Viviane Pons

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Permutahedron and Associahedron

Combinatorics and geometry

Permutations

size 2: 12 21

size 3: 123 132 213 231 312 321

size 4: 1234 1243 1324 1342 1423 1432 2134 2143 2314 2341 2413 2431 3124 3142 3214 3241 3412 3421 4123 4132 4213 4231 4312 4321

Permutations

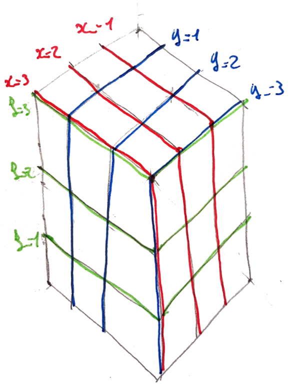
size 2: 12 21

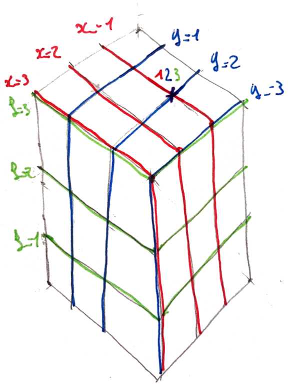
size 3: 123 132 213 231 312 321

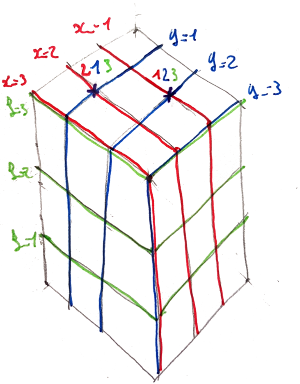
size 4: 1234 1243 1324 1342 1423 1432 2134 2143 2314 2341 2413 2431 3124 3142 3214 3241 3412 3421 4123 4132 4213 4231 4312 4321

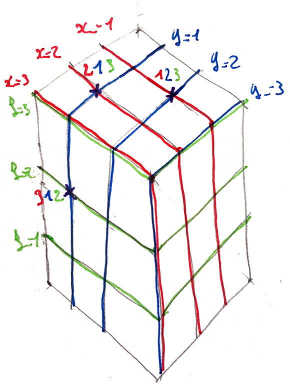
As vectors of $\mathbb{R}^n$

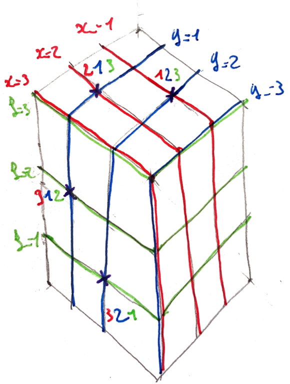
$123 \rightarrow (1, 2, 3)$

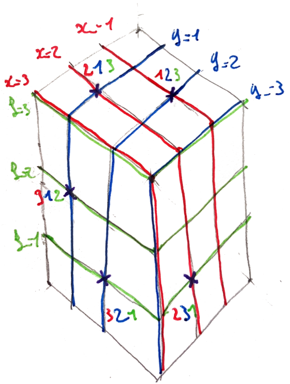

x=1
x=2
x=3
y=1
y=2
y=3
z=3
z=2
z=1


x=1
x=2
x=3
y=1
y=2
y=3
123
z=3
z=2
z=1


x=1
x=2
x=3
y=1
y=2
y=3
213
123
z=3
z=2
z=1


x=1
x=2
x=3
y=1
y=2
y=3
z=3
z=2
z=1
213
123
312


x=1
x=2
x=3
y=1
y=2
y=3
z=3
z=2
z=1
213
123
312
321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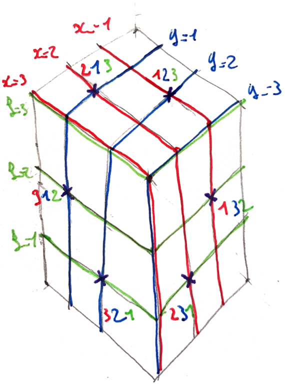
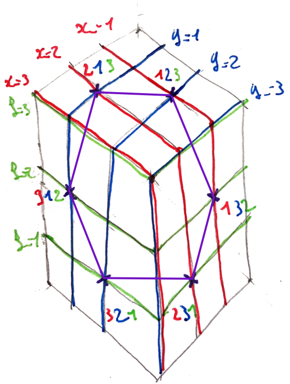
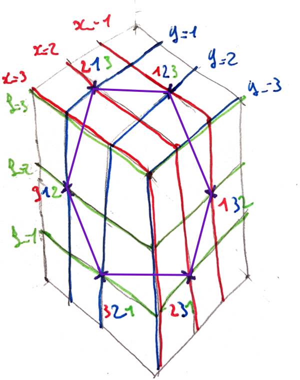
x=2
x=3
y=1
y=2
y=3
z=3
z=2
z=1
213
123
312
321
231


x=1
x=2
x=3
y=1
y=2
y=3
z=3
z=2
z=1
213
123
312
132
321
231


x=1
x=2
x=3
y=1
y=2
y=3
213
123
312
132
321
231

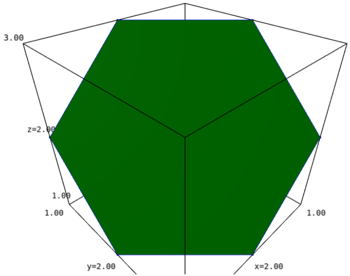


$$x + y + z = 6$$

Using SageMath

Entrée [1]:
```
P = Polyhedron(list(Permutations(3)))
P
```
Out[1]: A 2-dimensional polyhedron in ZZ^3 defined as the convex hull of 6 vertices (use the .plot() method to plot)

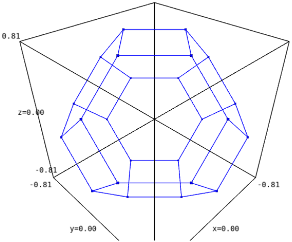
Entrée [2]:
```
P.plot()
```
Out[2]:



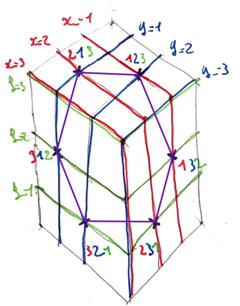
For size 4

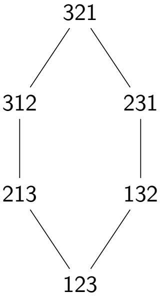
Entrée [3]:
```
P = Polyhedron(list(Permutations(4)))
P
```
Out[3]: A 3-dimensional polyhedron in ZZ^4 defined as the convex hull of 24 vertices (use the .plot() method to plot)

Entrée [4]:
```
P.plot()
```
Out[4]:

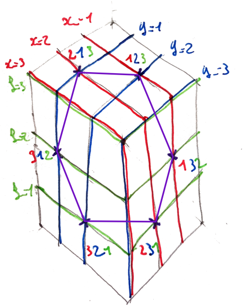


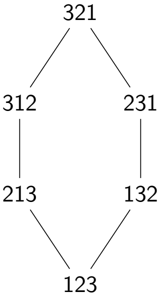
Looking at the "skeleton" of the polytope"





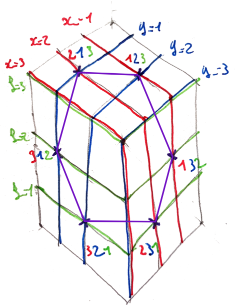
Looking at the “skeleton” of the polytope”

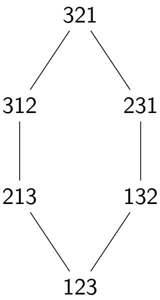


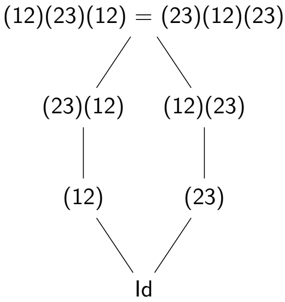


The *the left weak order* on permutations

Looking at the “skeleton” of the polytope”

