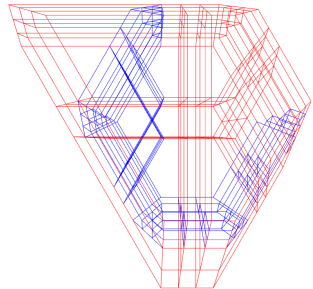
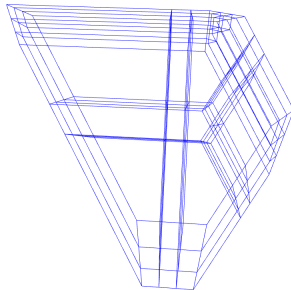
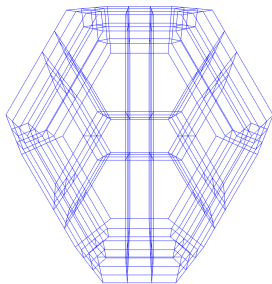
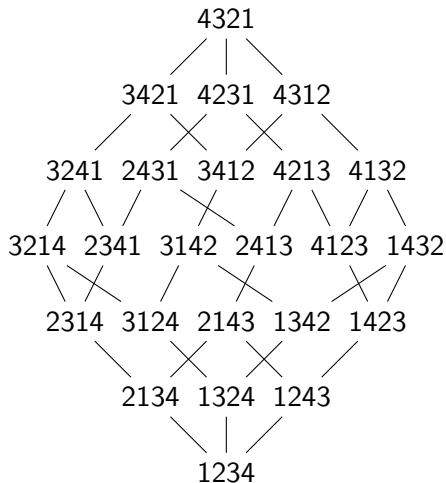
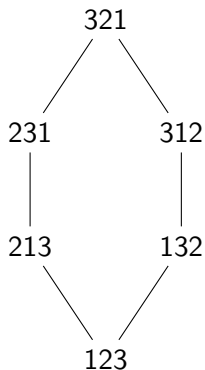


s -weak order and s -permutahedra

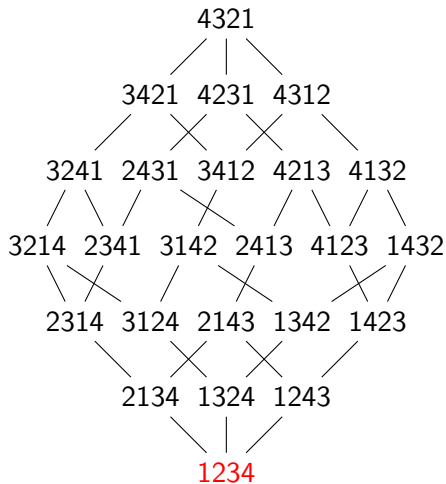
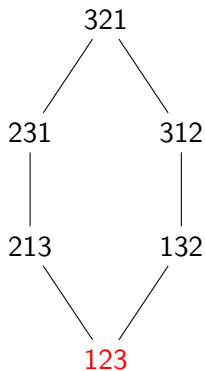
Cesar Ceballos – Viviane Pons
TU Graz – LISN, Univ. Paris-Saclay et IRL CRM Montréal



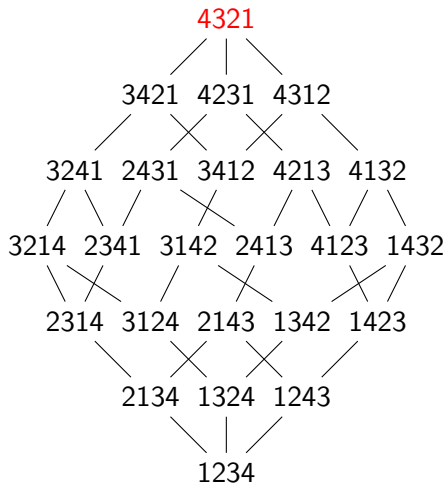
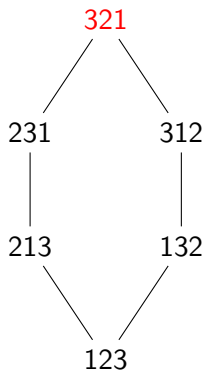
Weak Order



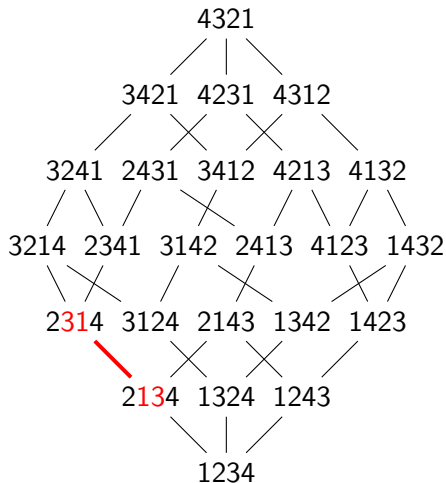
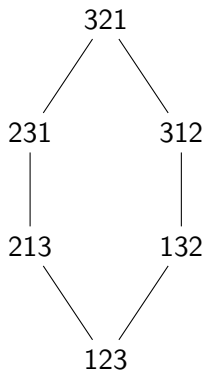
Weak Order



