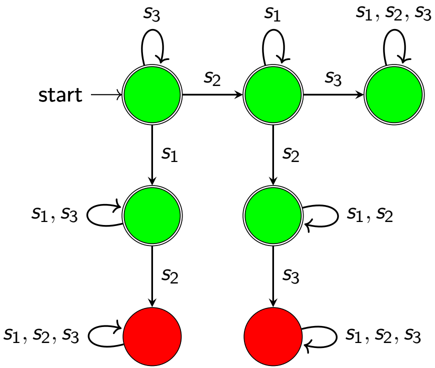
Permutree Sorting (Permutri)

Daniel Tamayo – Vincent Pilaud – **Viviane Pons**
CNRS & Ecole Polytechnique – LISN, Univ. Paris-Saclay



Sorting

3241

Sorting

3241

1234

Sorting

3241

1234

Sorting

3241

$$\begin{tikzcd} 2134 \\ 1234 \arrow[u, "s_1"'] \end{tikzcd}$$

Sorting

$$\begin{tikzcd}
3241 \\
\\
2134 \\
1234 \arrow[u, "s_1"']
\end{tikzcd}$$

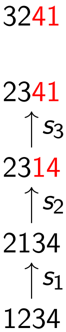
Sorting

$$\begin{tikzcd} 3241 \\ \\ 2314 \\ 2134 \arrow[u, "s_2"'] \\ 1234 \arrow[u, "s_1"'] \end{tikzcd}$$

Sorting

$$\begin{tikzcd}
3241 \\
\\
2314 \\
2134 \arrow[u, "s_2"'] \\
1234 \arrow[u, "s_1"']
\end{tikzcd}$$

Sorting



3241

2341

↑ s_3

2314

↑ s_2

2134

↑ s_1

1234

Sorting

$$\begin{tikzcd}
3241 \\
2341 \\
2314 \arrow[u, "s_3"'] \\
2134 \arrow[u, "s_2"'] \\
1234 \arrow[u, "s_1"']
\end{tikzcd}$$

Sorting

$$\begin{tikzcd}
3241 \\
2341 \arrow[u, "s_1"'] \\
2314 \arrow[u, "s_3"'] \\
2134 \arrow[u, "s_2"'] \\
1234 \arrow[u, "s_1"']
\end{tikzcd}$$

Sorting

$$\begin{tikzcd}
3241 \\
2341 \arrow[u, "s_1"'] \\
2314 \arrow[u, "s_3"'] \\
2134 \arrow[u, "s_2"'] \\
1234 \arrow[u, "s_1"']
\end{tikzcd}$$

Reduced word

$$3241 = s_1 s_2 s_3 s_1$$

Sorting

3142

1234

Sorting

3142

1234

Sorting

3142

1234

Sorting

$$\begin{tikzcd} 3142 \\ \\ 1324 \\ 1234 \arrow[u, "s_2"'] \end{tikzcd}$$

Sorting

3142

$$\begin{tikzcd} 1324 \\ 1234 \arrow[u, "s_2"'] \end{tikzcd}$$

Sorting

$$\begin{tikzcd}
3142 \\
1342 \\
1324 \arrow[u, "s_3"'] \\
1234 \arrow[u, "s_2"']
\end{tikzcd}$$

Sorting

$$\begin{tikzcd}
3142 \\
1342 \\
1324 \arrow[u, "s_3"'] \\
1234 \arrow[u, "s_2"']
\end{tikzcd}$$

Sorting

$$\begin{tikzcd}
3142 \\
1342 \arrow[u, "s_1"'] \\
1324 \arrow[u, "s_3"'] \\
1234 \arrow[u, "s_2"']
\end{tikzcd}$$

Sorting

$$\begin{tikzcd}
3142 \\
1342 \arrow[u, "s_1"'] \\
1324 \arrow[u, "s_3"'] \\
1234 \arrow[u, "s_2"']
\end{tikzcd}$$

Reduced word

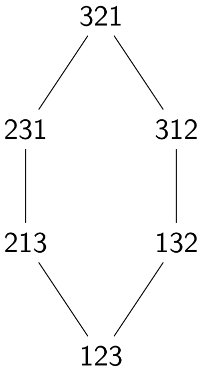
$$3142 = s_2 s_3 s_1$$

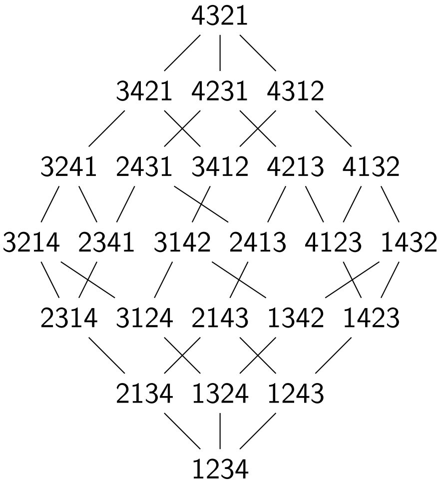
Stack sorting

3241 is (reversed) stack sortable

3142 is not (reversed) stack sortable

Weak Order


321
231
312
213
132
123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3124
2143
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1423
2134
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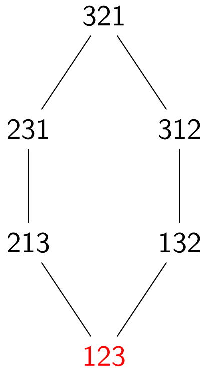
1243
1234

