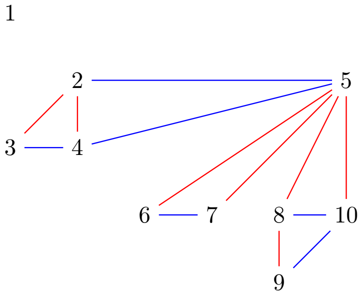
The Rise-Contact involution on Tamari intervals

Viviane Pons
Paris-Saclay University



Motivations: Tamari intervals (they're cool)

2005 **Chapoton** proves a nice formula counting intervals of the Tamari lattice (also counting triangular maps!)

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$$

F. Chapoton. Sur le nombre d'intervalles dans les treillis de Tamari. *Sém. Lothar. Combin.*, 2005.

Motivations: Tamari intervals (they're cool)

2010 **Bergeron** and **Préville-Ratelle** give the definition of m-Tamari lattices and conjecture that the number of intervals is counted by

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n + m}{n-1}$$

F. Bergeron and L.-F. Préville-Ratelle. Higher trivariate diagonal harmonics via generalized Tamari posets. *J. Comb.*, 2012.

Motivations: Tamari intervals (they're cool)

2011 **Bousquet-Mélou, Fusy, and Préville-Ratelle** prove the formula for m-Tamari intervals.

The generating function shows a symmetry between the *the number of contacts* and the *initial rises* but they have no combinatorial explanation.

M. Bousquet-Mélou, E. Fusy, and L.-F. Préville-Ratelle. The number of intervals in the m-Tamari lattices. *Electron. J. Combin.*, 2011.

Motivations: Tamari intervals (they're cool)

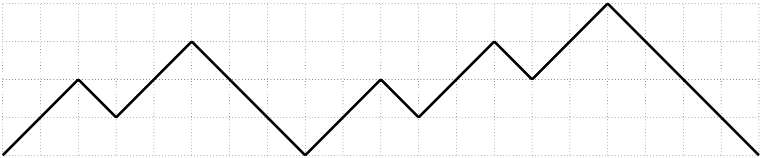
2012 **Préville-Ratelle** leaves a conjecture at this end of his thesis about the symmetry of two series of statistics *"contacts"* and *"rises"*

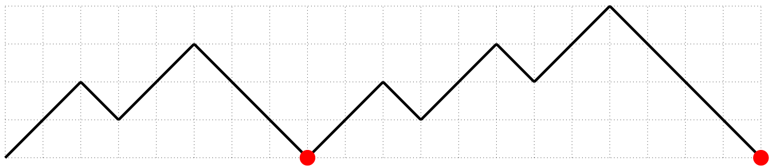
L.-F. Préville-Ratelle. *Combinatoire des espaces coinvariants trivariés du groupe symétrique.* Thèse de Doctorat, Université du Québec à Montréal, 2012.

Motivations: Tamari intervals (they're cool)

Today, I give you the involution that proves this symmetry.

V. Pons. The Rise-Contact involution on Tamari intervals. *The Electronic Journal of Combinatorics*, 2019





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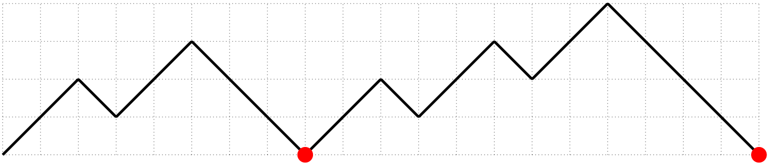
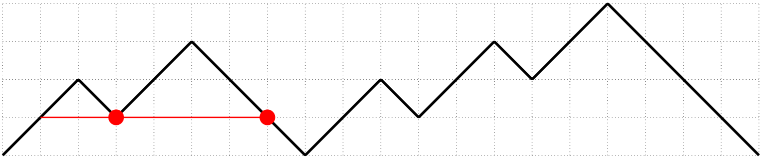


Contacts : 2

Contact vector: 2,



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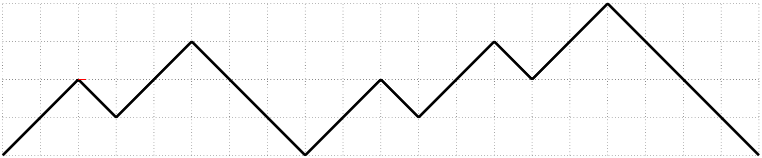
Contact vector: 2, 2,



