}^n \longrightarrow \text{Permutrees}$$
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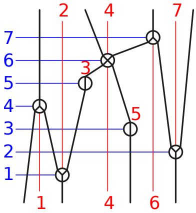
$$\dots \overline{b} \dots ac \dots \equiv_\delta \dots \overline{b} \dots ca \dots \qquad (\delta_b \in \{⦹, \otimes\})$$

Property

$$\sigma \equiv_\delta \tau \Leftrightarrow \mathbf{P}_\delta(\sigma) = \mathbf{P}_\delta(\tau)$$



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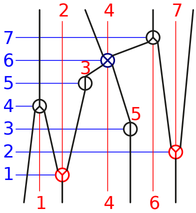

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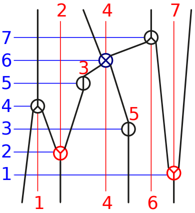
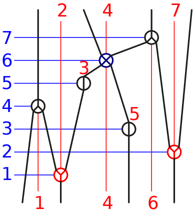


$\overline{7}\overline{2}5\underline{1}3\underline{\overline{4}}\underline{6} \equiv$



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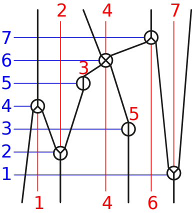

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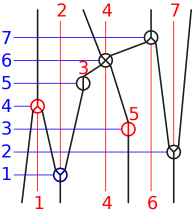

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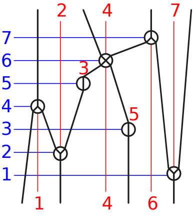

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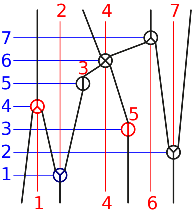
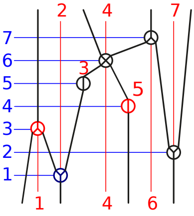


$\overline{2}75\underline{1}3\underline{\overline{4}6}$



$\equiv \overline{2}7\underline{1}53\underline{\overline{4}6}$

Numerology

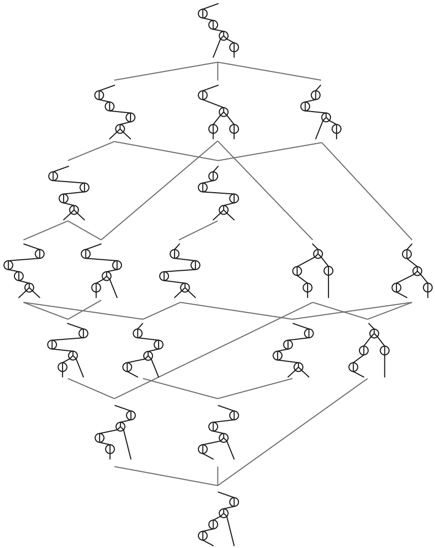
⦶⊗⊗⦶ 8					
⦶☮⊗⦶ 10	⦶Ⓨ⊗⦶ 10	⦶⊗☮⦶ 10	⦶⊗Ⓨ⦶ 10		
⦶⦶⊗⦶ 12	⦶☮☮⦶ 14	⦶☮Ⓨ⦶ 14	⦶Ⓨ☮⦶ 14	⦶ⓎⓎ⦶ 14	⦶⊗⦶⦶ 12
⦶⦶☮⦶ 18	⦶⦶Ⓨ⦶ 18	⦶☮⦶⦶ 18	⦶Ⓨ⦶⦶ 18		
⦶⦶⦶⦶ 24					