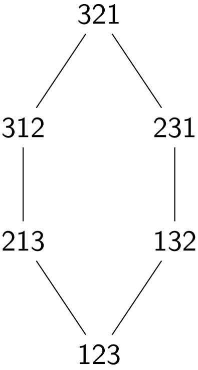




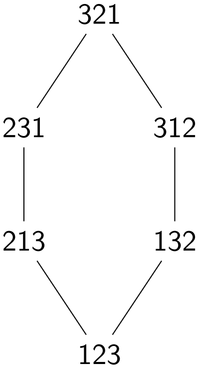


The *the left weak order* on permutations

The left and right weak orders

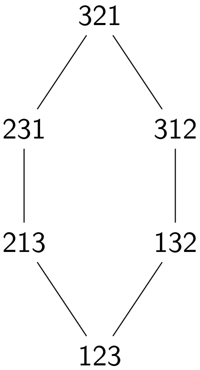


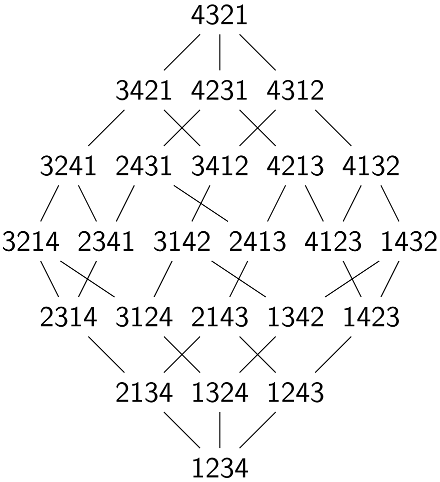
left weak order



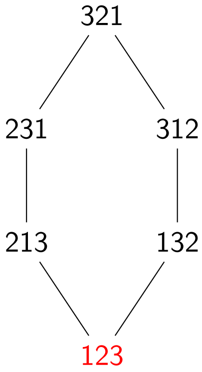
right weak order

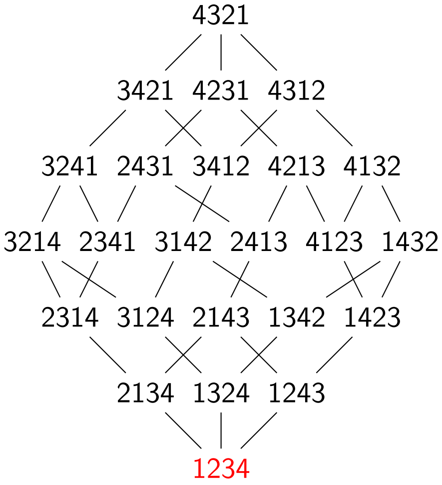
Right weak Order



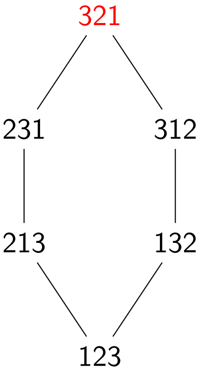


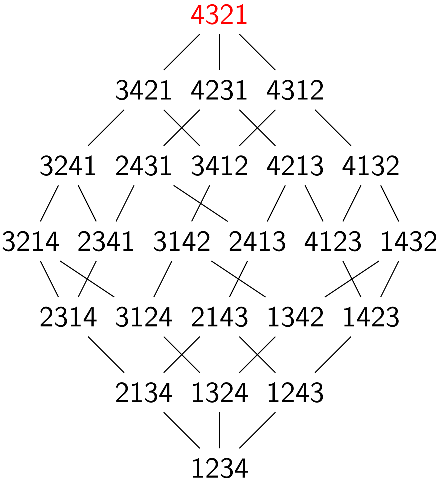
Right weak Order



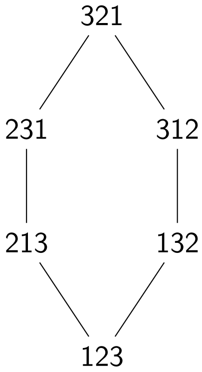


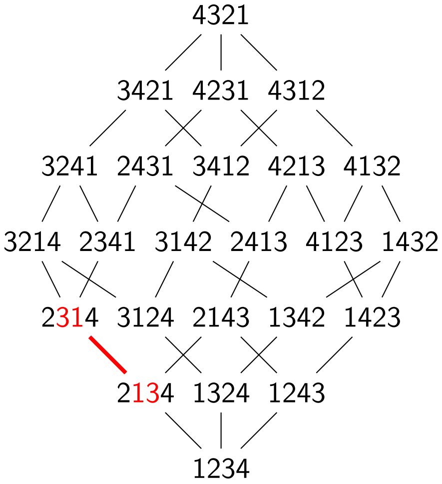
Right weak Order



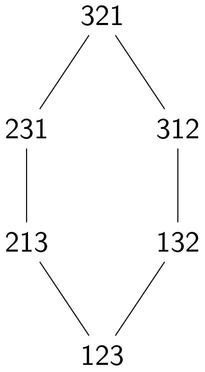


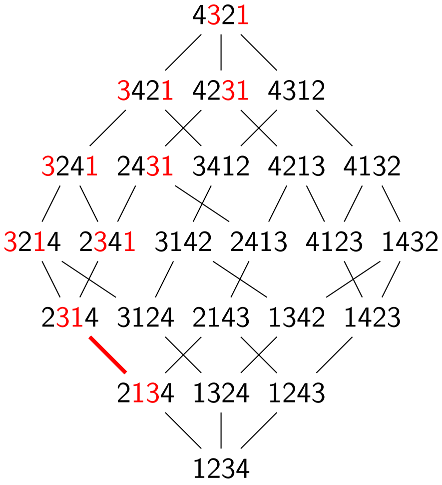
Right weak Order



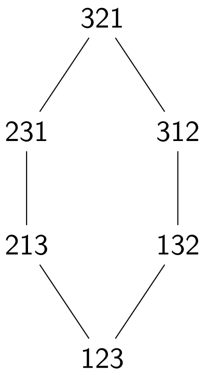


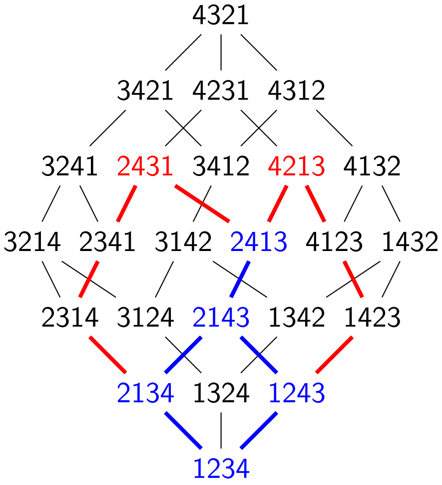
Right weak Order





Right weak Order

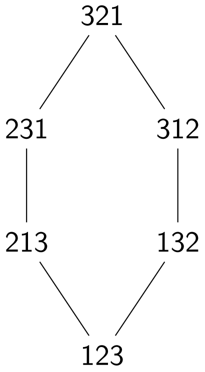


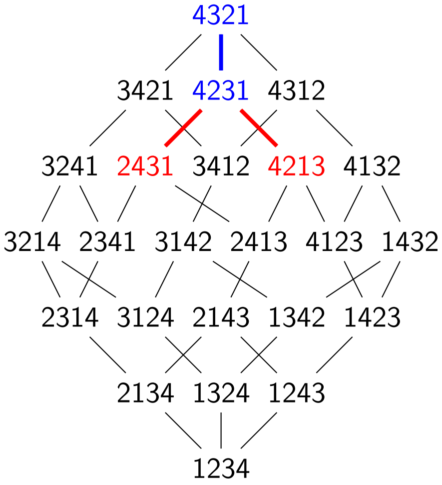


$2413 \wedge 4213 = 2413$

$2413 \vee 4213 = 4231$

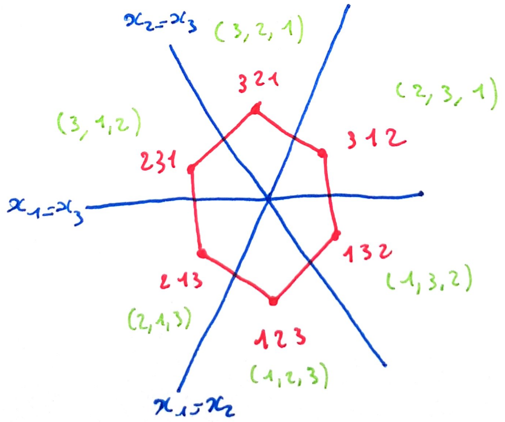
Right weak Order