Weak Order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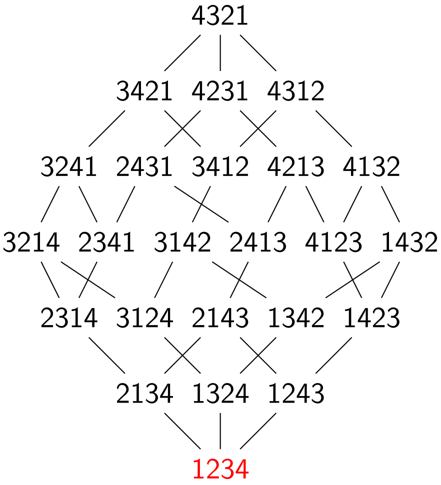
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3421 4231 4312
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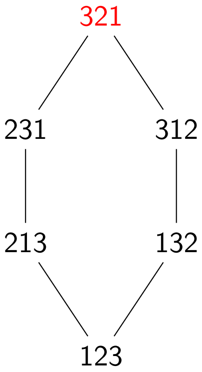
2314 3124 2143 1342 1423
2134 1324 1243
1234

Weak Order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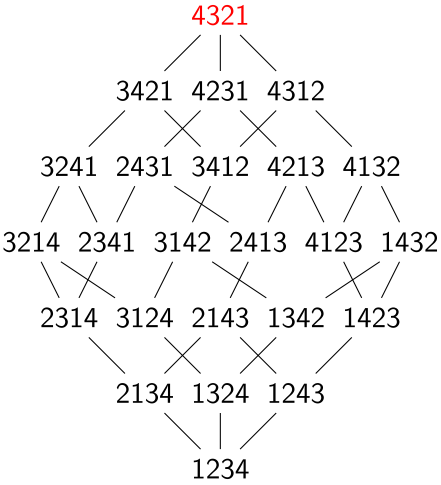
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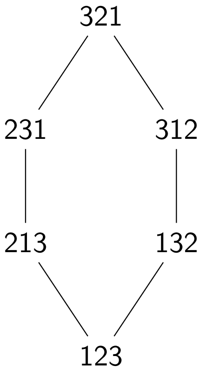
1243
1234

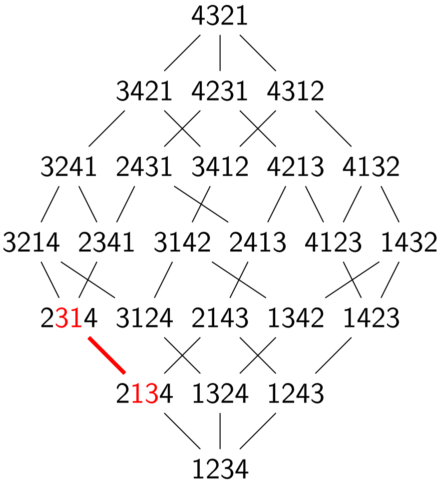
Weak Order


321

231 312

213 132

123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3421 4231 4312

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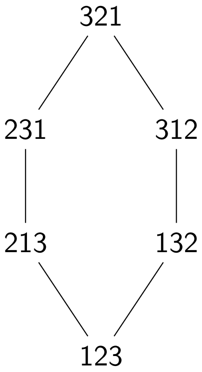
2314 3124 2143 1342 1423

2134 1324 1243

1234

Weak Order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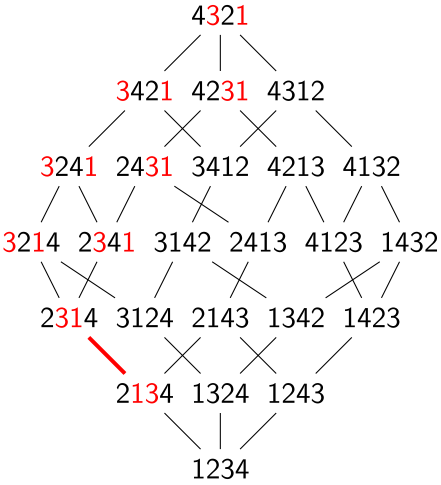
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2134
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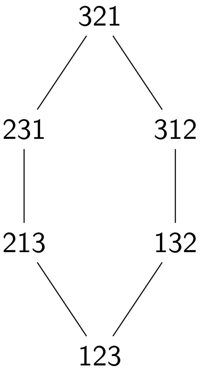
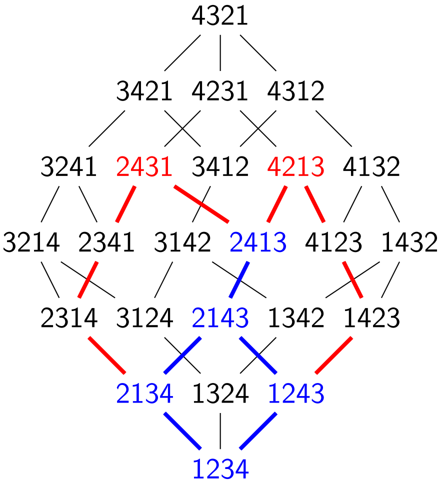
1243
1234