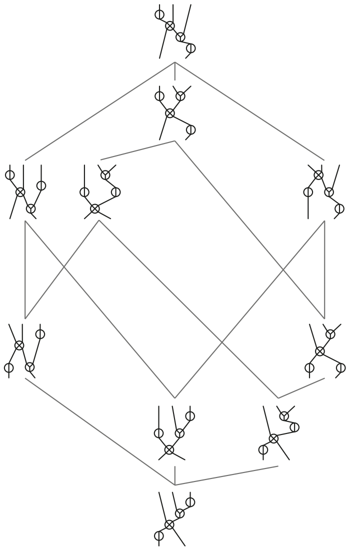
Factorial-Catalan numbers

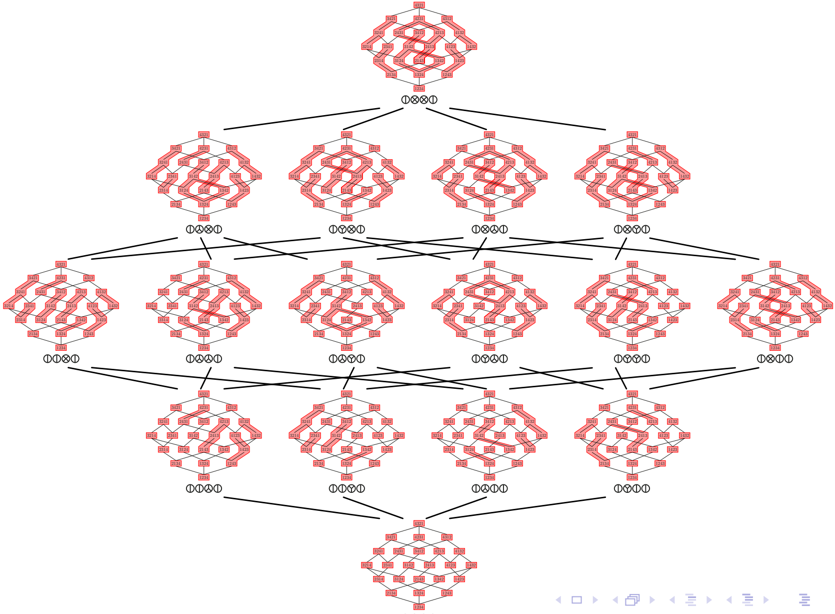
$\mathbf{C}(\delta) :=$ number of permutrees with decoration δ.

- δ_1 and δ_n do not affect the number of permutrees.
- If $\delta_i = \otimes$, $\mathbf{C}(\delta) = \mathbf{C}(\delta\,[1 \dots i]) \times \mathbf{C}(\delta\,[i \dots n])$.
- Any Ⓨ can be changed into a ☮ without changing $\mathbf{C}(\delta)$.
- If $\delta \in \{\text{⦶}, \text{☮}\}^n$, then $\mathbf{C}(\delta)$ is given by the recursive formula:

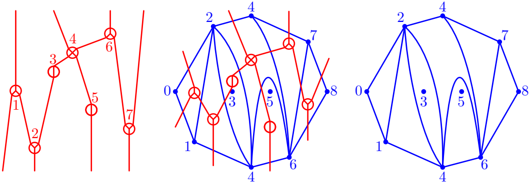
$$\mathbf{C}(\delta) = \sum_{i \in \delta^{-1}(\text{⦶})} \mathbf{C}(\delta \backslash i) + \sum_{i \in \delta^{-1}(\text{☮})} \mathbf{C}(\delta\,[1 \dots i-1])\mathbf{C}(\delta\,[i+1 \dots i])$$





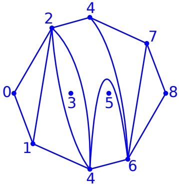


Tree-angulation

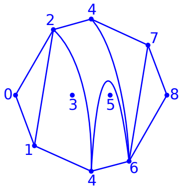


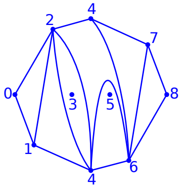
vertices above/below/inside $[0, 8]$	$\longleftrightarrow$	decoration
diangle enclosing j	$\longleftrightarrow$	node ⦶ labeled j
triangle $i < j < k$ with j below	$\longleftrightarrow$	node ☮ labeled j
triangle $i < j < k$ with j above	$\longleftrightarrow$	node Ⓨ labeled j
quadrangle $i < j^-, j^+ < k$	$\longleftrightarrow$	node $\otimes$ labeled j