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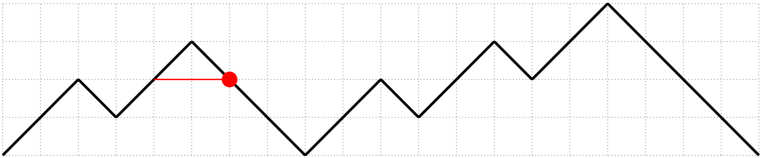

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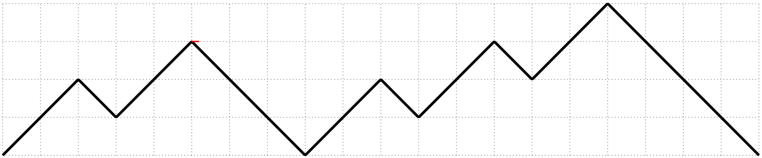

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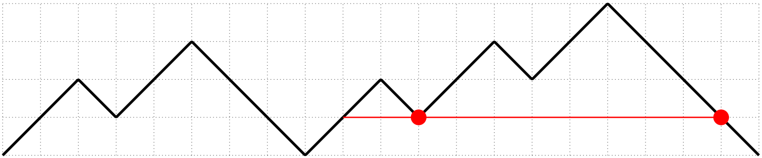

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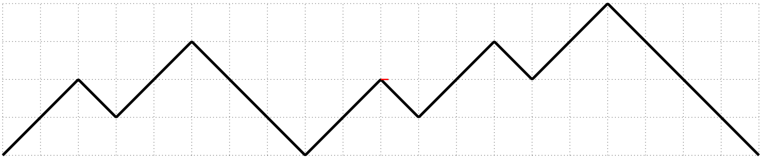

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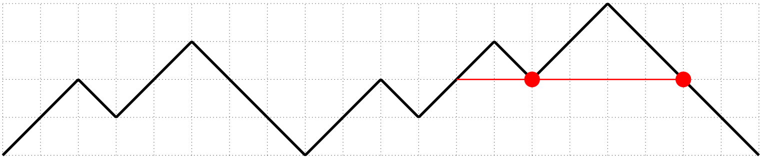

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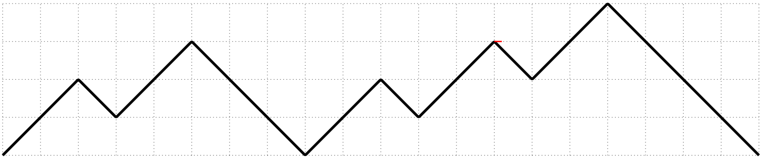

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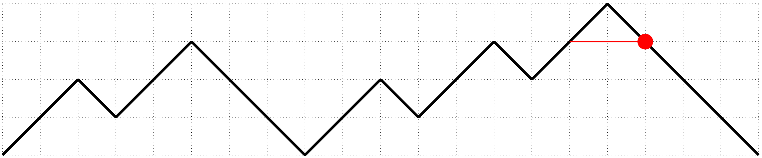

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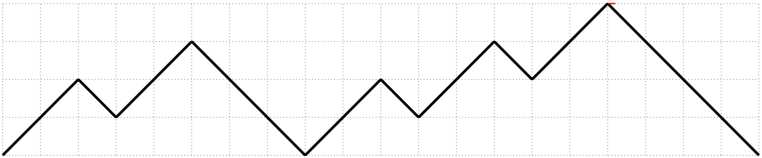

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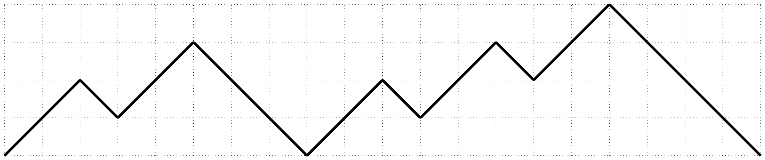