Weak Order

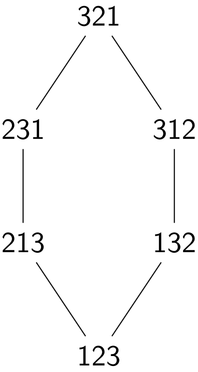


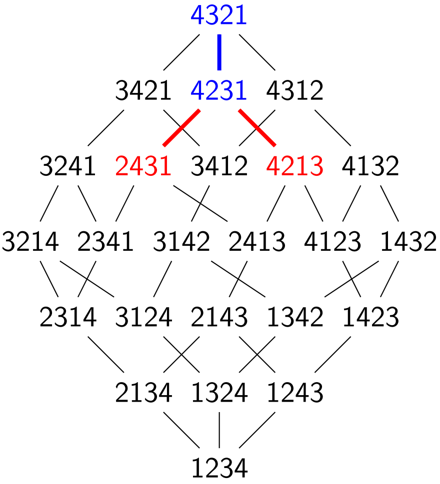


$2413 \wedge 4213 = 2413$

$2413 \vee 4213 = 4231$

Weak Order





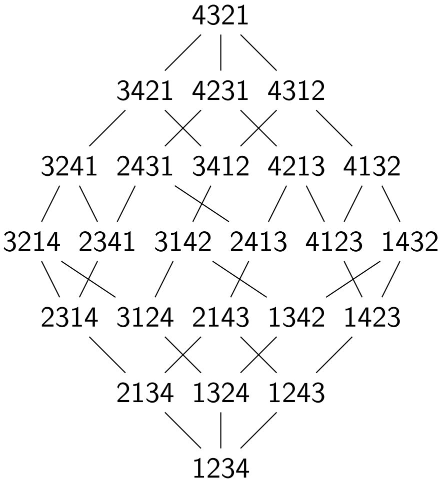
$2413 \wedge 4213 = 2413$

$2413 \vee 4213 = 4231$

Lattice congruence

The equivalence relation must be compatible with the lattice structure

$$\begin{array}{c} x_1 \equiv x_2 \\ y_1 \equiv y_2 \end{array} \Rightarrow \begin{array}{c} (x_1 \wedge y_1) \equiv (x_2 \wedge y_2) \\ (x_1 \vee y_1) \equiv (x_2 \vee y_2) \end{array}$$