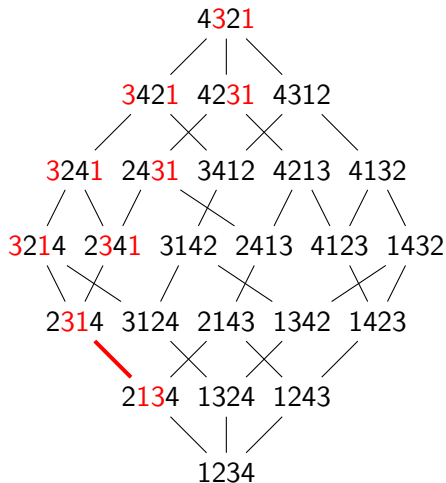
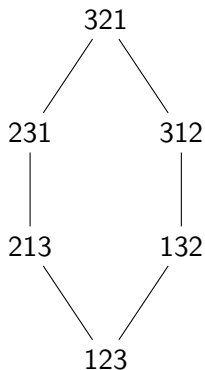
Weak Order



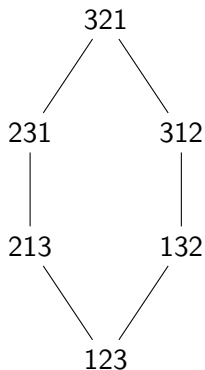
Weak Order



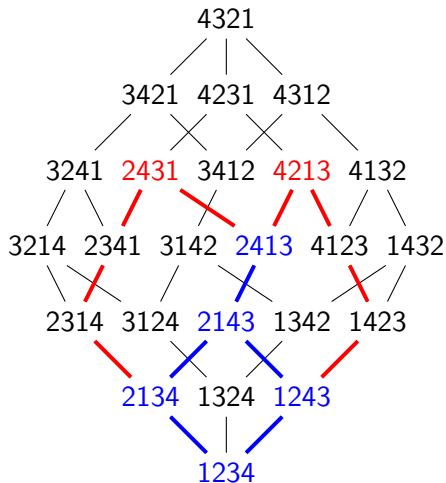
Weak Order



Weak Order

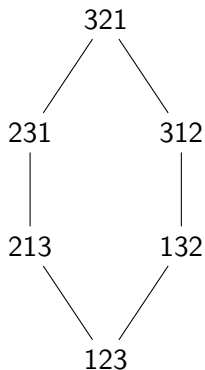


$$2413 \wedge 4213 = 2413$$

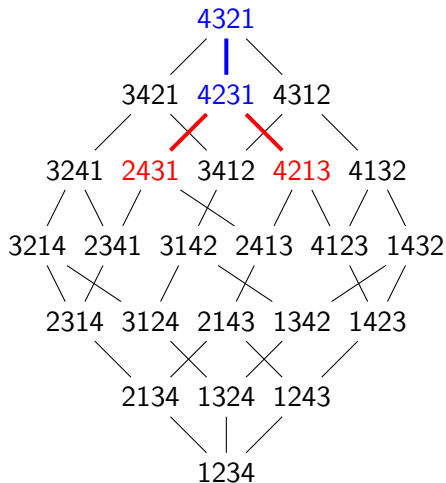


$$2413 \vee 4213 = 4231$$

Weak Order

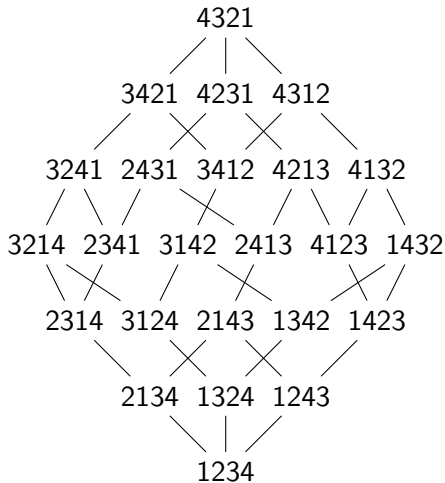
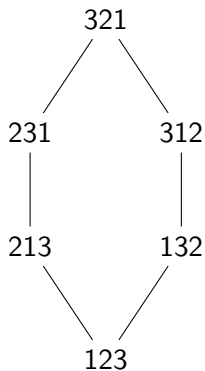