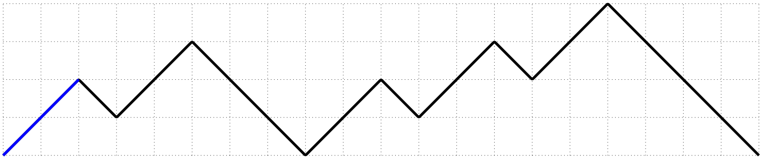

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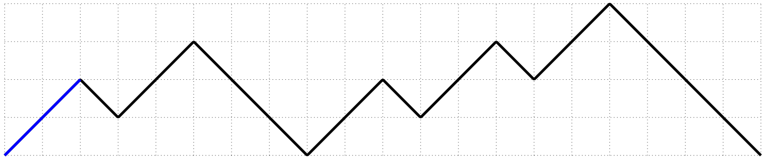
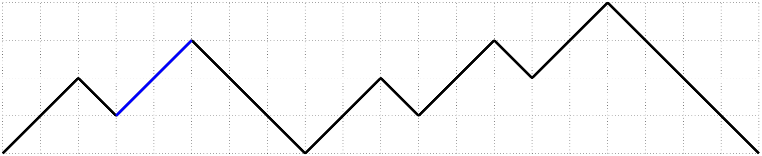


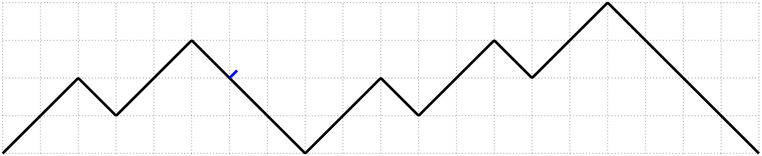
Rises : 2

Rises vector: 2,



Rises : 2

Rises vector: 2, 2,



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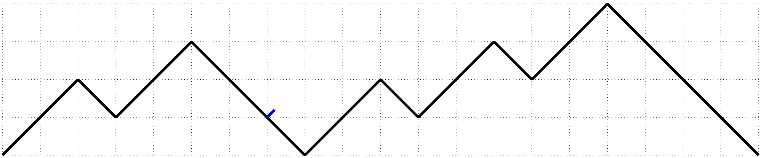
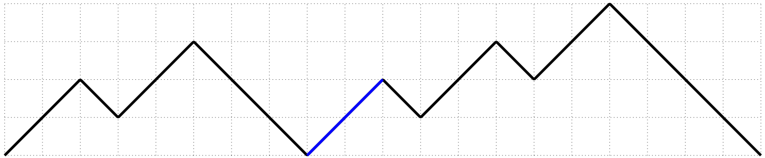
Rises vector: 2, 2, 0,



Rises : 2

Rises vector: 2, 2, 0, 0,



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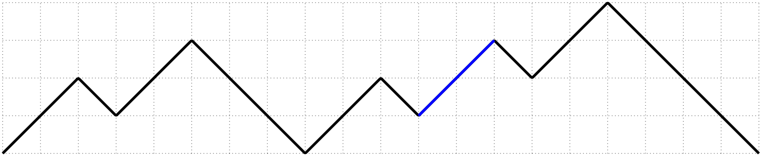

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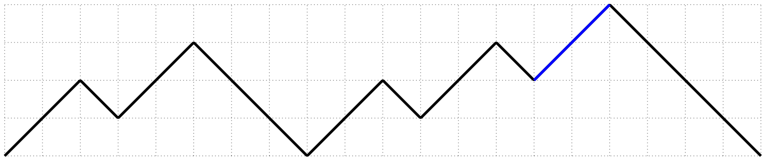

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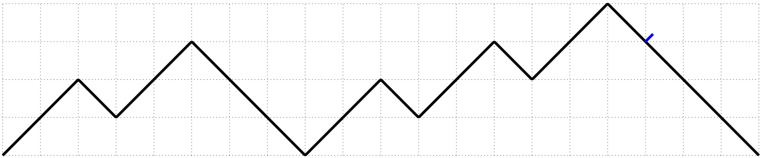

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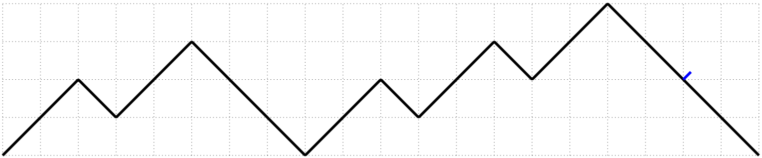