4321

3421 4231 4312

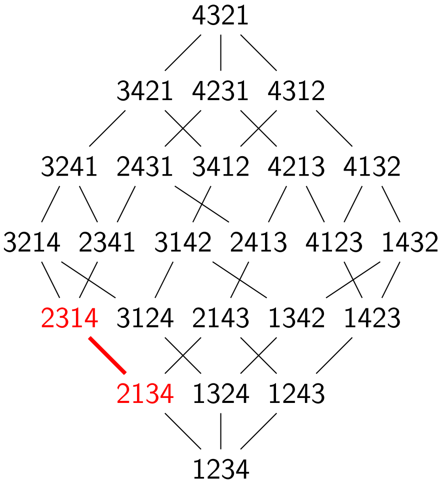
3241 2431 3412 4213 4132

3214 2341 3142 2413 4123 1432

2314 3124 2143 1342 1423

2134 1324 1243

1234



4321

3421 4231 4312

3241 2431 3412 4213 4132

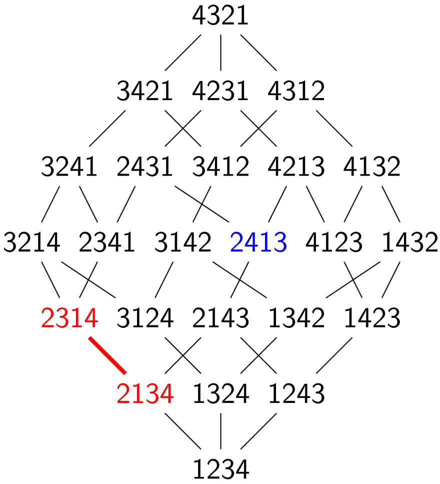
3214 2341 3142 2413 4123 1432

2314 3124 2143 1342 1423

2134 1324 1243

1234

$$2134 \equiv 2314$$



4321

3421 4231 4312

3241 2431 3412 4213 4132

3214 2341 3142 2413 4123 1432

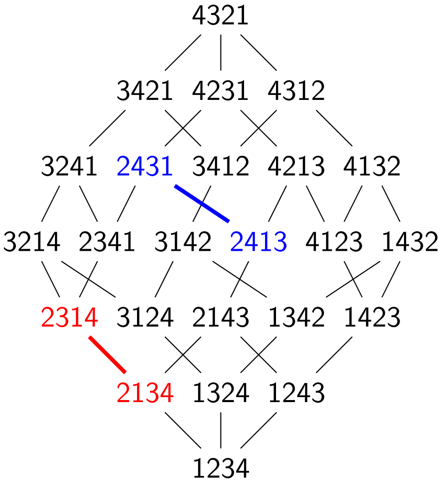
2314 3124 2143 1342 1423

2134 1324 1243

1234

$$2134 \equiv 2314$$

$$2134 \vee 2413 \equiv 2314 \vee 2413$$



4321

3421 4231 4312

3241 2431 3412 4213 4132

3214 2341 3142 2413 4123 1432

2314 3124 2143 1342 1423

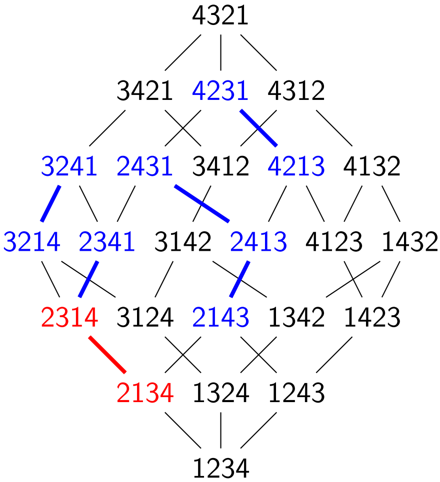
2134 1324 1243

1234

$$2134 \equiv 2314$$

$$2134 \vee 2413 \equiv 2314 \vee 2413$$

$$2431 \equiv 2413$$



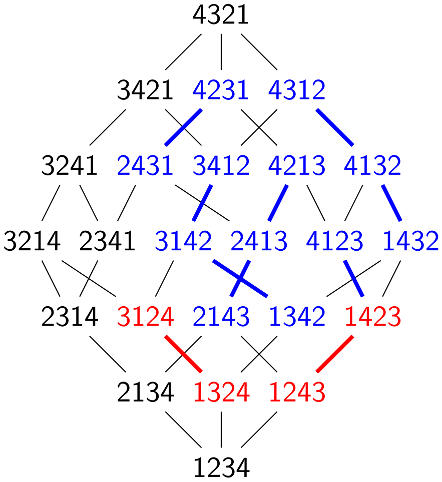
$2134 \equiv 2314 \equiv 2341$

$2143 \equiv 2413 \equiv 2431$

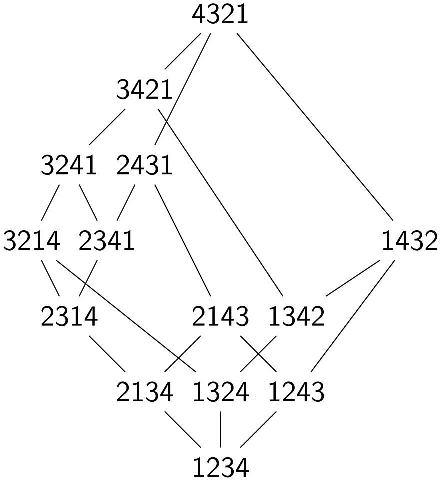
$3214 \equiv 3241$

$4213 \equiv 4231$

the sylvester congruence and the Tamari lattice



the sylvester congruence and the Tamari lattice

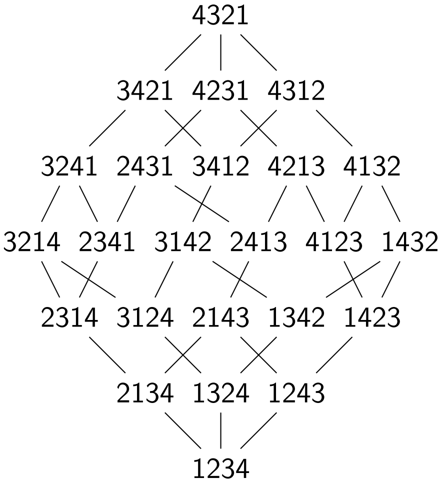


The (reversed) stack sortable permutations are the minimal elements of the congruence classes.

What are all the lattice congruence of the Weak order?

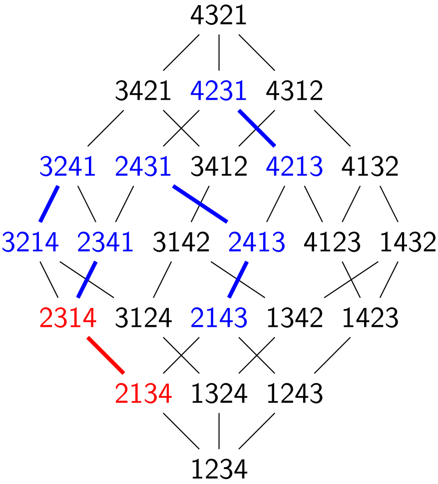
Lattice Congruences of the Weak Order, Nathan Reading, *Order*, 2005