$$2413 \wedge 4213 = 2413 \qquad 2413 \vee 4213 = 4231$$

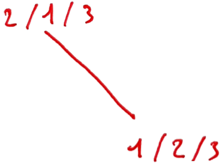
As a reflection group



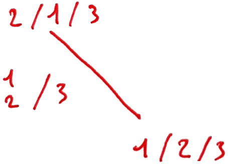
Combinatorics of faces

$1/2/3$

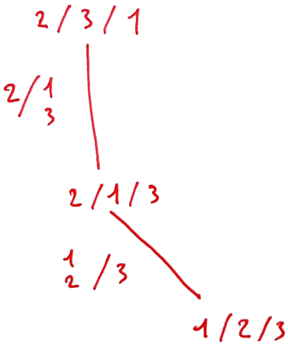
Combinatorics of faces



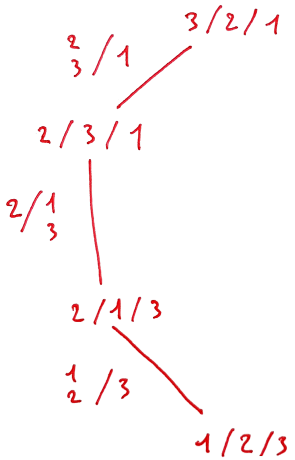
Combinatorics of faces



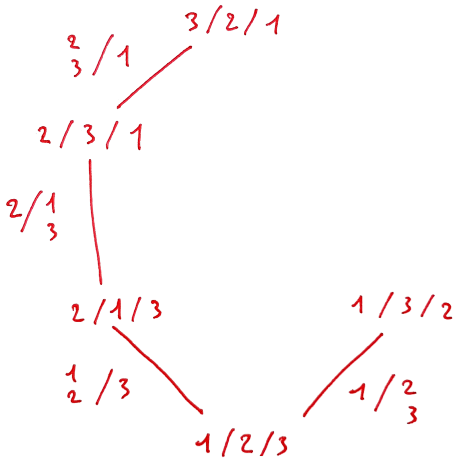
Combinatorics of faces



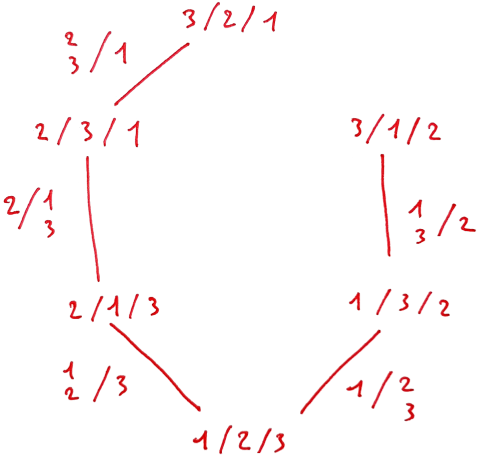
Combinatorics of faces



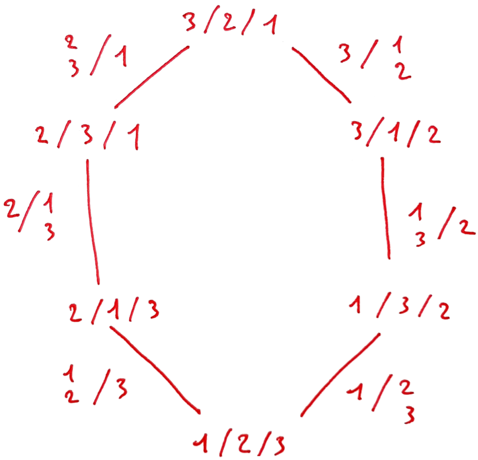
Combinatorics of faces



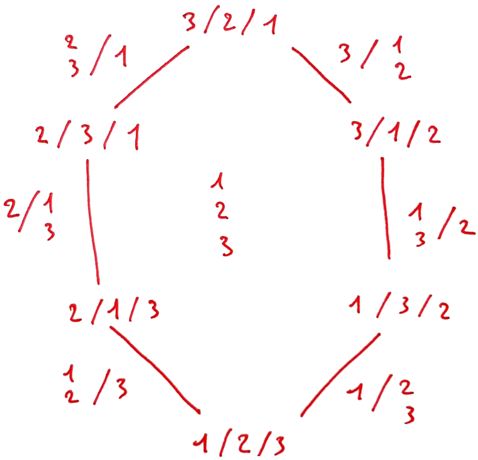
Combinatorics of faces



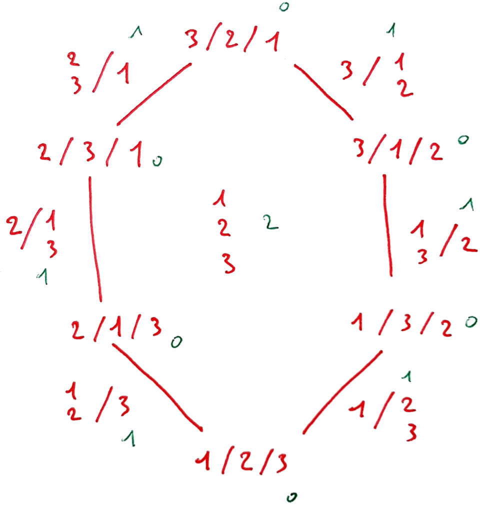
Combinatorics of faces



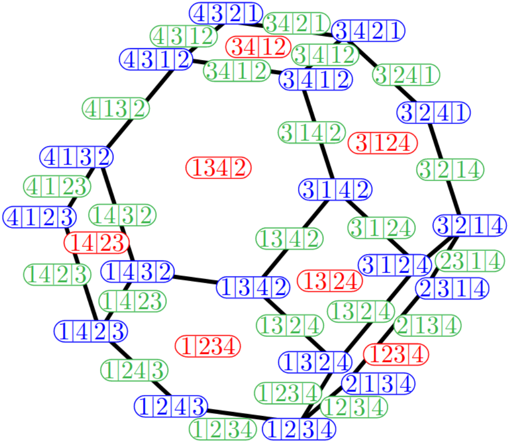
Combinatorics of faces



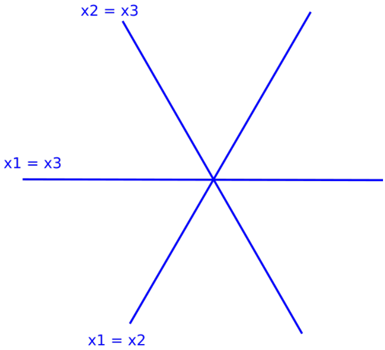
Combinatorics of faces

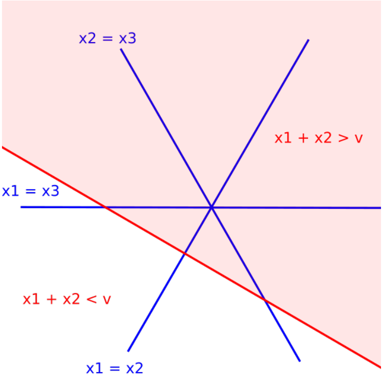


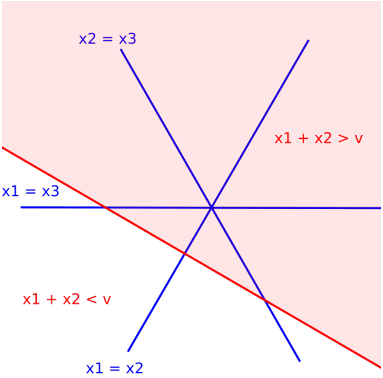
$$d = n - \#parts$$



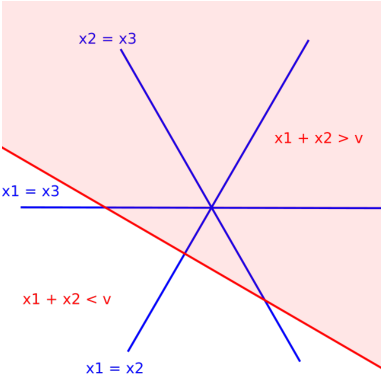
(image from V. Pilaud's talk "The Associahedron and its friends")