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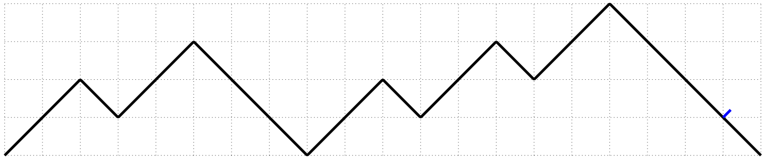

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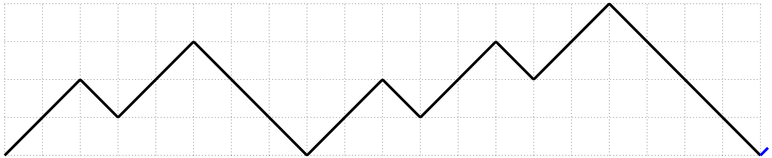
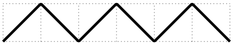
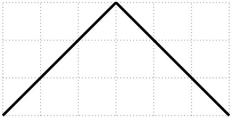


Rises : 2

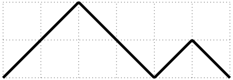
Rises vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0, 0



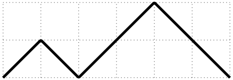
3,0,0 1,1,1



1,1,1 3,0,0



2,1,0 2,0,1



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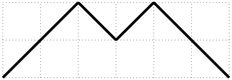

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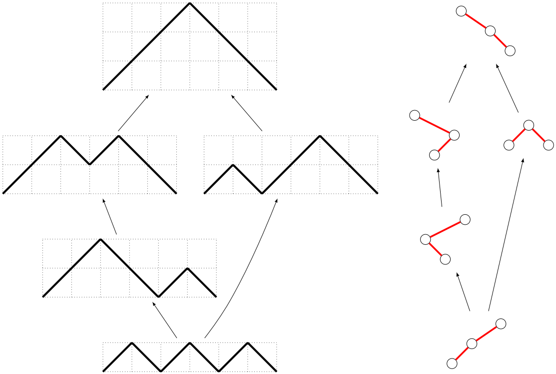
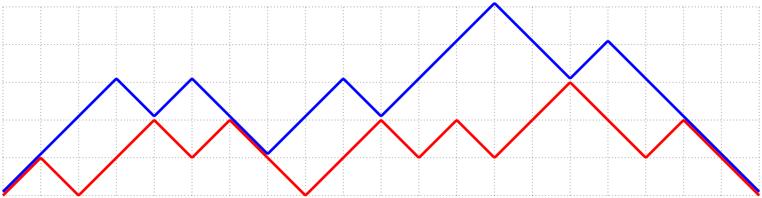
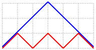




Contact vector: 3, 0, 2, 0, 0, 4, 0, 0, 1, 0

Rise vector: 3, 1, 0, 2, 3, 0, 1, 0, 0, 0



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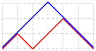

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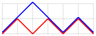

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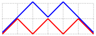

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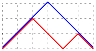

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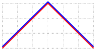

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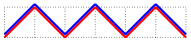

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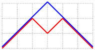

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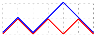

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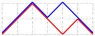

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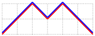

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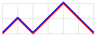

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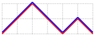
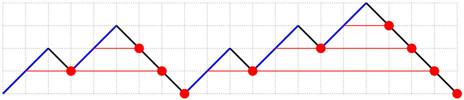
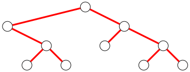


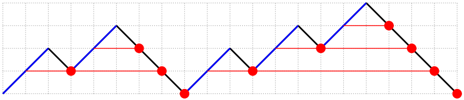
2, 1, 0 2, 0, 1

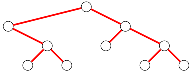




Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0, 1

Rise vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

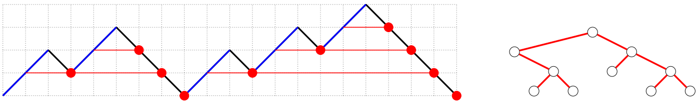




Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0, 1

Rise vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

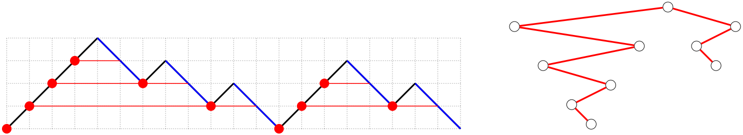
$\downarrow \Phi$



Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0, 1

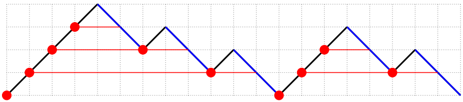
Rise vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