The Permutree congruences



→ ⦶⦶⦶⦶ 24 classes

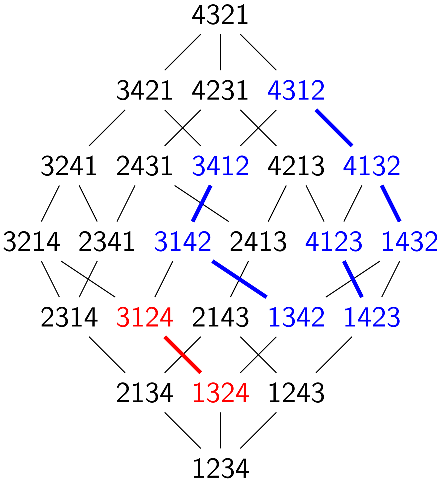
The Permutree congruences



→

⦶⦶⦶⦶ 24 classes
⦶Ⓨ⦶⦶ 18 classes

The Permutree congruences



4321

3421 4231 4312

3241 2431 3412 4213 4132

3214 2341 3142 2413 4123 1432

2314 3124 2143 1342 1423

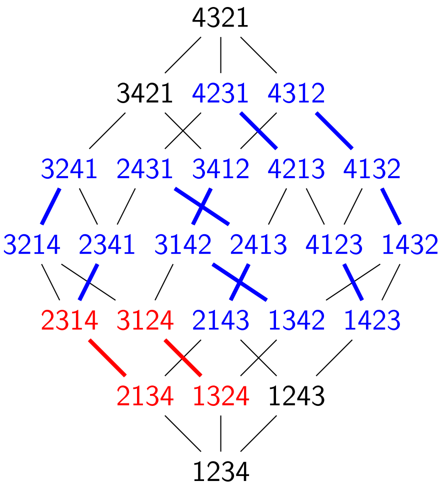
2134 1324 1243

1234

→

⦶⦶⦶⦶ 24 classes
⦶Ⓨ⦶⦶ 18 classes
⦶☮⦶⦶ 18 classes

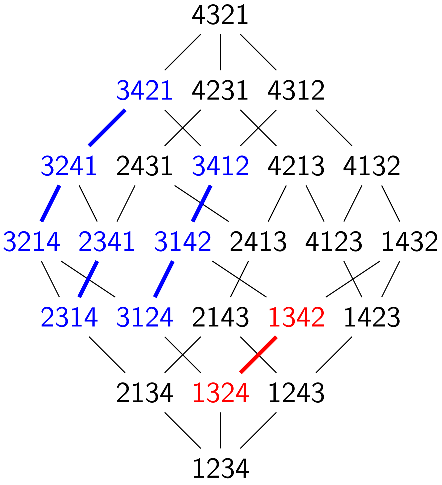
The Permutree congruences



→

⦶⦶⦶⦶	24 classes
⦶Ⓨ⦶⦶	18 classes
⦶☮⦶⦶	18 classes
⦶⊗⦶⦶	12 classes

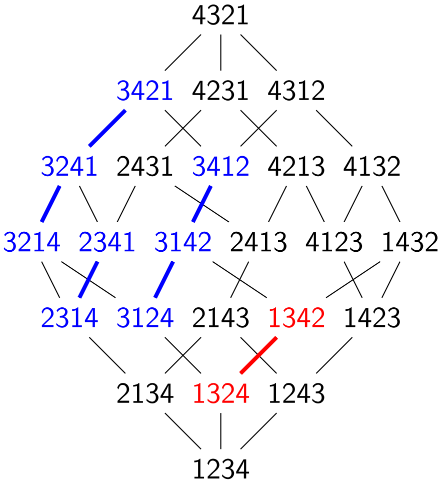
The Permutree congruences



→

⦶⦶⦶⦶	24 classes
⦶Ⓨ⦶⦶	18 classes
⦶☮⦶⦶	18 classes
⦶⊗⦶⦶	12 classes
⦶⦶Ⓨ⦶	18 classes

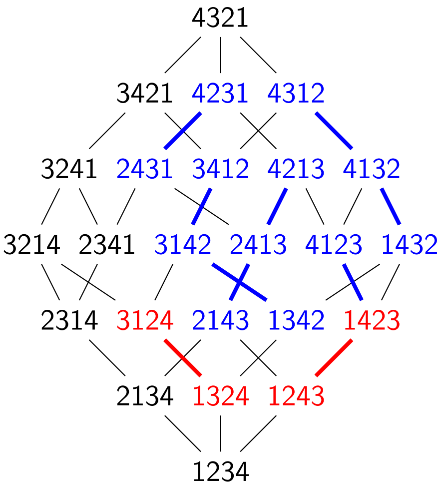
The Permutree congruences



→

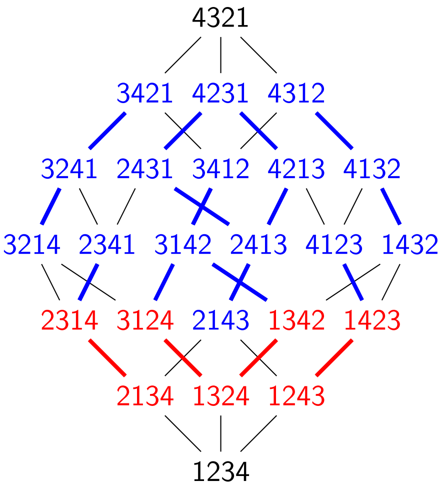
⦶⦶⦶⦶	24 classes
⦶Ⓨ⦶⦶	18 classes
⦶☮⦶⦶	18 classes
⦶⊗⦶⦶	12 classes
⦶⦶Ⓨ⦶	18 classes
⦶⦶☮⦶	18 classes
⦶⦶⊗⦶	12 classes

The Permutree congruences



	⦶⦶⦶⦶	24 classes
	⦶Ⓨ⦶⦶	18 classes
	⦶☮⦶⦶	18 classes
	⦶⊗⦶⦶	12 classes
	⦶⦶Ⓨ⦶	18 classes
	⦶⦶☮⦶	18 classes
	⦶⦶⊗⦶	12 classes
→	⦶☮☮⦶	14 classes

The Permutree congruences



	⦶⦶⦶⦶	24 classes
	⦶Ⓨ⦶⦶	18 classes
	⦶☮⦶⦶	18 classes
	⦶⊗⦶⦶	12 classes
	⦶⦶Ⓨ⦶	18 classes
	⦶⦶☮⦶	18 classes
	⦶⦶⊗⦶	12 classes
	⦶☮☮⦶	14 classes
→	⦶⊗⊗⦶	8 classes

Permutrees

The permutrees are combinatorial objects which encode those lattice congruences. They interpolate between permutations, binary trees and binary sequence.

→ V. Pilaud, V. Pons. Permutrees. *Algebraic Combinatorics* (2018).

The Permutree Recipe

- Take a word in {⦶, ☮, ⍦, ⊗}^{n}
- Take a permutation
- Do the insertion: get a Leveled Permutree (bijection)
- Remove the levels: get a Permutree (surjection)

The Permutree Recipe

- Take a word in $\{⦶, ☮, Ⓨ, \otimes\}^n$
- Take a permutation
- Do the insertion: get a Leveled Permutree (bijection)
- Remove the levels: get a Permutree (surjection)

Example

$⦶^n$	$\longleftrightarrow$	permutations of $[n]$
$☮^n$	$\longleftrightarrow$	standard binary search trees
$\{☮, Ⓨ\}^n$	$\longleftrightarrow$	Cambrian trees (Reading)
$\otimes^n$	$\longleftrightarrow$	binary sequences

The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⌽⊗⌽☮Ⓨ

The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

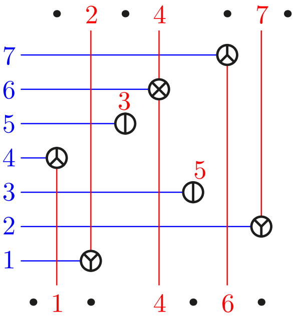
Decorated permutation: $\overline{2}\overline{7}5\underline{1}3\underline{\overline{4}}\underline{6}$

The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated permutation: $\overline{2}\,\overline{7}\,5\,\underline{1}\,3\,\underline{\overline{4}}\,\underline{6}$

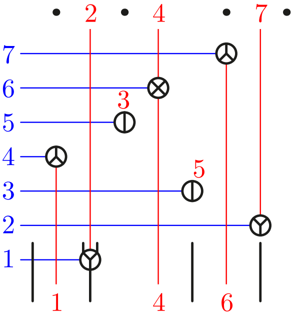


The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated permutation: $\overline{2}\overline{7}5\underline{1}3\underline{\overline{4}}\underline{6}$

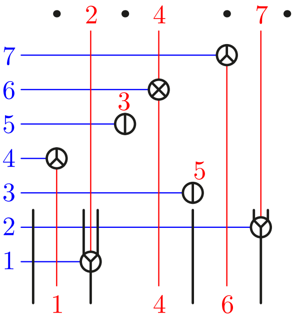


The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated permutation: $\overline{27}5\underline{1}3\underline{\overline{4}6}$