x2 = x3
x1 = x3
x1 = x2

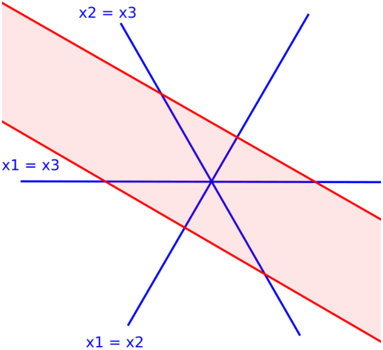

x2 = x3
x1 + x2 > v
x1 = x3
x1 + x2 < v
x1 = x2



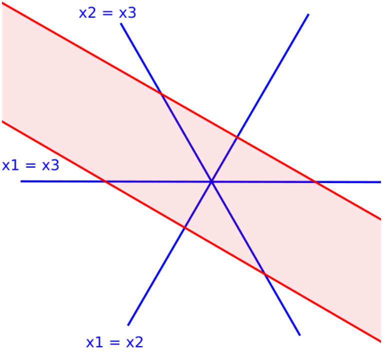
$$x_1 + x_2 + x_3 = 6$$



$$x_1 + x_2 = 6 - x_3$$

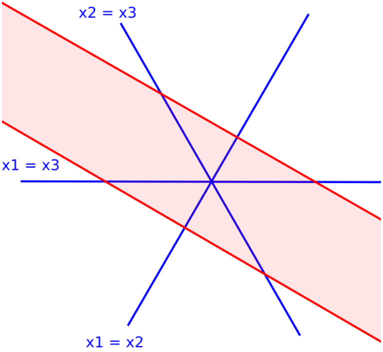


$$3 \leq x_1 + x_2 \leq 5$$



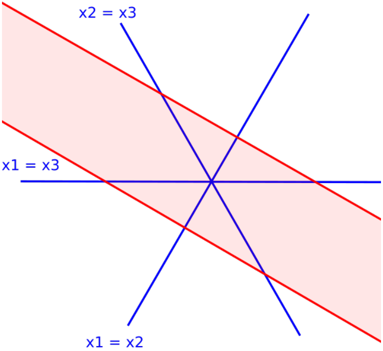
$$x_1 + x_2 \geq 3$$

$$x_1 + x_2 \leq 5$$



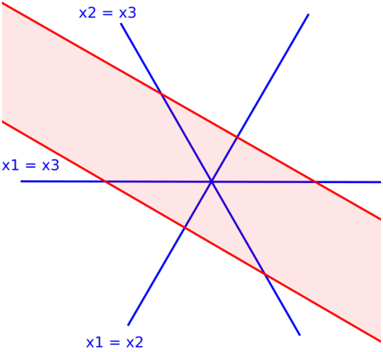
$$x_1 + x_2 \geq 3$$

$$x_3 \geq 1$$



$12|3 \quad x_1 + x_2 \geq 3$

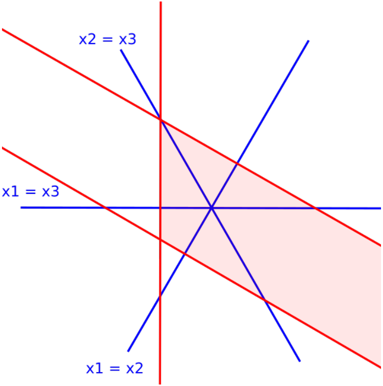
$3|12 \quad x_3 \geq 1$



$$J \subseteq [n] \rightarrow \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$

$12|3 \quad x_1 + x_2 \geq 3$

$3|12 \quad x_3 \geq 1$

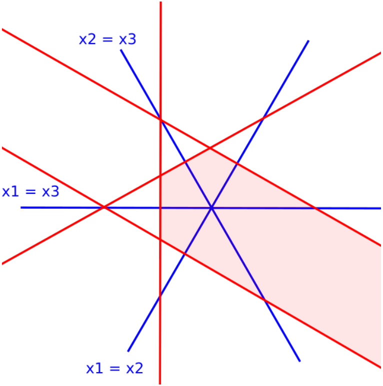


$$J \subseteq [n] \to \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$

$12|3 \quad x_1 + x_2 \geq 3$

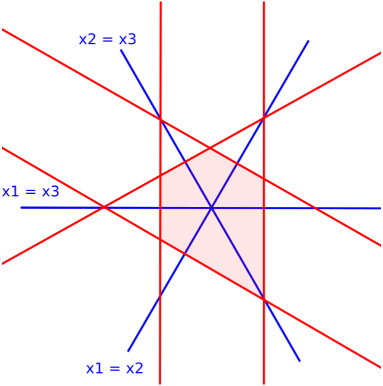
$2|13 \quad x_2 \geq 1$

$3|12 \quad x_3 \geq 1$



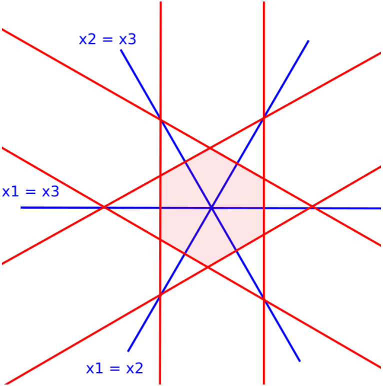
$$J \subseteq [n] \rightarrow \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$

$12 \vert 3$	$x_1 + x_2 \geq 3$
$2 \vert 13$	$x_2 \geq 1$
$23 \vert 1$	$x_2 + x_3 \geq 3$
$3 \vert 12$	$x_3 \geq 1$



$$J \subseteq [n] \rightarrow \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$

$12\vert 3$	$x_1 + x_2 \geq 3$
$2\vert 13$	$x_2 \geq 1$
$23\vert 1$	$x_2 + x_3 \geq 3$
$3\vert 12$	$x_3 \geq 1$
$13\vert 2$	$x_1 + x_3 \geq 3$



$$J \subseteq [n] \rightarrow \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$

$12\|3$	$x_1 + x_2 \geq 3$
$2\|13$	$x_2 \geq 1$
$23\|1$	$x_2 + x_3 \geq 3$
$3\|12$	$x_3 \geq 1$
$13\|2$	$x_1 + x_3 \geq 3$
$1\|23$	$x_1 \geq 1$

From the weak order to the Tamari lattice

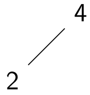
We define a *surjection* from permutations to *binary trees* which gives us a new lattice.