$$2413 \wedge 4213 = 2413$$

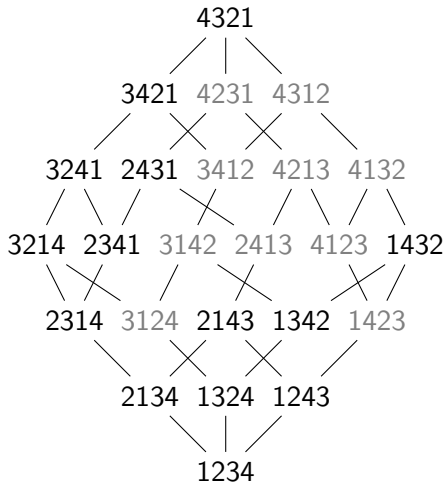
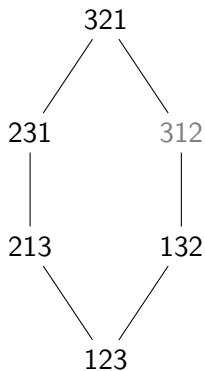


$$2413 \vee 4213 = 4231$$

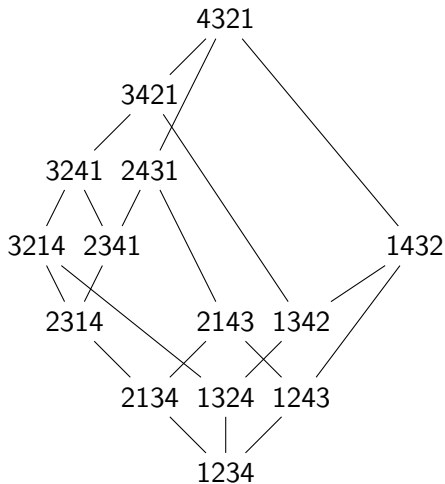
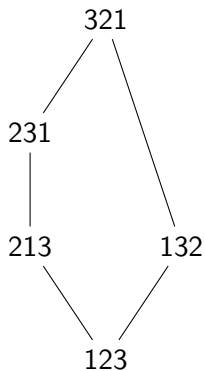
From the Weak Order to the Tamari lattice



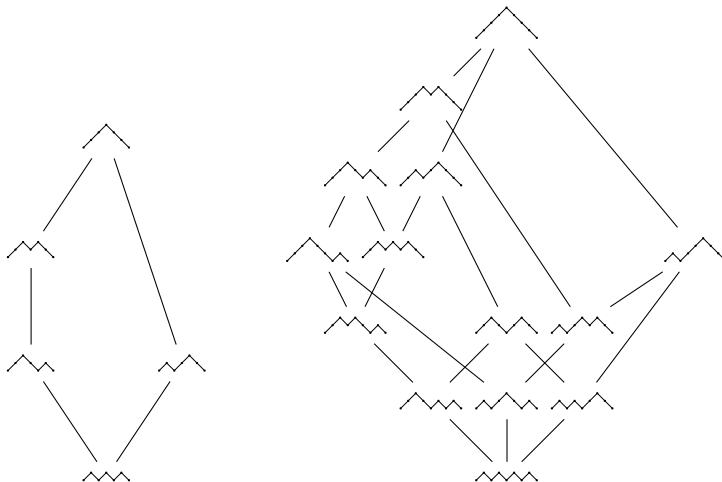
From the Weak Order to the Tamari lattice



From the Weak Order to the Tamari lattice

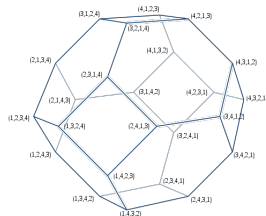


From the Weak Order to the Tamari lattice

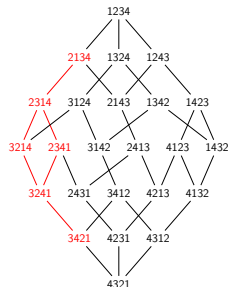


Context

Geometry

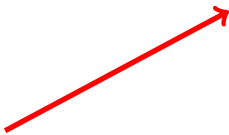


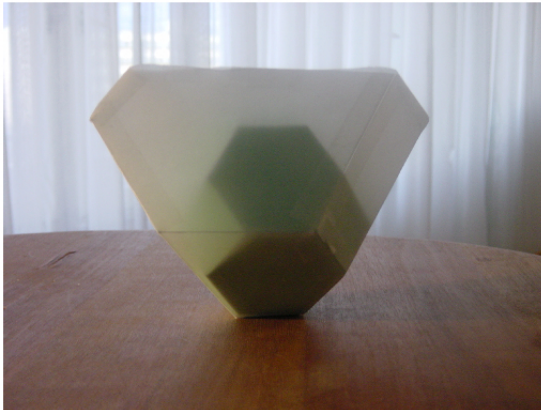
Combinatorics



Algebra

$$\begin{aligned}
 F_{21} \cdot F_{12} &= F_{21 \sqcup 12} \\
 &= F_{2134} + F_{2314} + F_{2341} + F_{3214} + F_{3241} + F_{3421}
 \end{aligned}$$