Tree-angulation flip

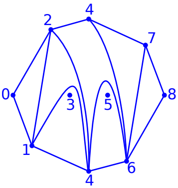


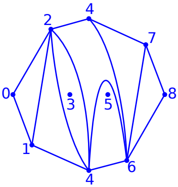
Tree-angulation flip



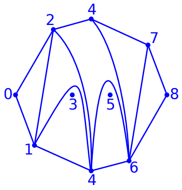


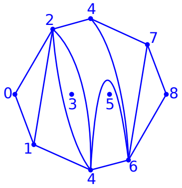
Tree-angulation flip

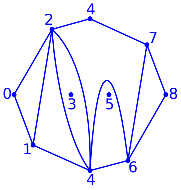




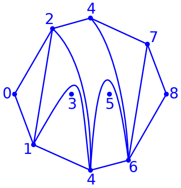
Tree-angulation flip

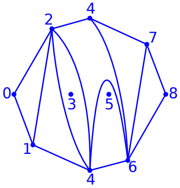


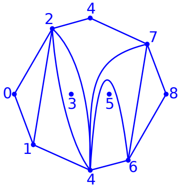




Tree-angulation flip



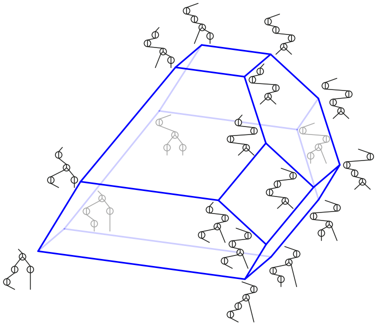




The Permutreehedron

Thm (Pilaud, P.) for every $\delta \in \{⦶, ☮, Ⓨ, \otimes\}^n$, there is an explicit construction of a

- a complete simplicial fan, the δ-permutree fan $\mathcal{F}(\delta)$;
- a polytope, the Permutreehedron $\mathbb{PT}(\delta)$, whose normal fan is $\mathcal{F}(\delta)$.



The Permutreehedron

Thm (Pilaud, P.) for every $\delta \in \{⦶, ☮, \text{⦽}, \otimes\}^n$, there is an explicit construction of a

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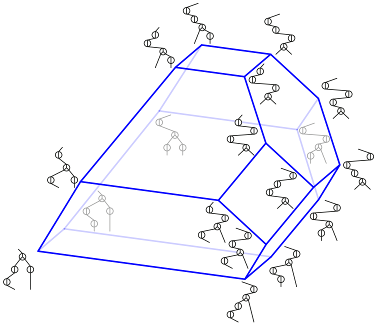

The Permutreehedron can be constructed by convex hull or hyperplane intersection.

The Permutreehedron

Thm (Pilaud, P.) for every $\delta \in \{$⦶, ☮, Ⓨ, ⊗$\}^n$, there is an explicit construction of a

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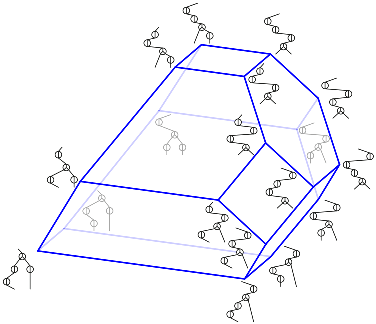
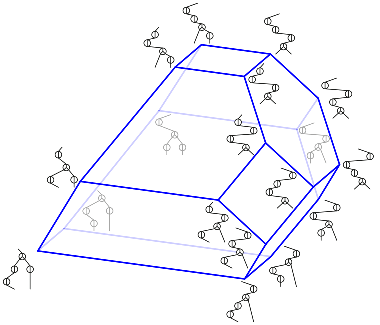


The vertices of $\mathbb{PT}(\delta)$ are the δ-permutrees.

The Permutreehedron

Thm (Pilaud, P.) for every $\delta \in \{⦶, ☮, \rotatebox[origin=c]{180}{☮}, \otimes\}^n$, there is an explicit construction of a

- a complete simplicial fan, the <u>δ-permutree fan</u> $\mathcal{F}(\delta)$;
- a polytope, the <u>Permutreehedron</u> $\mathbb{PT}(\delta)$, whose normal fan is $\mathcal{F}(\delta)$.