Binary search tree insertion

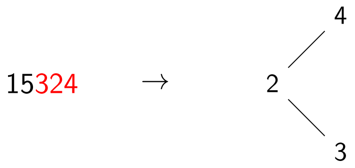
$$15324 \quad \rightarrow \quad 4$$

Binary search tree insertion

15324 $\rightarrow$

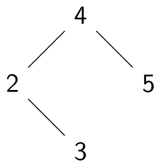


Binary search tree insertion



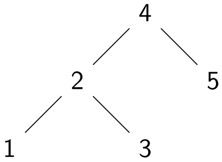
Binary search tree insertion

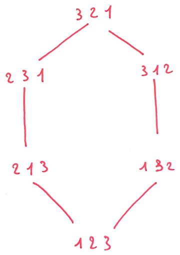
15324 $\rightarrow$

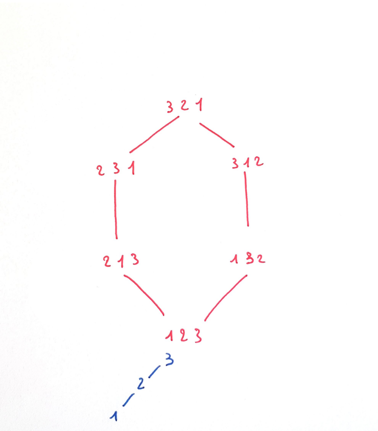


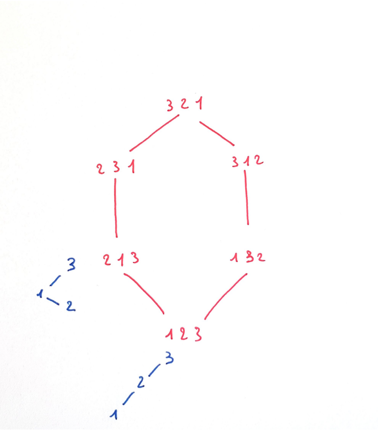
Binary search tree insertion

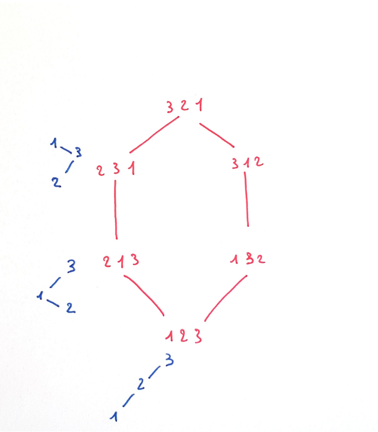
15324 $\rightarrow$

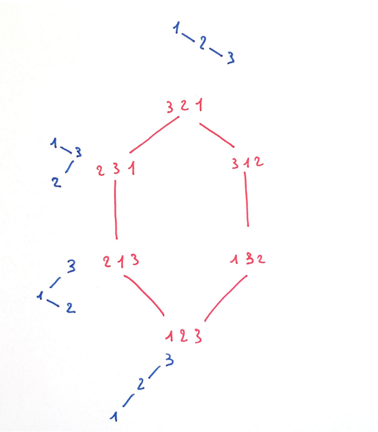



321
231
312
213
132
123

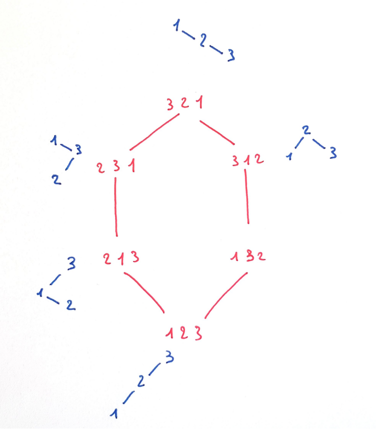

321
231
312
213
132
123
1
2
3

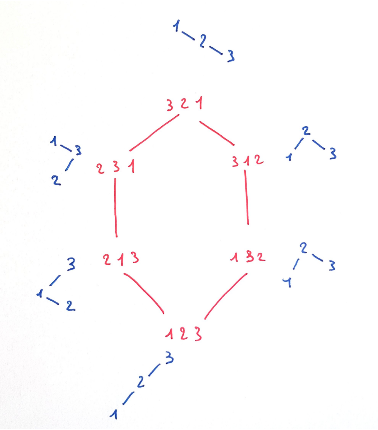

3 2 1
2 3 1
3 1 2
2 1 3
1 3 2
1 2 3
3
1
2
3
2
1

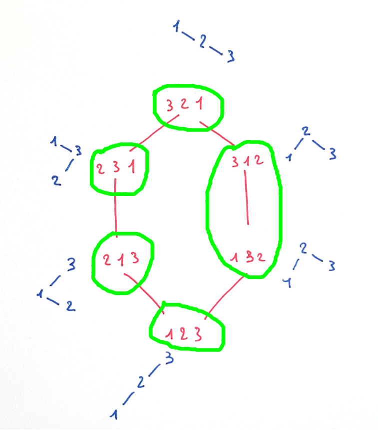

3 2 1
2 3 1
3 1 2
2 1 3
1 3 2
1 2 3
1 3 2
3 1 2
1 2 3



1 2 3
3 2 1
1 3 2
2 3 1
3 1 2
3 1 2
2 1 3
1 3 2
1 2 3
3 2 1


3 2 1
2 3 1
3 1 2
2 1 3
1 3 2
1 2 3


1 2 3
321
231
312
213
132
123


1
2
3
3 2 1
1
3
2
2 3 1
2
1
3
3 1 2
2 1 3
3
1
2
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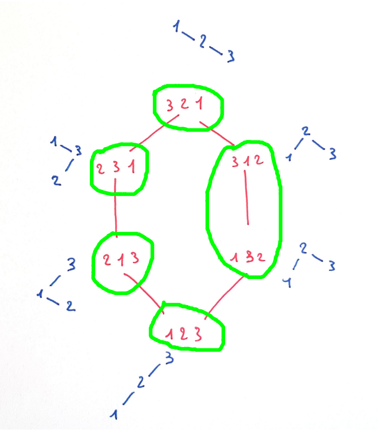
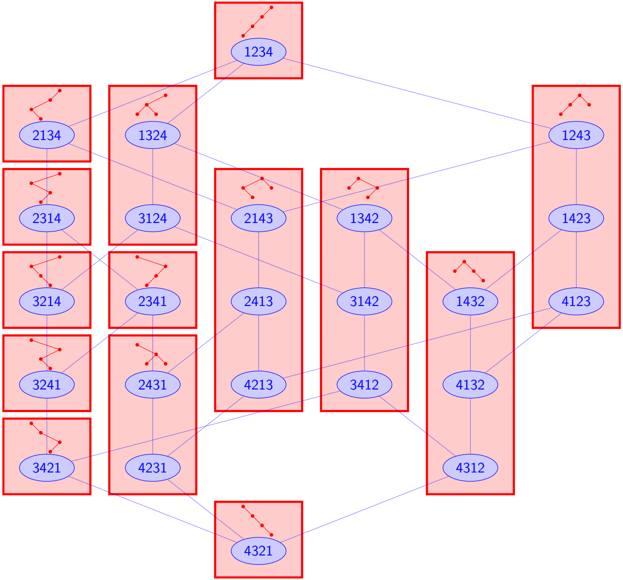
2
1
3
1 2 3
3
2
1



The **Tamari lattice** is a lattice on binary trees. It is a **quotient lattice** of the weak order.


1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

More about the Tamari lattice

Binary trees are counted by the Catalan numbers

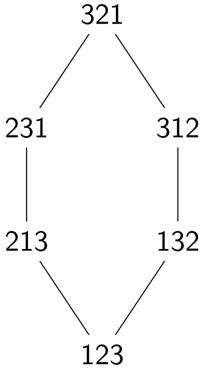
$$\frac{1}{n+1}\binom{2n}{n}$$

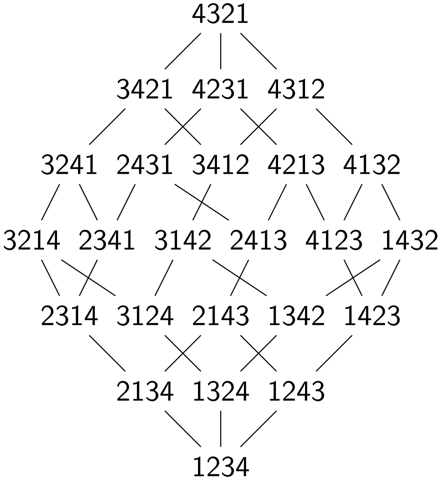
.

The Tamari lattice can be defined on **many** families of combinatorial objects, such as

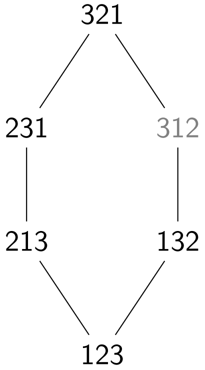
- Triangulations of regular polygons
- Dyck paths
- Ordered forests
- certain pattern avoiding permutations (312 avoiding and 231 avoiding)
- ways to parenthesize an expression (original definition of Tamari)
- ...