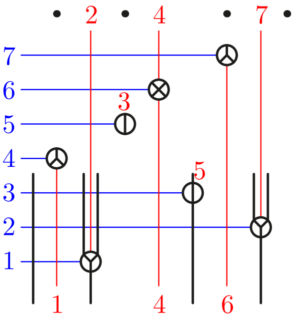


The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated permutation: $\overline{2}\overline{7}5\underline{1}3\overline{\underline{4}}\underline{6}$

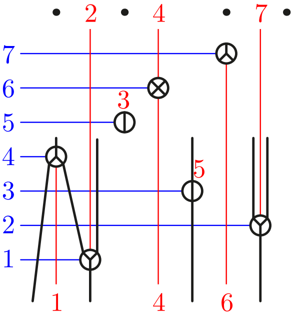


The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated permutation: $\overline{27}5\underline{1}3\underline{\overline{4}6}$

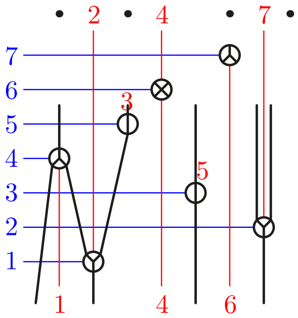


The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated permutation: $\overline{2}\overline{7}5\underline{1}3\underline{\overline{4}}\underline{6}$

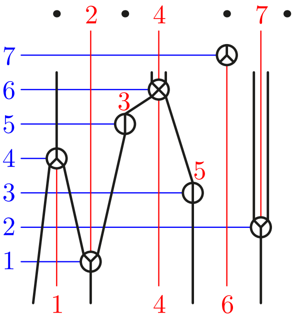


The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated permutation: $\overline{2}\overline{7}5\underline{1}3\underline{\overline{4}}\underline{6}$

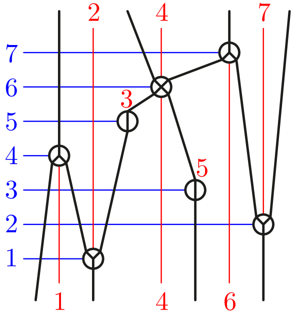


The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated permutation: $\overline{2}\overline{7}5\underline{1}3\underline{\overline{4}}\underline{6}$

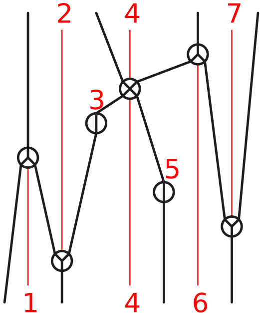


The Permutree insertion

Permutation: 2751346

Decoration: ☮Ⓨ⦶⊗⦶☮Ⓨ

Decorated permutation: $\overline{27}5\underline{1}3\underline{\overline{4}6}$



Insertion

$$\begin{aligned} \mathfrak{S}_n \times \{\text{⦶}, \text{☮}, \text{Ⓨ}, \otimes\}^n &\longrightarrow \text{Permutrees} \\ (\sigma, \delta) &\longrightarrow \mathbf{P}_\delta(\sigma) \end{aligned}$$

Property

$$\sigma \equiv_\delta \tau \Leftrightarrow \mathbf{P}_\delta(\sigma) = \mathbf{P}_\delta(\tau)$$

Insertion

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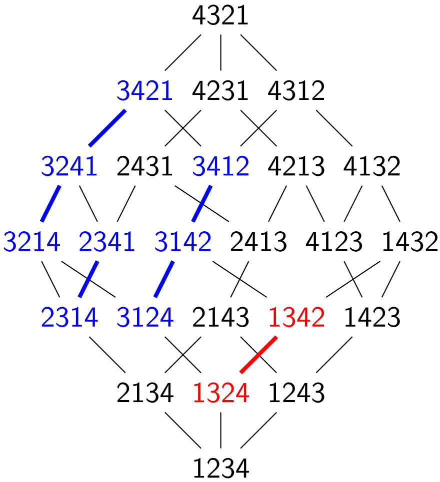
Property

$$\sigma \equiv_\delta \tau \Leftrightarrow \mathbf{P}_\delta(\sigma) = \mathbf{P}_\delta(\tau)$$

Congruence

$$\dots ac \dots \underline{b} \dots \equiv_\delta \dots ca \dots \underline{b} \dots \qquad (\delta_b \in \{\text{☮}, \otimes\})$$

$$\dots \overline{b} \dots ac \dots \equiv_\delta \dots \overline{b} \dots ca \dots \qquad (\delta_b \in \{\text{Ⓨ}, \otimes\})$$



4321

3421 4231 4312

3241 2431 3412 4213 4132

3214 2341 3142 2413 4123 1432

2314 3124 2143 1342 1423

2134 1324 1243

1234

Decoration: ⦶⦶Ⓨ⦶

$\overline{3}214 \equiv \overline{3}241 \equiv \overline{3}421$

$2\overline{3}14 \equiv 2\overline{3}41$

$\overline{3}124 \equiv \overline{3}142 \equiv \overline{3}412$

$1\overline{3}24 \equiv 1\overline{3}42$

Classical patterns

Example: bca $a < b < c$

(some) permutations containing bca: 4231, 2341, 2413, 3421

Classical patterns

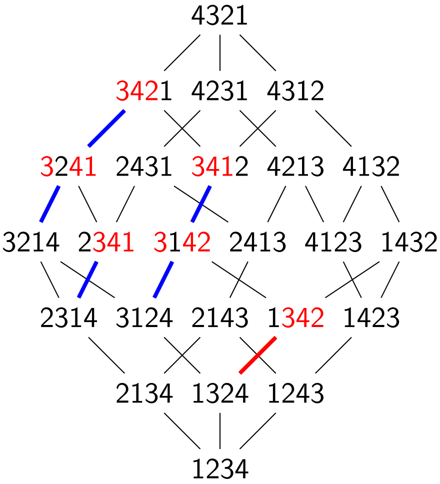
Example: bca $a < b < c$

(some) permutations containing bca: 4231, 2341, 2413, 3421

Permutree patterns

We fix the value of b

All size 4 permutations containing $3ca$: 3421 , 3241, 2341, 3142, 3412, 1342


4321
3421
4231
4312
3241
2431
3412
4213
4132
3214
2341
3142
2413
4123
1432
2314
3124
2143
1342
1423
2134
1324
1243
1234

Minimal permutations:

Decoration	avoided pattern	
⦶☮⦶⦶	$ca2$	$a < 2 < c$
⦶Ⓨ⦶⦶	$2ca$	$a < 2 < c$
⦶⦶☮⦶	$ca3$	$a < 3 < c$
⦶⦶Ⓨ⦶	$3ca$	$a < 3 < c$

But what about "sorting"?

