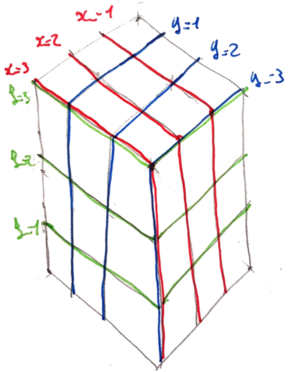
Viviane Pons

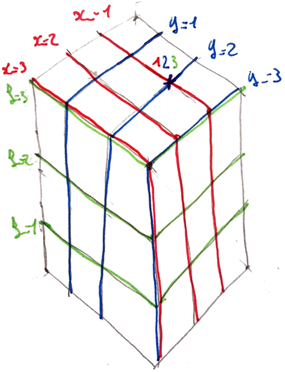
Maîtresse de conférences, Univ. Paris-Saclay
viviane.pons@lri.fr

Combinatorial aspects of the Tamari lattices

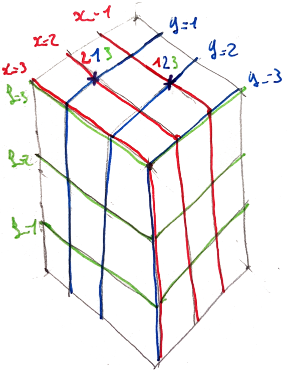
The Permutahedron



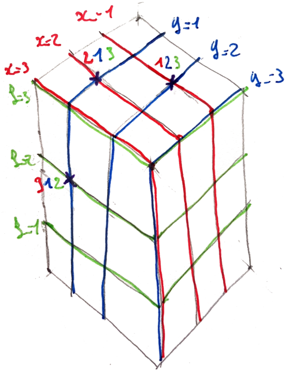
The Permutahedron



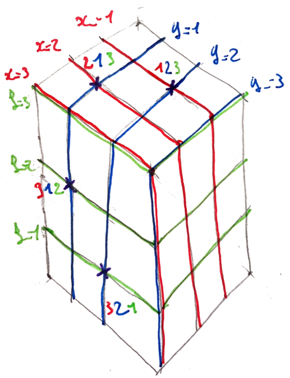
The Permutahedron



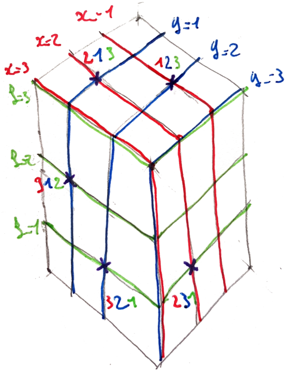
The Permutahedron



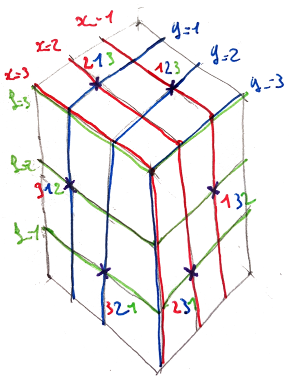
The Permutahedron



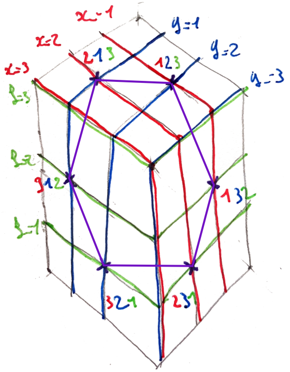
The Permutahedron



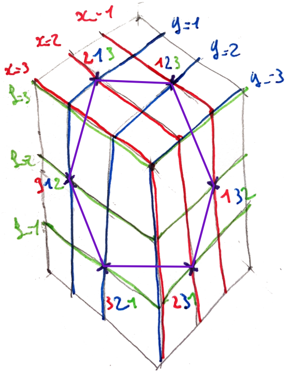
The Permutahedron



The Permutahedron



The Permutahedron



$$x + y + z = 6$$

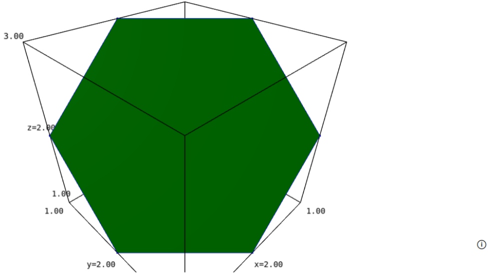
Using SageMath

Entrée [1]:
```
P = Polyhedron(list(Permutations(3)))
P
```

Out[1]: A 2-dimensional polyhedron in ZZ^3 defined as the convex hull of 6 vertices (use the .plot() method to plot)