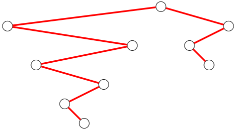
$\downarrow \Phi$



Contact vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0

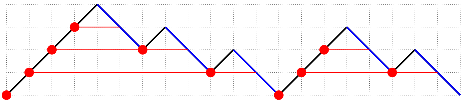
rotcev esiR: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

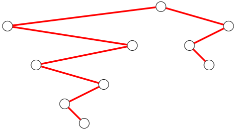




Contact vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0

rotcev esiR: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

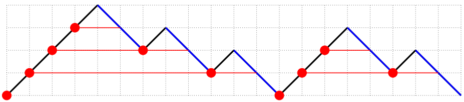


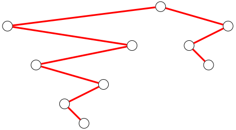


Contact vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0

rotcev esiR: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

$\downarrow \Psi$

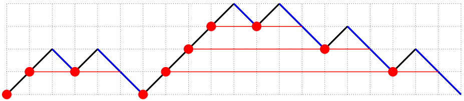


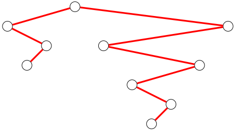


Contact vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0

rotcev esiR: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

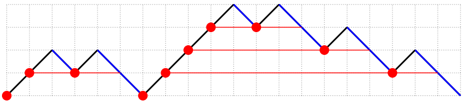
$\downarrow \Psi$

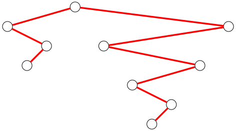




Contact vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

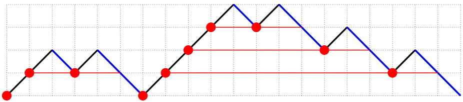
rotcev esiR: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0

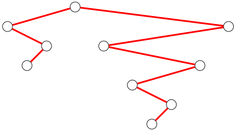




Contact vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

rotcev esiR: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0

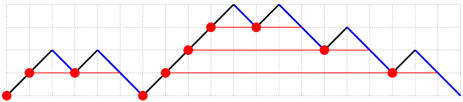


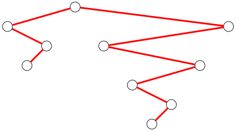


Contact vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

rotcev esiR: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0

$\downarrow \Phi$

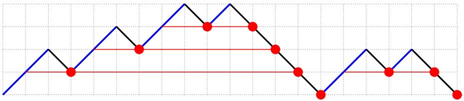


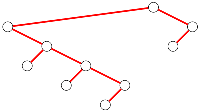


Contact vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

rotcev esiR: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0

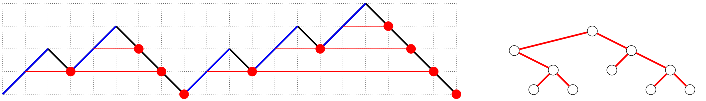
$\downarrow \Phi$





Contact vector: 2, 2, 0, 2, 0, 2, 0, 0, 2, 0

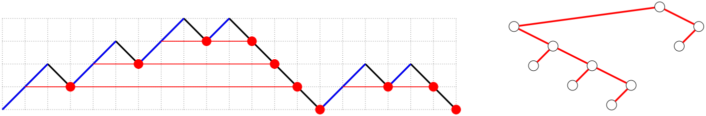
Rise vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0



Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0, 1 – $\underline{2^4, 1^2, 0^4}$

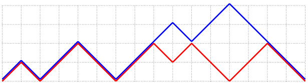
Rise vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0 – $\underline{2^5, 0^5}$

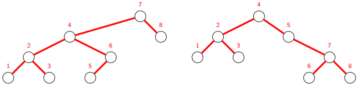
$$\downarrow \Phi \circ \Psi \circ \Phi$$

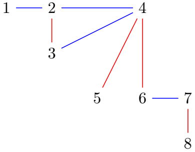


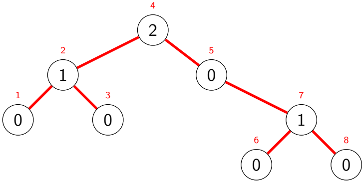
Contact vector: 2, 2, 0, 2, 0, 2, 0, 0, 2, 0 – $\underline{2^5, 0^5}$