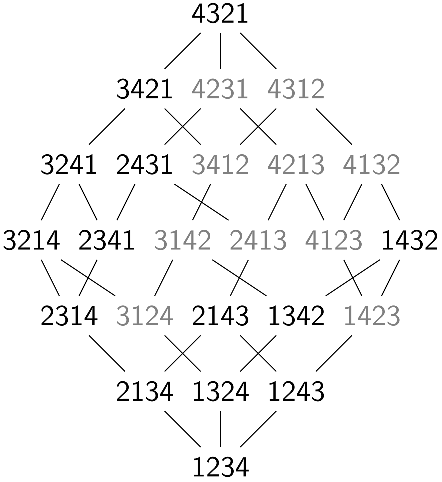
312 - avoiding permutations



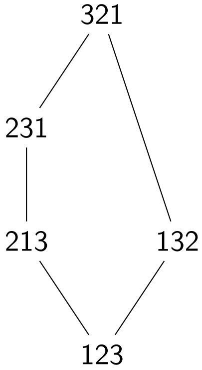


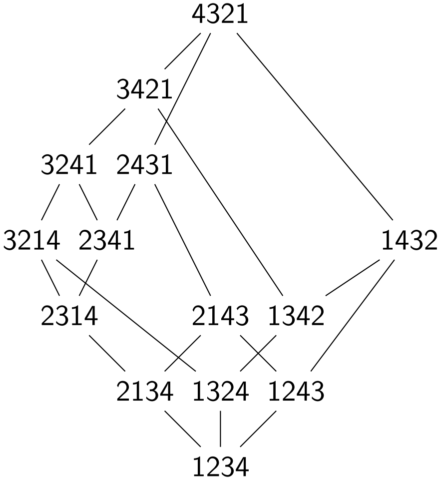
312 - avoiding permutations



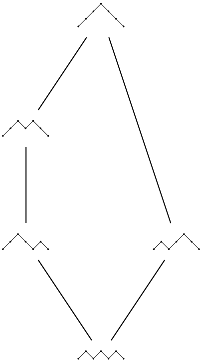


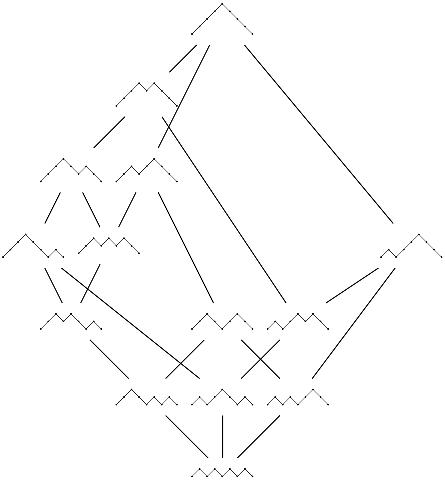
312 - avoiding permutations





Dyck paths





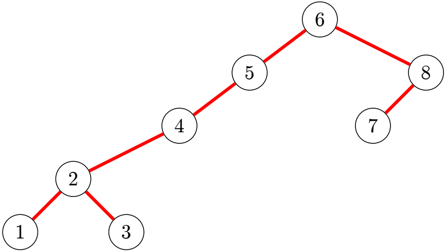
The Associahedron – Stasheff polytope

Different constructions

- Loday 2004
- Billera-Filliman-Sturmfels 1990
- Gelfand-Kapranov-Zelevinsky 1994
- Chapoton-Fomin-Zelevinsky 2002
- Hohlweg-Lange 2007
- Ceballos-Santos-Ziegler 2011
- Hohlweg-Lange-Thomas 2012

Loday's Associahedron

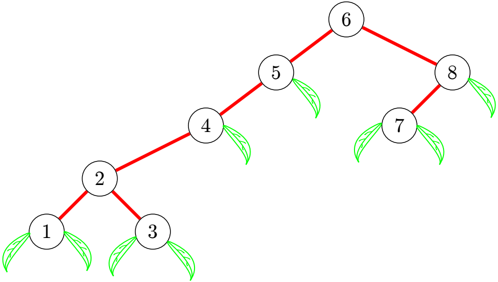
$i \to$ (# of left leaves) $\times$ (# of right leaves)



$$(\ , \ , \ , \ , \ , \quad , \ , \ )$$

Loday's Associahedron

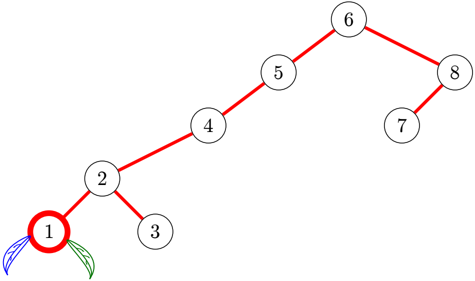
$i \rightarrow$ (# of left leaves) $\times$ (# of right leaves)



$$(\; , \; , \; , \; , \; , \quad , \; , \;)$$

Loday's Associahedron

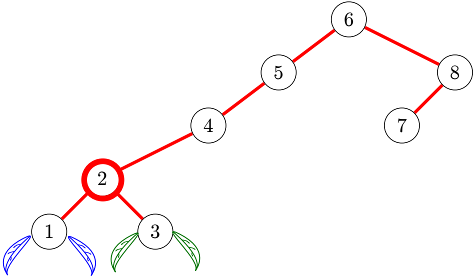
$i \to$ (# of left leaves) $\times$ (# of right leaves)



$$(1, \; , \; , \; , \; , \quad , \; , \;)$$

Loday's Associahedron

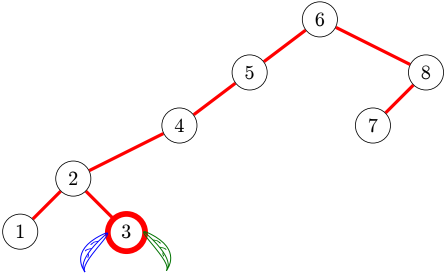
$i \rightarrow$ (# of left leaves) $\times$ (# of right leaves)



$$(1, 4, \;,\;,\;,\quad,\;,\;)$$

Loday's Associahedron

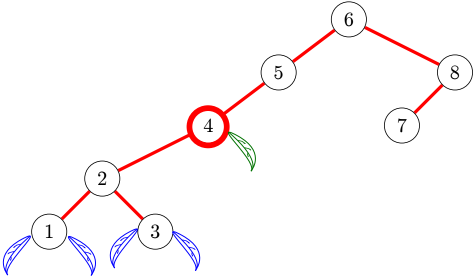
$i \rightarrow$ (# of left leaves) $\times$ (# of right leaves)



$$(1, 4, 1, \quad , \quad , \quad \quad , \quad , \quad)$$

Loday's Associahedron

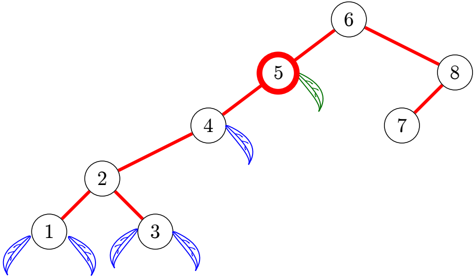
$i \rightarrow$ (# of left leaves) $\times$ (# of right leaves)