Entrée [2]:
```
P.plot()
```

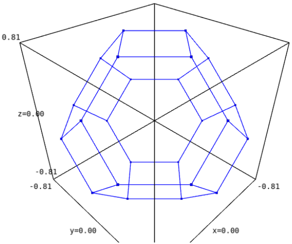
Out[2]:



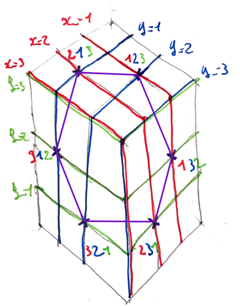
For size 4

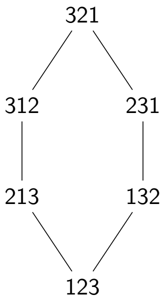
Entrée [3]:
```
P = Polyhedron(list(Permutations(4)))
P
```
Out[3]: A 3-dimensional polyhedron in ZZ^4 defined as the convex hull of 24 vertices (use the .plot() method to plot)

Entrée [4]:
```
P.plot()
```
Out[4]:

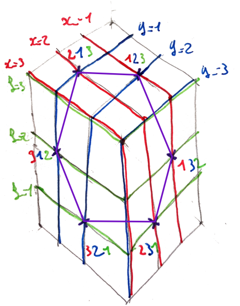


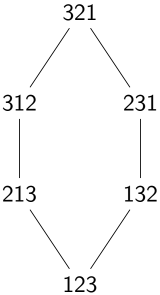
Looking at the “skeleton” of the polytope”





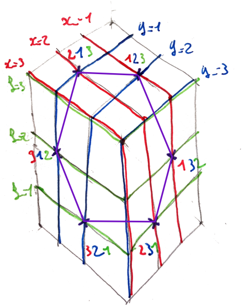
Looking at the “skeleton” of the polytope”

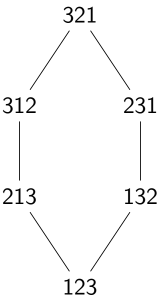


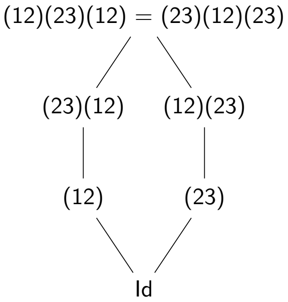


The *the left weak order* on permutations

Looking at the “skeleton” of the polytope”