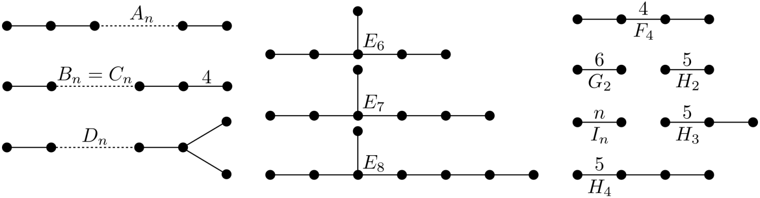
Can we find a "sorting" algorithm that selects the minimal permutation (like for stack sorting)?

Why do we want that?

We want an algorithm based on the reduced word to generalize it to other types.

Coxeter group

A group generated by some operators (s_i) such that:
$(s_i s_j)^{m_{ij}} = Id$



The weak order lattice as well as the permutree congruences can be defined on all finite Coxeter groups but we don't have the combinatorial interpretation and the pattern avoidance rule.

Cambrian lattices

→ N. Reading. Cambrian lattices. *Advances in Mathematics* (2006)

Cambrian congruences correspond to a special case of Permutree decoration {☮, Ⓨ}. The c-sorting algorithm described by Reading works in all types.

Permutree sorting: Oriented edges

Permutree decorations correspond to “partial edge orientations” of the Coxeter graph

⦶☮⦶⦶	$1 \longrightarrow 2 \relbar\joinrel\relbar 3$
⦶Ⓨ⦶⦶	$1 \longleftarrow 2 \relbar\joinrel\relbar 3$
⦶⦶☮⦶	$1 \relbar\joinrel\relbar 2 \longrightarrow 3$
⦶⦶Ⓨ⦶	$1 \relbar\joinrel\relbar 2 \longleftarrow 3$
⦶⊗Ⓨ⦶	$1 \longleftrightarrow 2 \longleftarrow 3$

Permutree sorting: Oriented edges

The orientation translates into rules on the reduced word

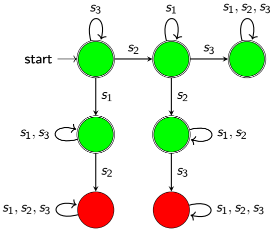
$1 \longleftarrow 2 \longrightarrow 3$

s_2 needs to be applied before s_1
and then s_3 needs to be applied before s_2.

Automaton

This translates into an automaton

$1 \longleftarrow 2 \text{ —— } 3$

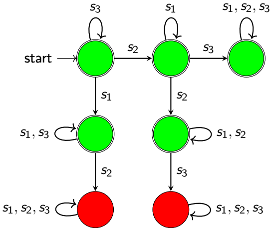


Theorem (Pilaud, P., Tamayo)

A permutation is minimal in its Permutree class if at least one of its reduced words is accepted by the automaton.

Example

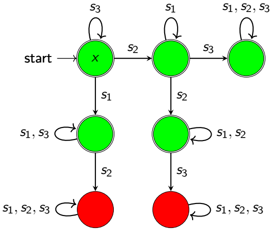
⦶ⓨ⦶⦶ $1 \longleftarrow 2 \,\rule[0.5ex]{2em}{0.4pt}\, 3$ selecting permutations avoiding $2ca$



Example

⦶Ⓨ⦶⦶ $1 \longleftarrow 2 \text{ --- } 3$ selecting permutations avoiding $2ca$

4213

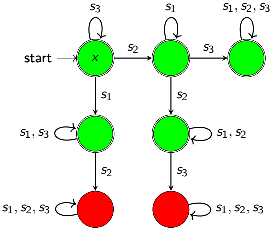


1234

Example

⦶Ⓨ⦶⦶ $1 \longleftarrow 2 \text{ —— } 3$ selecting permutations avoiding $2ca$

4213



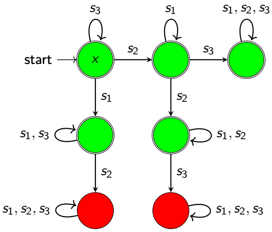
1234

Example

⦶Ⓨ⦶⦶ $1 \longleftarrow 2 \;\text{——}\; 3$ selecting permutations avoiding $2ca$

4213

1234



Example

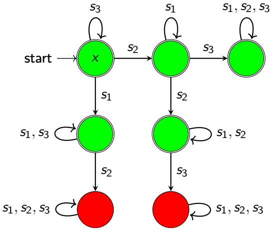
⦶Ⓨ⦶⦶ $1 \longleftarrow 2 \relbar\joinrel\relbar 3$ selecting permutations avoiding $2ca$

4213

1243

$\uparrow s_3$

1234



Example

⦶Ⓨ⦶⦶ $1 \longleftarrow 2 \text{ --- } 3$ selecting permutations avoiding $2ca$

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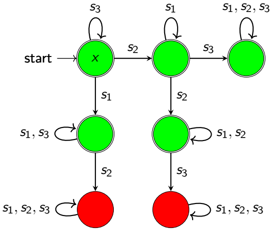
1243