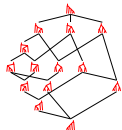




Weak order



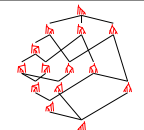
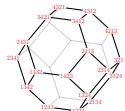
Tamari lattice



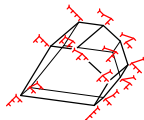
Weak order



Permutahedron



Tamari lattice

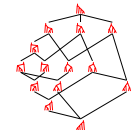
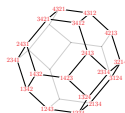


Associahedron

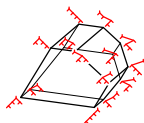
Weak order



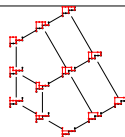
Permutahedron



Tamari lattice



Associahedron

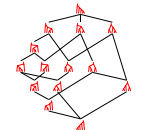
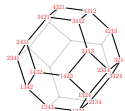
 ν -Tamari

Préville-Ratelle, Viennot

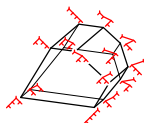
Weak order



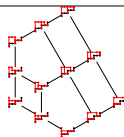
Permutahedron



Tamari lattice



Associahedron



ν -Tamari
Préville-Ratelle, Viennot

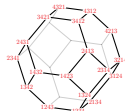


ν -Associahedron
Ceballos, Padrol, Sarmiento

Weak order

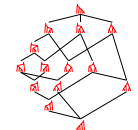


Permutahedron

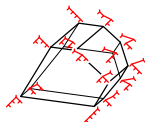


?

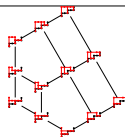
?



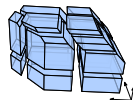
Tamari lattice



Associahedron



ν -Tamari
Préville-Ratelle, Viennot

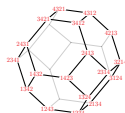
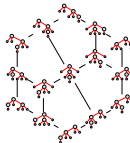
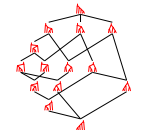
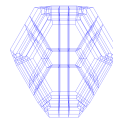


ν -Associahedron
Ceballos, Padrol, Sarmiento

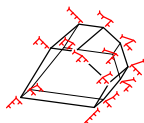
Weak order



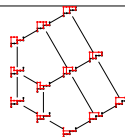
Permutahedron

 s -Weak order s -Permutahedron

Tamari lattice



Associahedron

 ν -Tamari
Préville-Ratelle, Viennot ν -Associahedron
Ceballos, Padrol, Sarmiento

s -decreasing trees

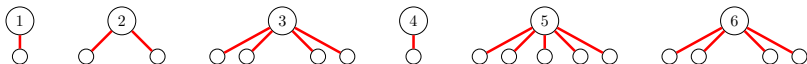
Let s be a sequence of n non-negative integers. An s -decreasing tree is a planar tree labeled with $1 \dots n$ such that each node i has $s(i) + 1$ children and labels are decreasing from root to leaves.

$$s = (0, 1, 3, 0, 4, 3)$$

s -decreasing trees

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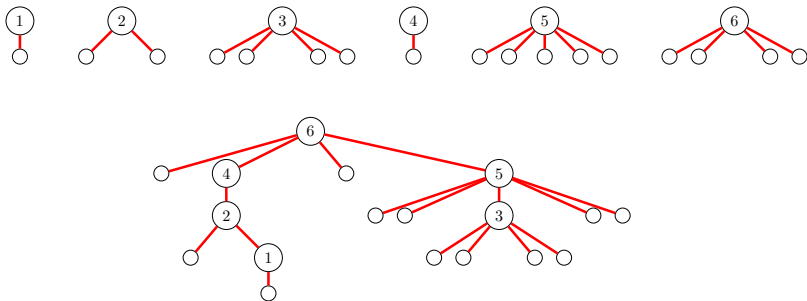
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How many trees?

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Number of s -decreasing trees:

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$$s = (0, 1, 3, 0, 4, 3)$$

Number of s -decreasing trees:

$$(1+3)$$

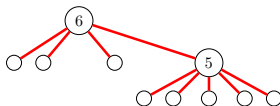


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s -decreasing trees:

$$(1+3) \times (1+3+4)$$

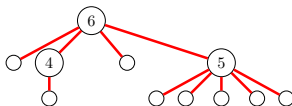


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s -decreasing trees:

$$(1+3) \times (1+3+4) \times (1+3+4+0)$$

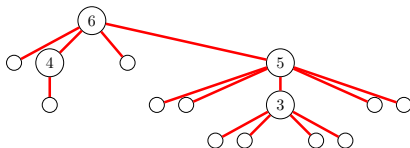


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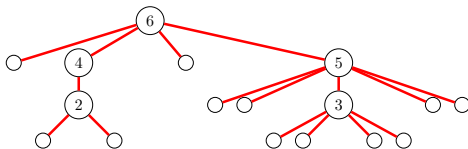