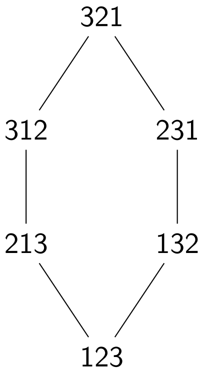




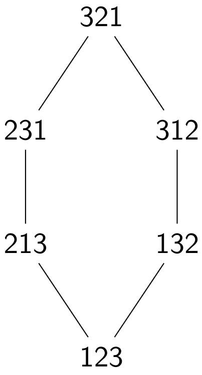


The *the left weak order* on permutations

The left and right weak orders

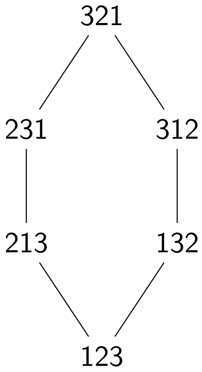


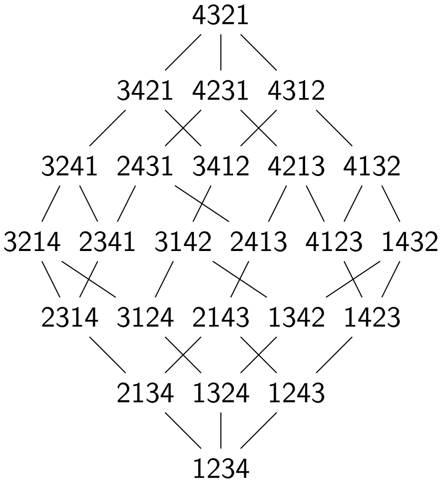
left weak order



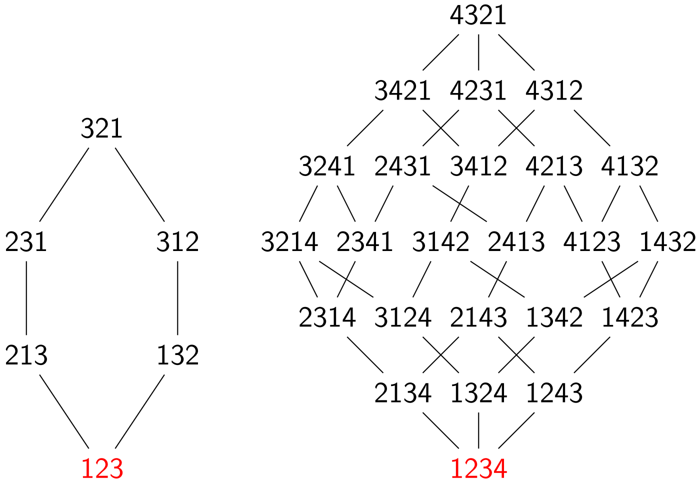
right weak order

Right weak Order

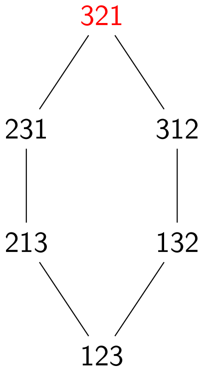


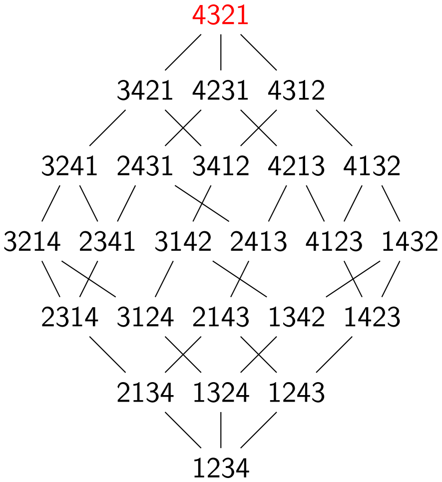


Right weak Order

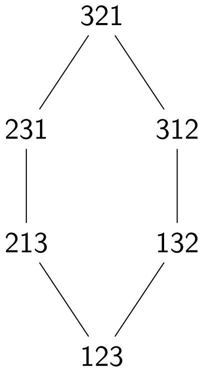


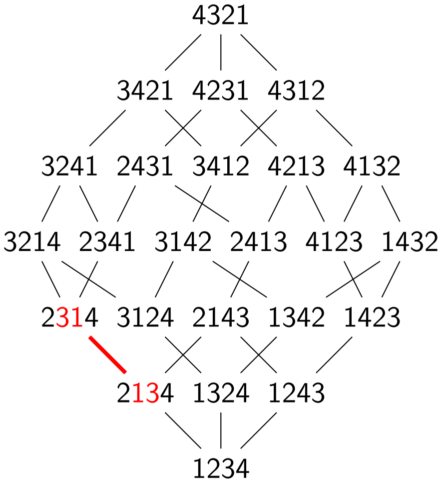
Right weak Order