The oriented graph of $\mathbb{PT}(\delta)$ is the Hasse diagram of the δ-permutree lattice.

The Permutreehedron

Thm (Pilaud, P.) for every $\delta \in \{⦶, ☮, ⊗\}^n$, there is an explicit construction of a

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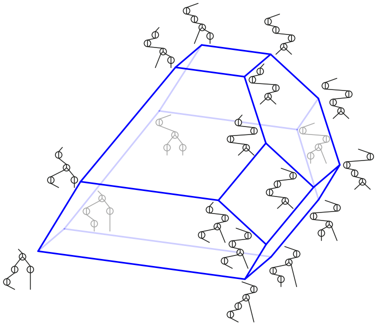

Combinatorics of the faces: Schröder permutrees.

The Permutreehedron

Thm (Pilaud, P.) for every $\delta \in \{⦶, ☮, \oplus, \otimes\}^n$, there is an explicit construction of a

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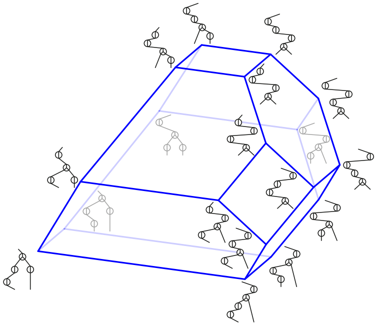
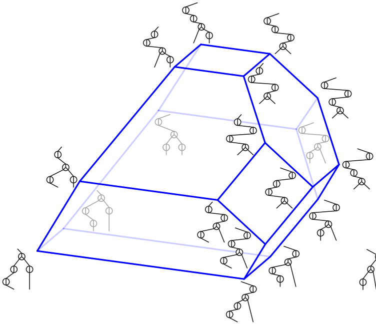
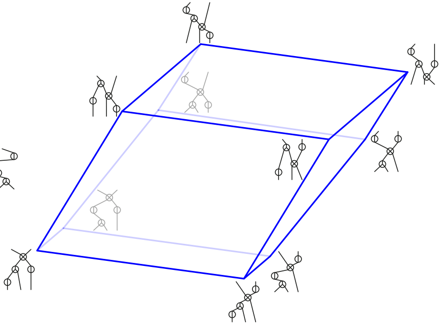


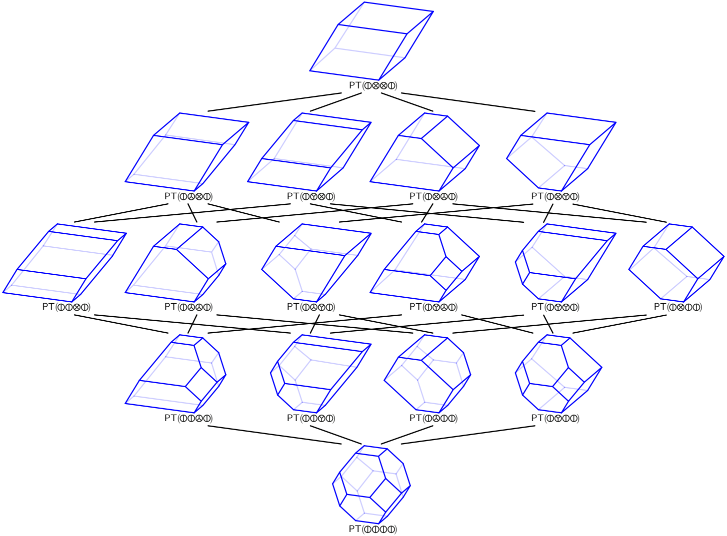
Show the 3D-surprise!

Matriochka Permutreehedra

refinement $\delta \preccurlyeq \delta'$ $\Longrightarrow$ inclusion $\mathbb{PT}(\delta) \subset \mathbb{PT}(\delta')$



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