Rise vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0 – $\underline{2^4, 1^2, 0^4}$



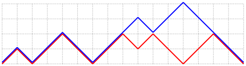


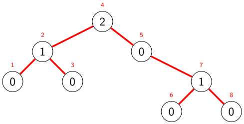


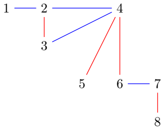


Contact vector: 4, 0, 1, 0, 2, 0, 0, 1

Rise vector: 1, 2, 0, 3, 2, 0, 0, 0

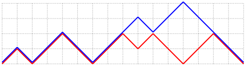


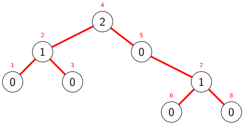


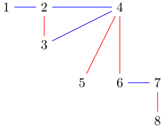


Contact vector: 4, 0, 1, 0, 2, 0, 0, 1

Rise vector: 1, 2, 0, 3, 2, 0, 0, 0



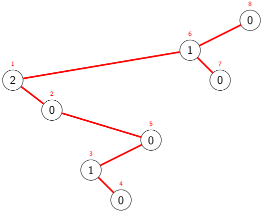


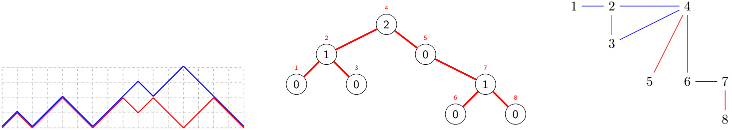


Contact vector: 4, 0, 1, 0, 2, 0, 0, 1

Rise vector: 1, 2, 0, 3, 2, 0, 0, 0

$\downarrow \Phi$

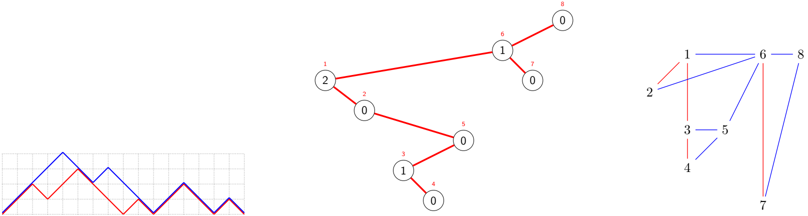




Contact vector: 4, 0, 1, 0, 2, 0, 0, 1

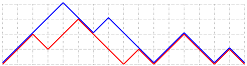
Rise vector: 1, 2, 0, 3, 2, 0, 0, 0

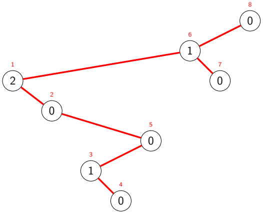
$\downarrow \Phi$

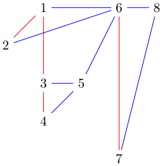


Contact vector: 4, 2, 0, 1, 0, 0, 1, 0

rotcev esiR: 1, 2, 0, 3, 2, 0, 0, 0

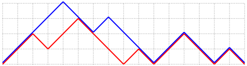


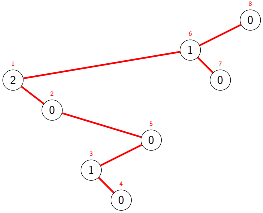


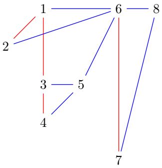


Contact vector: 4, 2, 0, 1, 0, 0, 1, 0

rotcev esiR: 1, 2, 0, 3, 2, 0, 0, 0



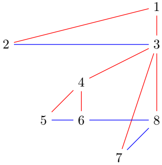


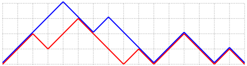
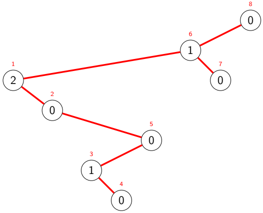
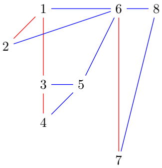