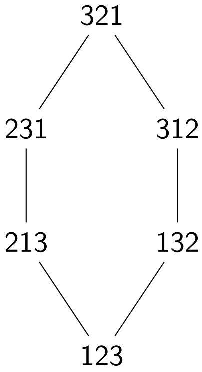


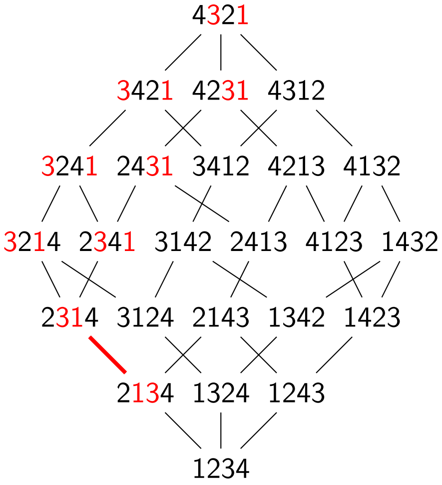
Right weak Order



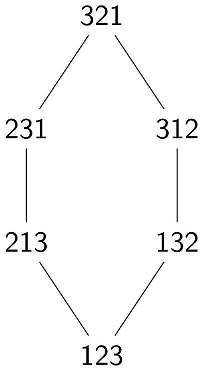


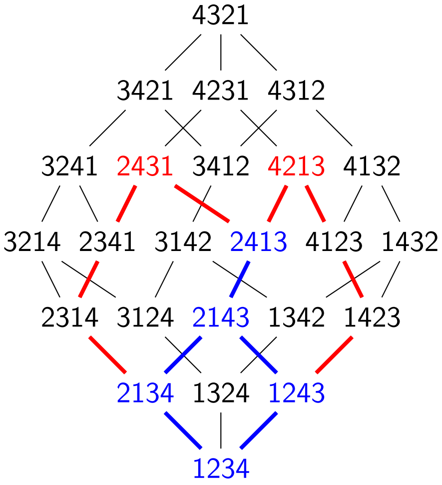
Right weak Order





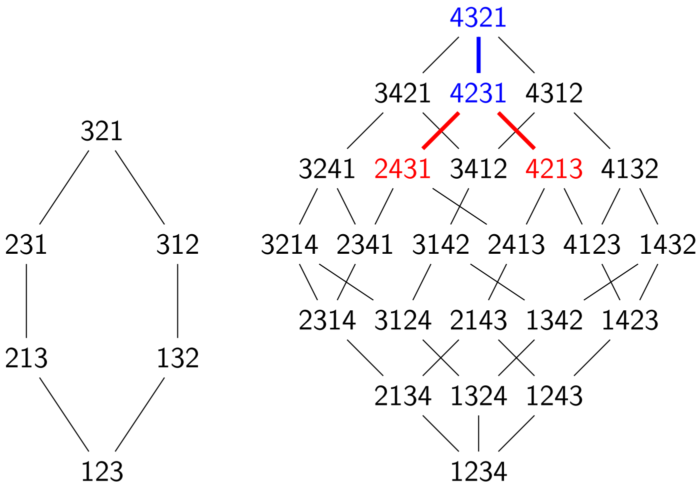
Right weak Order





$2413 \wedge 4213 = 2413$ $\qquad$ $2413 \vee 4213 = 4231$

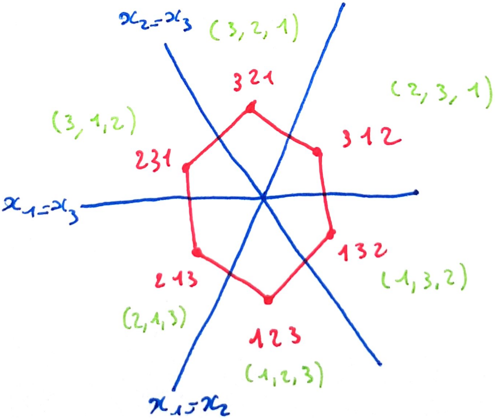
Right weak Order



$2413 \wedge 4213 = 2413$

$2413 \vee 4213 = 4231$

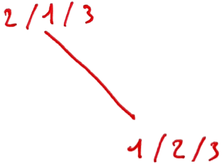
As a reflection group



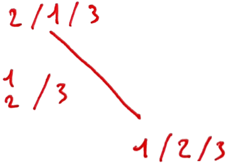
Combinatorics of faces