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$$(1+3) \times (1+3+4) \times (1+3+4+0) \times (1+3+4+0+3) \times (1+3+4+0+3+1)$$

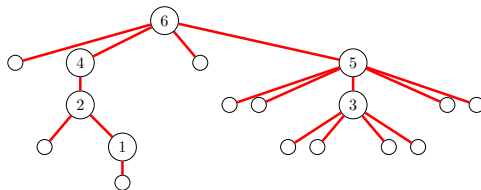


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s -decreasing trees:

$$(1+3) \times (1+3+4) \times (1+3+4+0) \times (1+3+4+0+3) \times (1+3+4+0+3+1)$$

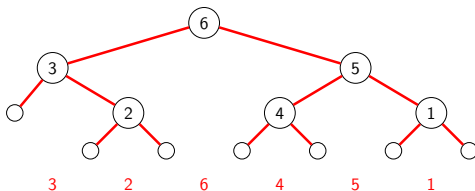


Permutations

$$s = (1, 1, 1, 1, 1, 1)$$

Number of s -decreasing trees:

$$(1+1) \times (1+1+1) \times (1+1+1+1) \times (1+1+1+1+1) \times (1+1+1+1+1+1)$$

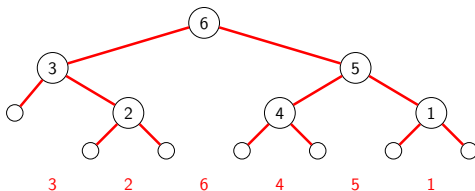


Permutations

$$s = (1, 1, 1, 1, 1, 1)$$

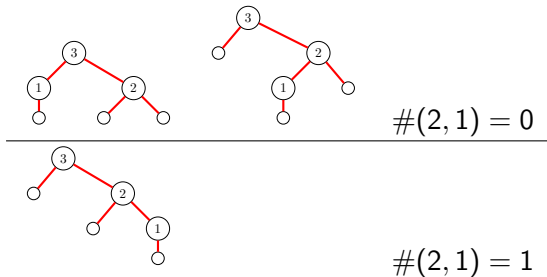
Number of s -decreasing trees: $6!$

$$(1+1) \times (1+1+1) \times (1+1+1+1) \times (1+1+1+1+1) \times (1+1+1+1+1+1)$$



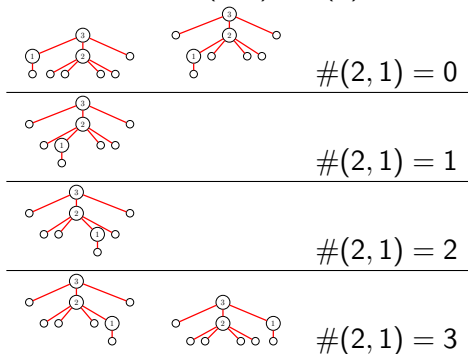
Tree-inversions

For all $b > a$, we define $0 \leq \#(b, a) \leq s(b)$.



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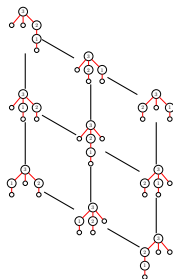
The s -weak order

R, T , s -decreasing trees:

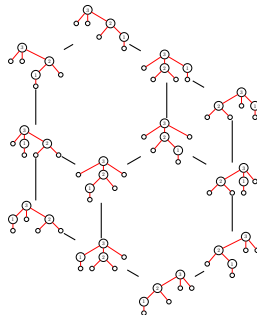
$$R \preceq T \Leftrightarrow \forall b > a, \#_R(b, a) \leq \#_T(b, a)$$

Theorem

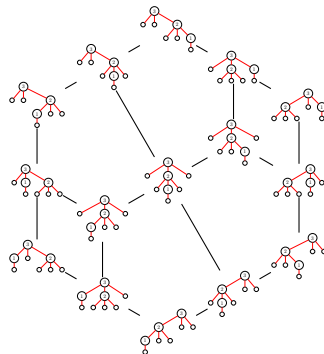
The s -weak order is always a lattice.