$$(1, 4, 1, 4, \ \ , \ \ \ , \ \ , \ \ )$$

Loday's Associahedron

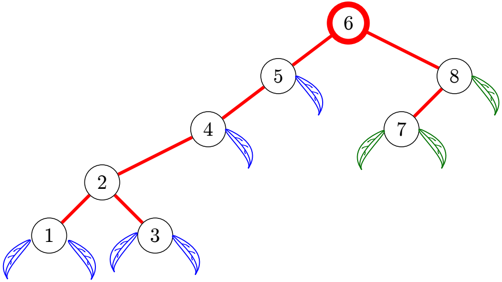
$i \rightarrow$ (# of left leaves) $\times$ (# of right leaves)



$$(1, 4, 1, 4, 5, \quad , \quad , \quad)$$

Loday's Associahedron

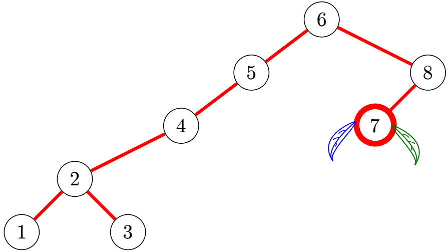
$i \rightarrow$ (# of left leaves) $\times$ (# of right leaves)



$$(1, 4, 1, 4, 5, 18, \quad , \quad)$$

Loday's Associahedron

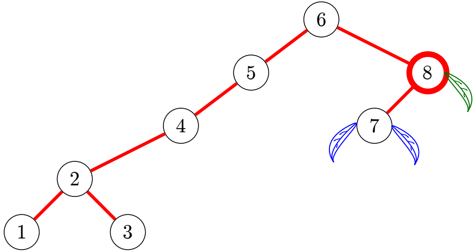
$i \to$ (# of left leaves) $\times$ (# of right leaves)



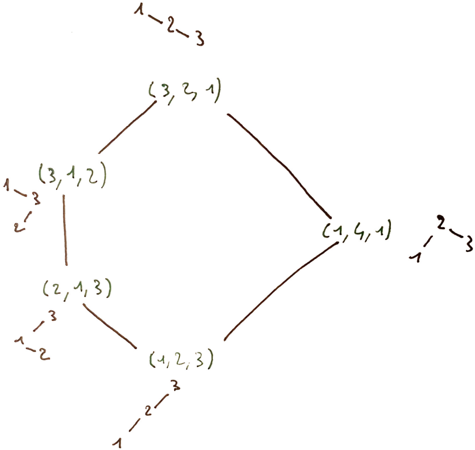
$$(1, 4, 1, 4, 5, 18, 1, \quad)$$

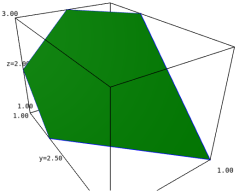
Loday's Associahedron

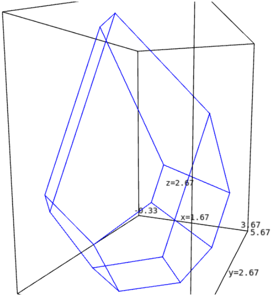
$i \to (\#\text{ of left leaves}) \times (\#\text{ of right leaves})$

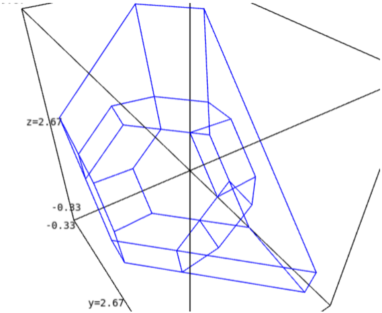


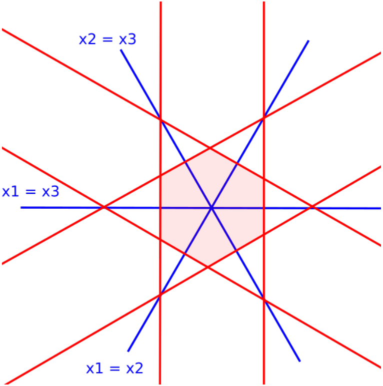
$$(1, 4, 1, 4, 5, 18, 1, 8)$$


(3, 2, 1)
(3, 1, 2)
(1, 4, 1)
(2, 1, 3)
(1, 2, 3)

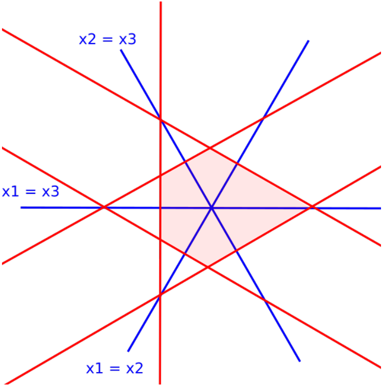

3.00
z=2.00
1.00
1.00
y=2.50
1.00


z=2.67
-0.33
x=1.67
3.67
5.67
y=2.67


z=2.67
-0.33
-0.33
y=2.67



12|3	$x_1 + x_2 \geq 3$
2|13	$x_2 \geq 1$
23|1	$x_2 + x_3 \geq 3$
3|12	$x_3 \geq 1$
13|2	$x_1 + x_3 \geq 3$
1|23	$x_1 \geq 1$



12|3	$x_1 + x_2 \geq 3$
2|13	$x_2 \geq 1$
23|1	$x_2 + x_3 \geq 3$
3|12	$x_3 \geq 1$
13|2	
1|23	$x_1 \geq 1$

More...

On the weak order

- Alain Lascoux and Marcel-Paul Schützenberger, Treillis et bases des groupes de Coxeter, Electron. J. Combin. 1996.
- A. Björner, M.L. Wachs, Permutation statistics and linear extensions of posets, J. Combin. Theory Ser. A, 1991

And more...

Relations between the weak order and the Tamari lattice + Hopf algebras

- F. Hivert, J.-C. Novelli, J.-Y. Thibon, The algebra of binary search trees, Theoret. Comput. Sci. 2005
- Andy Tonks, Relating the associahedron and the permutohedron, in: Proceedings of Renaissance Conferences, 1995
- Nathan Reading, Lattice congruences, fans and Hopf algebras, J. Combin. Theory Ser. A, 2005.

And more...

Generalizations of the Tamari lattice

- Nathan Reading, Cambrian lattices, Adv. Math. 2006
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- L.-F. Préville-Ratelle and Viennot X. An extension of Tamari lattices. DMTCS Proceedings, FPSAC, 2015
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