1 / 2 / 3

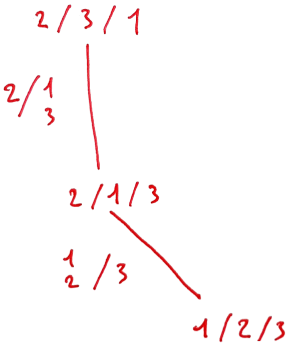
Combinatorics of faces



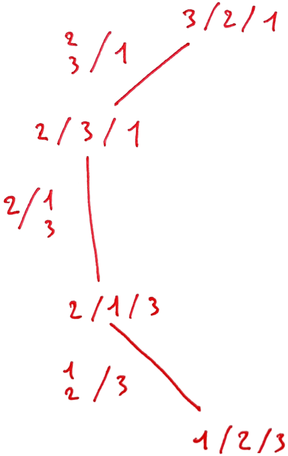
Combinatorics of faces



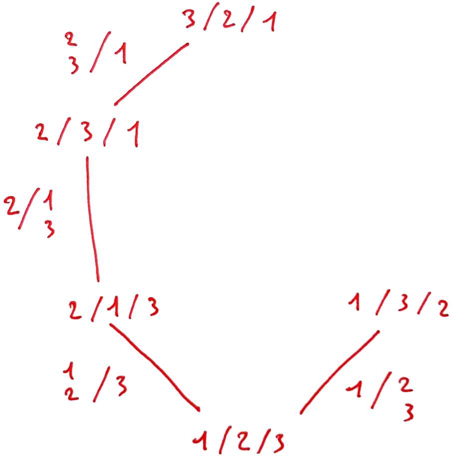
Combinatorics of faces



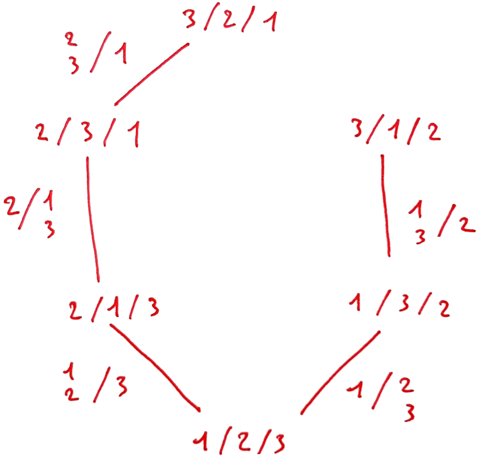
Combinatorics of faces



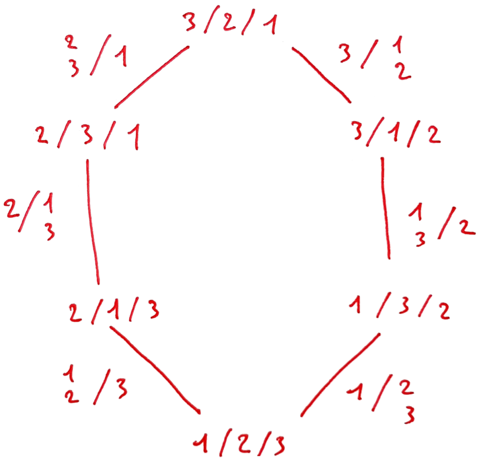
Combinatorics of faces



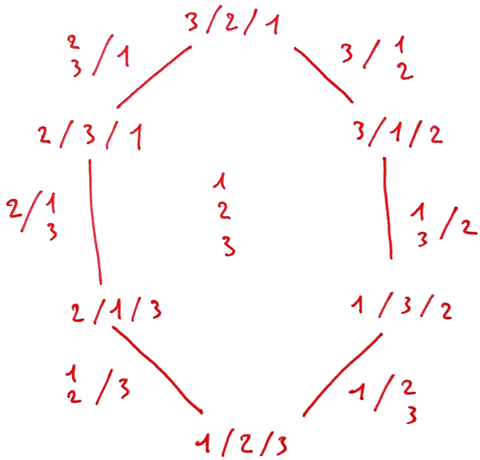
Combinatorics of faces



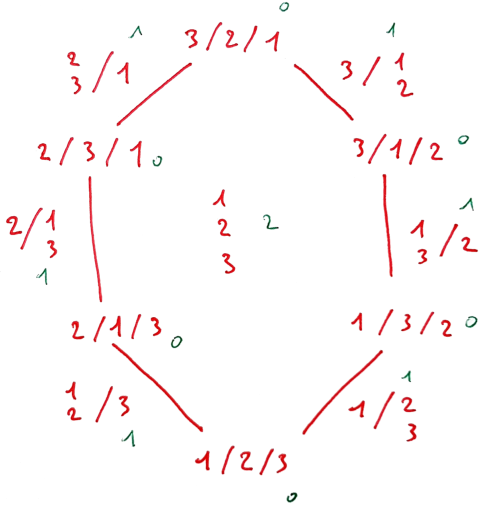
Combinatorics of faces



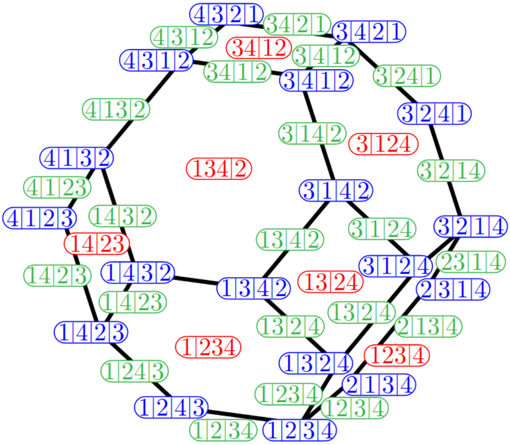
Combinatorics of faces