Contact vector: 4, 2, 0, 1, 0, 0, 1, 0

rotcev esiR: 1, 2, 0, 3, 2, 0, 0, 0

$\downarrow \Psi$

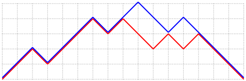
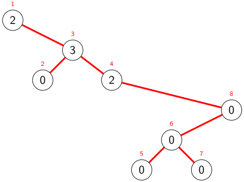
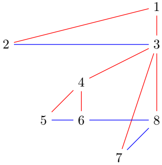


Contact vector: 4, 2, 0, 1, 0, 0, 1, 0

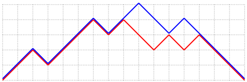
rotcev esiR: 1, 2, 0, 3, 2, 0, 0, 0

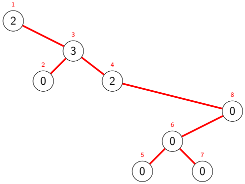
$\downarrow \Psi$

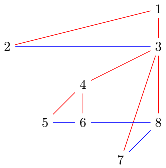




Contact vector: 1, 2, 0, 3, 2, 0, 0, 0

rotcev esiR: 4, 2, 0, 1, 0, 0, 1, 0

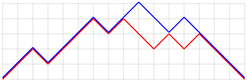


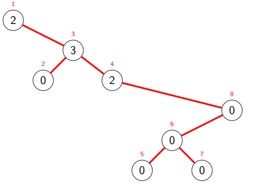


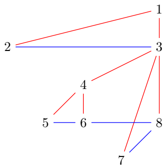


Contact vector: 1, 2, 0, 3, 2, 0, 0, 0

rotcev esiR: 4, 2, 0, 1, 0, 0, 1, 0



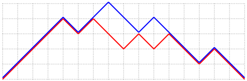


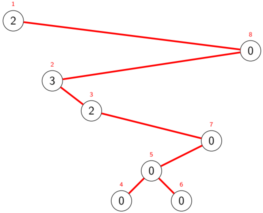


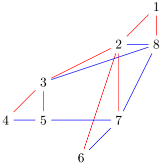
Contact vector: 1, 2, 0, 3, 2, 0, 0, 0

rotcev esiR: 4, 2, 0, 1, 0, 0, 1, 0

$\downarrow \Phi$

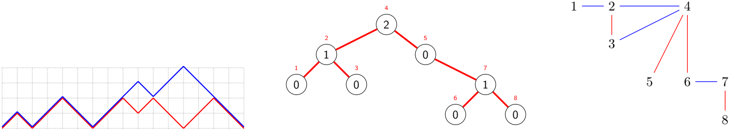






Contact vector: 1, 2, 3, 2, 0, 0, 0, 0

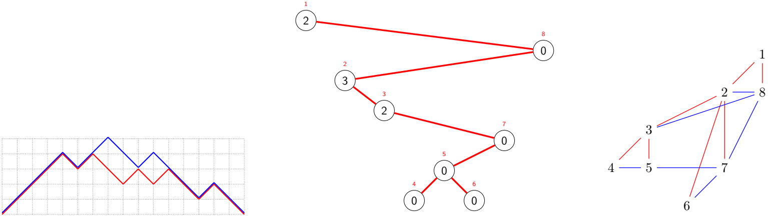
Rise vector: 4, 2, 0, 1, 0, 0, 1, 0



Contact vector: 4, 0, 1, 0, 2, 0, 0, 1 – $\underline{4, 2, 1^2, 0^4}$

Rise vector: 1, 2, 0, 3, 2, 0, 0, 0 – $\underline{3, 2^2, 1, 0^4}$

$$\downarrow \Phi \circ \Psi \circ \Phi$$

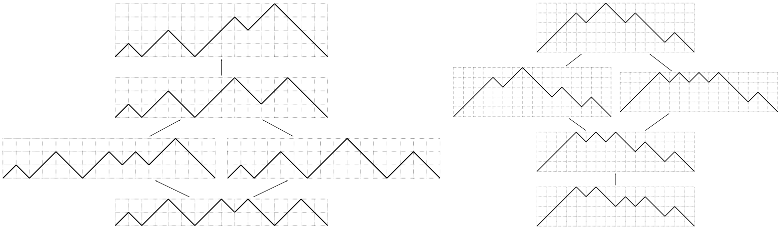


Contact vector: 1, 2, 3, 2, 0, 0, 0, 0 – $\underline{3, 2^2, 1, 0^4}$

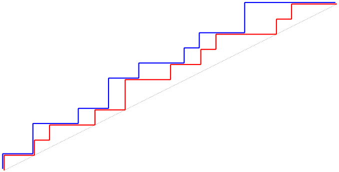
Rise vector: 4, 2, 0, 1, 0, 0, 1, 0 – $\underline{4, 2, 1^2, 0^4}$