$(0, 0, 2)$



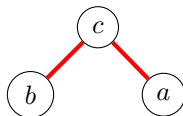
$(0, 1, 2)$



$(0, 2, 2)$

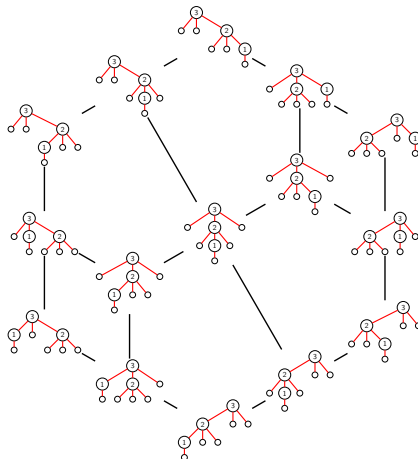
s -Tamari lattice

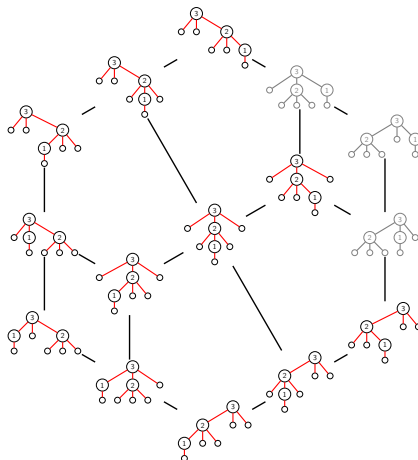
Select trees which avoid “pattern 231”: $a < b < c$

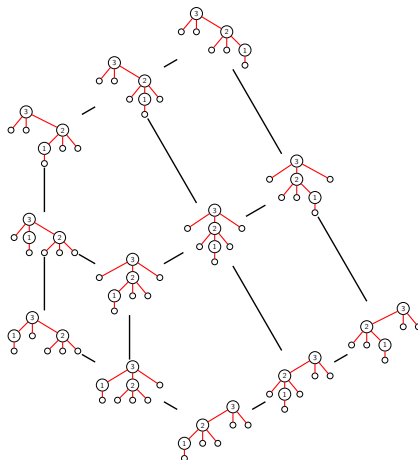


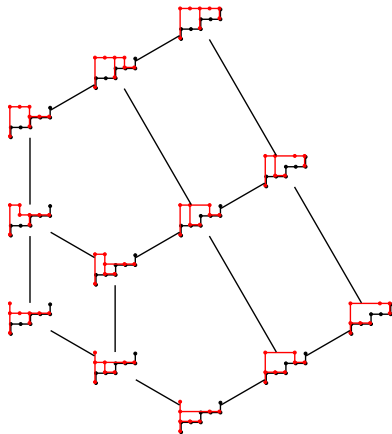
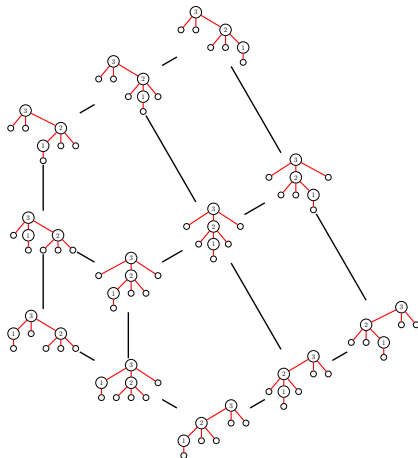
Theorem

The set of 231-avoiding s -decreasing trees form a sublattice, the s -Tamari lattice, isomorphic to the ν -Tamari lattice.