$\uparrow s_3$

1234

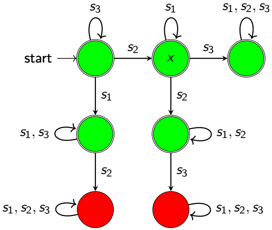


Example

⦶Ⓨ⦶⦶ $1 \longleftarrow 2 \;\text{---}\; 3$ selecting permutations avoiding $2ca$

4213

1423
$\uparrow s_2$
1243
$\uparrow s_3$
1234



Example

⦶Ⓨ⦶⦶ $1 \longleftarrow 2 \longrightarrow 3$ selecting permutations avoiding $2ca$

4213

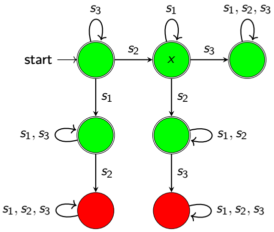
1423

$\uparrow s_2$

1243

$\uparrow s_3$

1234

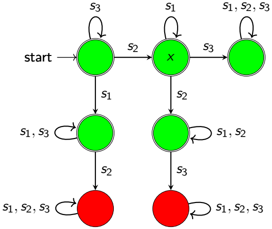


Example

⦶Ⓨ⦶⦶ $1 \longleftarrow 2 \;\text{——}\; 3$ selecting permutations avoiding $2ca$

4213

4123
$\uparrow s_1$
1423
$\uparrow s_2$
1243
$\uparrow s_3$
1234

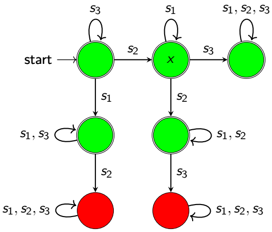


Example

⦶Ⓨ⦶⦶ $1 \longleftarrow 2 \text{ —— } 3$ selecting permutations avoiding $2ca$

$$4213$$

$$\begin{tikzcd}
4123 \\
1423 \arrow[u, "s_1"'] \\
1243 \arrow[u, "s_2"'] \\
1234 \arrow[u, "s_3"']
\end{tikzcd}$$



Example

⦶Ⓨ⦶⦶ $1 \longleftarrow 2 \text{ --- } 3$ selecting permutations avoiding $2ca$

4213

4123

$\uparrow s_1$

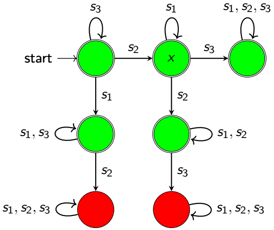
1423

$\uparrow s_2$

1243

$\uparrow s_3$

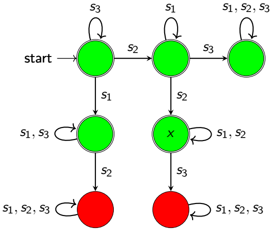
1234



Example

⦶Ⓨ⦶⦶ $1 \longleftarrow 2 \;\text{———}\; 3$ selecting permutations avoiding $2ca$

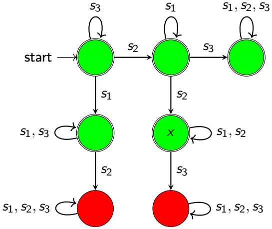
$$\begin{tikzcd} 4213 \\ 4123 \arrow[u, "s_2"'] \\ 1423 \arrow[u, "s_1"'] \\ 1243 \arrow[u, "s_2"'] \\ 1234 \arrow[u, "s_3"'] \end{tikzcd}$$



Example

⦶Ⓨ⦶⦶ $1 \longleftarrow 2 \relbar\joinrel\relbar 3$ selecting permutations avoiding $2ca$

$$\begin{array}{c} 4231 \\ \\ 4213 \\ \uparrow s_2 \\ 4123 \\ \uparrow s_1 \\ 1423 \\ \uparrow s_2 \\ 1243 \\ \uparrow s_3 \\ 1234 \end{array}$$

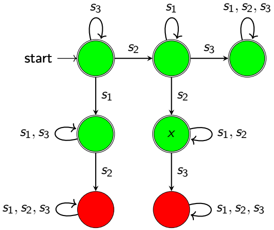


Example

⦶Ⓨ⦶⦶ 1 ⟵ 2 —— 3 selecting permutations avoiding $2ca$

4231

4213
↑ s_2
4123
↑ s_1
1423
↑ s_2
1243
↑ s_3
1234



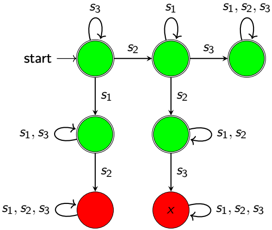
Example

⦶Ⓨ⦶⦶

$1 \longleftarrow 2 \longrightarrow 3$

selecting permutations avoiding $2ca$

$$\begin{tikzcd}
4231 \\
4213 \arrow[u, "s_3"'] \\
4123 \arrow[u, "s_2"'] \\
1423 \arrow[u, "s_1"'] \\
1243 \arrow[u, "s_2"'] \\
1234 \arrow[u, "s_3"']
\end{tikzcd}$$

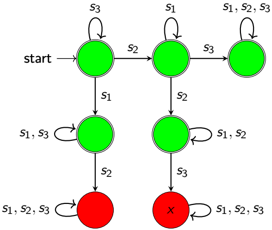


Example

$1 \longleftarrow 2 \longrightarrow\!\!\!\!\!\!\!-\; 3$

selecting permutations avoiding $2ca$

4231
$\uparrow s_3$
4213
$\uparrow s_2$
4123
$\uparrow s_1$
1423
$\uparrow s_2$
1243
$\uparrow s_3$
1234



What next?

Do other types! (In progress...)

Main reference

V. Pilaud, V. Pons. Permutrees. *Algebraic Combinatorics* (2018).
V. Pilaud, V. Pons, and D. Tamayo Jiménez. Permutree sorting. (2020+)

Some background

N. Reading. Cambrian lattices. *Advances in Mathematics* (2006)
G. Châtel, V. Pilaud, Cambiran Hopf algebras, *Advances in Mathematics* (2017)