- A combinatorial bijection which switches the contacts and the rises
- Can be extended to the m-Tamari case
- Preserve the “distance” of the interval
- **The paper** : The Rise-Contact involution on Tamari intervals. *The Electronic Journal of Combinatorics*, 26(2):P2.32, 2019. arXiv:1701.07995
- **SageMath Demo** : github.com/VivianePons/public-notebooks/

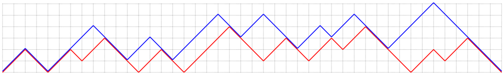
“distance” preserving

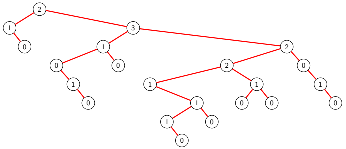


m-Tamari

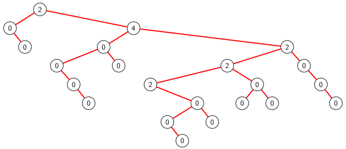


m-Contact vector: 5, 0, 0, 1, 0, 0, 0, 0, 0, 2, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0
m-Rise vector: 1, 0, 2, 0, 0, 1, 0, 2, 0, 1, 0, 0, 1, 1, 0, 0, 2, 0, 0, 0, 0, 0

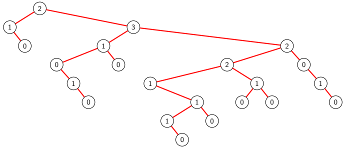




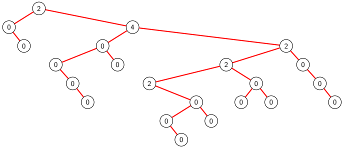
expand $\longrightarrow$



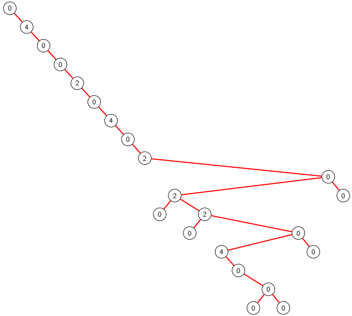
$\downarrow \Phi \circ \Psi \circ \Phi$



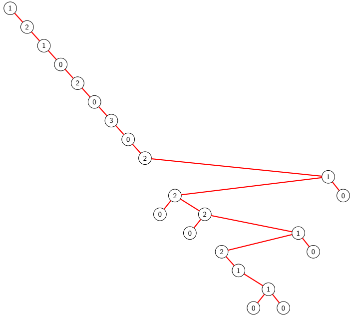
expand $\longrightarrow$

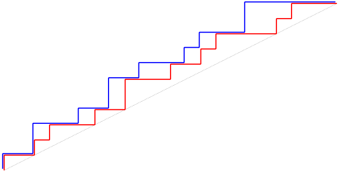


$\downarrow \Phi \circ \Psi \circ \Phi$



contract $\longrightarrow$

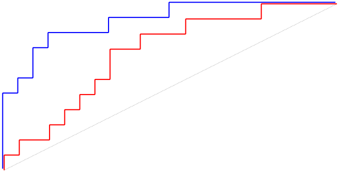




m-Contact vector: 5, 0, 0, 1, 0, 0, 0, 0, 0, 2, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0 – $5, 2, 1^4, 0^{16}$

m-Rise vector: 1, 0, 2, 0, 0, 1, 0, 2, 0, 1, 0, 0, 1, 1, 0, 0, 2, 0, 0, 0, 0, 0 – $2^3, 1^5, 0^{14}$

expand $\longrightarrow \Phi \circ \Psi \circ \Phi \longrightarrow$ contract



m-Contact vector: 1, 2, 0, 0, 0, 1, 0, 2, 0, 1, 0, 1, 0, 1, 2, 0, 0, 0, 0, 0, 0, 0 – $2^3, 1^5, 0^{14}$

m-Rise vector: 5, 1, 2, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 – $5, 2, 1^4, 0^{16}$