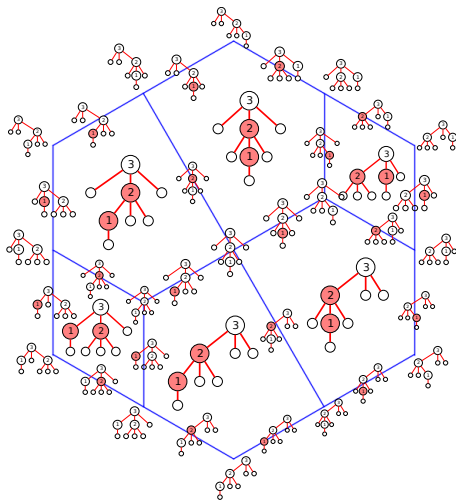


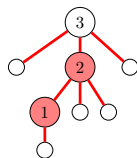


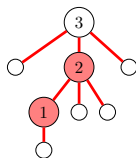


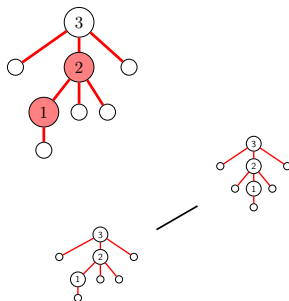


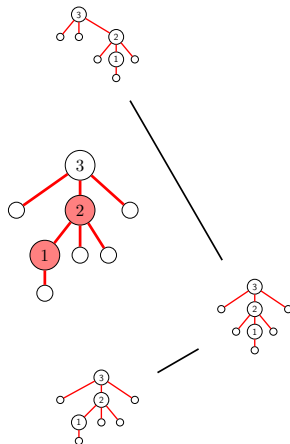
Geometry: the s -Permutahedron

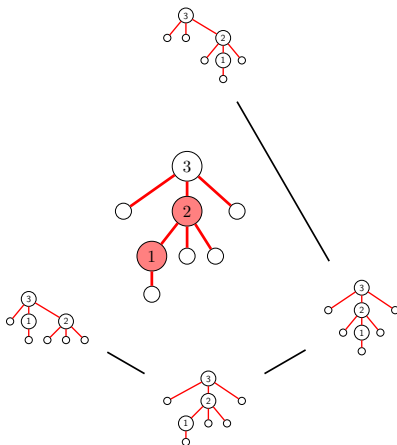


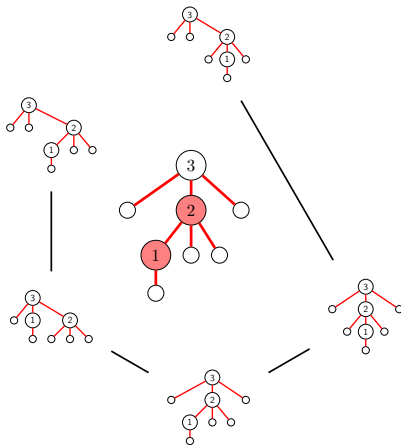


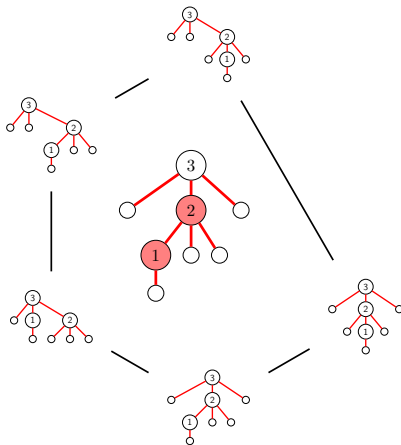


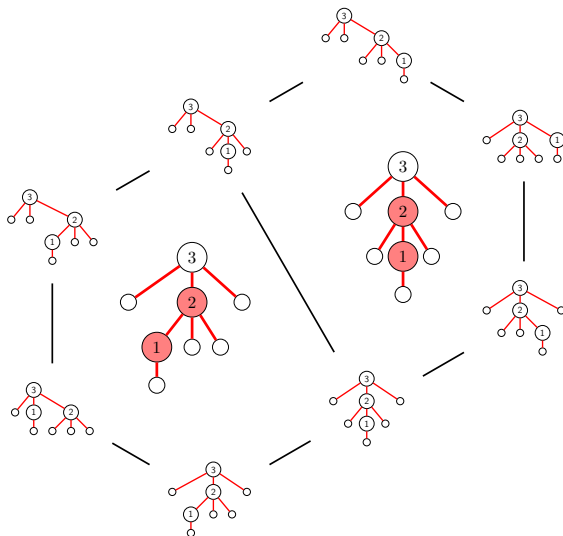


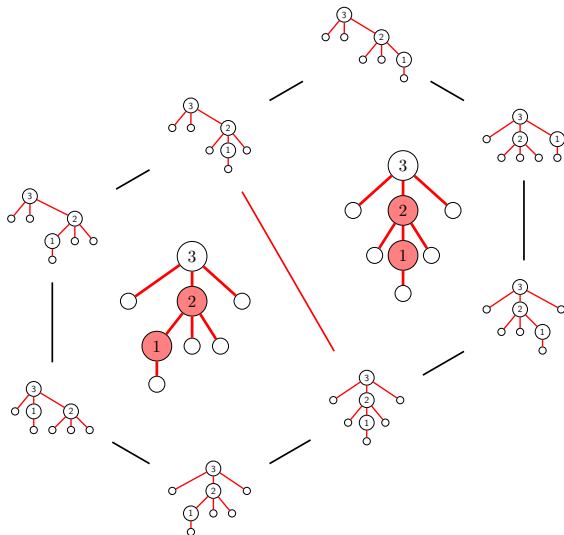


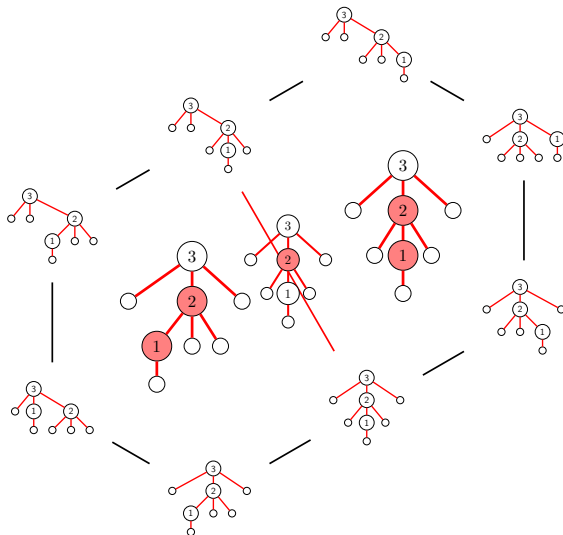












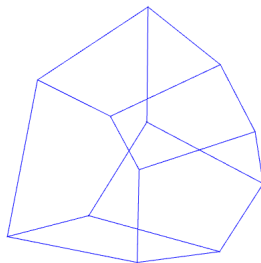
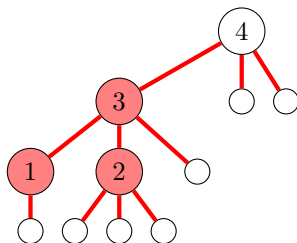
Polytopal complex? Ascentopes

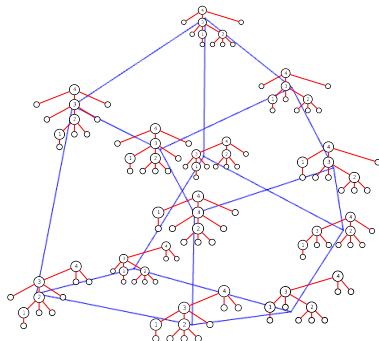
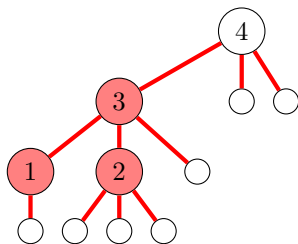
Can each face be realized as a polytope?

Polytopal complex? Ascentopes

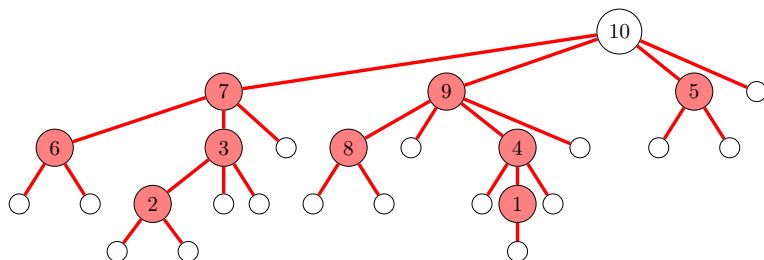
Can each face be realized as a polytope?

To each facet, we associate a **generalized permutahedron**: we define a injection between maximal faces included in a given facet and the facets of the permutahedron.





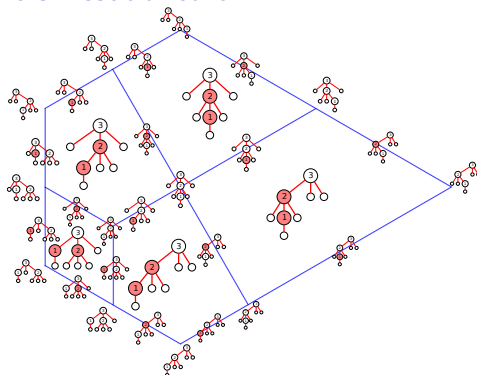
- ▶ 12 vertices
- ▶ 18 edges
- ▶ 8 facets



f vector =

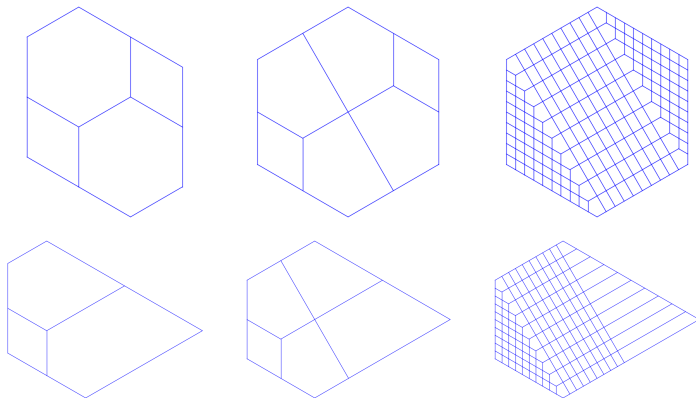
(1, 2178, 9801, 19008, 20790, 14082, 6099, 1680, 282, 26, 1)

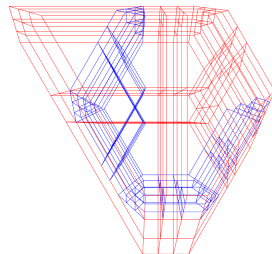
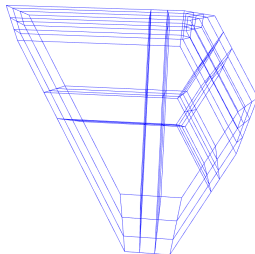
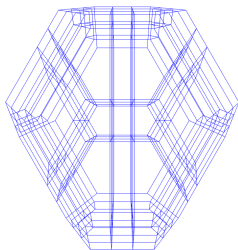
The s -Associahedron



Theorem: The s -associahedron is isomorphic to the ν -associahedron for $\nu = NE^{s(n)} \dots NE^{s(1)}$.

Polytopal subdivision





Conjecture 1

The s -permutahedron can be realized as a polytopal subdivision of the permutahedron.

Conjecture 2

One can obtain a realization of the s -associahedron by removing some facets of the s -permutahedron realization.