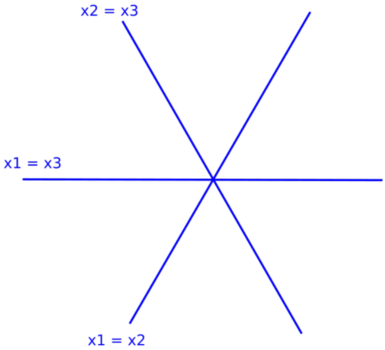
Combinatorics of faces

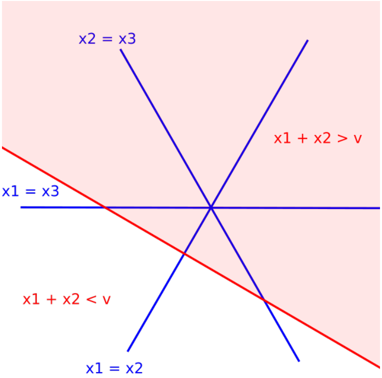


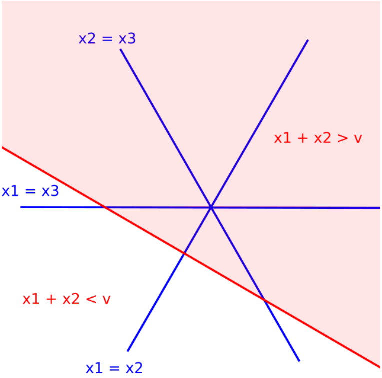
$$d = n - \#parts$$



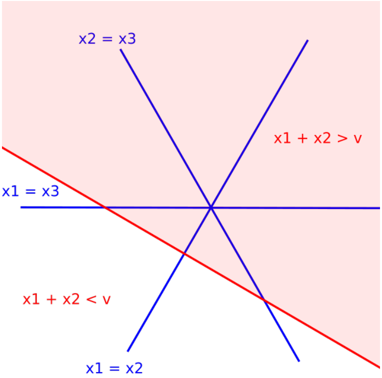
(image from V. Pilaud's talk "The Associahedron and its friends")


x2 = x3
x1 = x3
x1 = x2

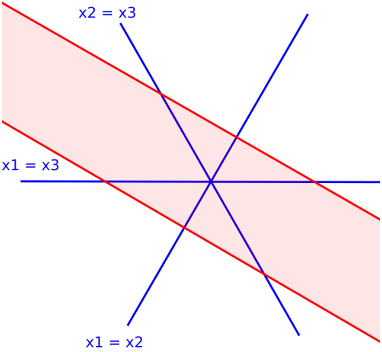

x2 = x3
x1 + x2 > v
x1 = x3
x1 + x2 < v
x1 = x2



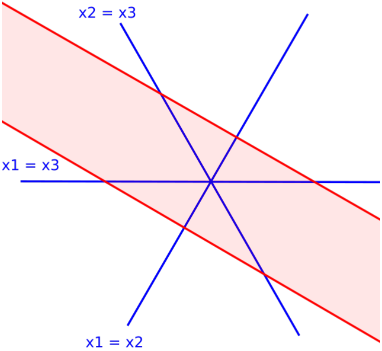
$$x_1 + x_2 + x_3 = 6$$



$$x_1 + x_2 = 6 - x_3$$

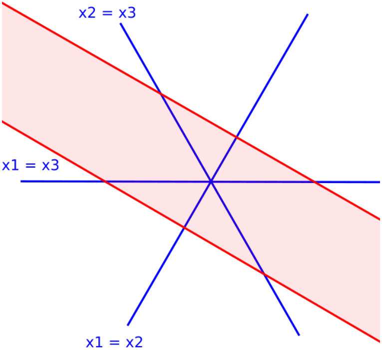


$$3 \leq x_1 + x_2 \leq 5$$



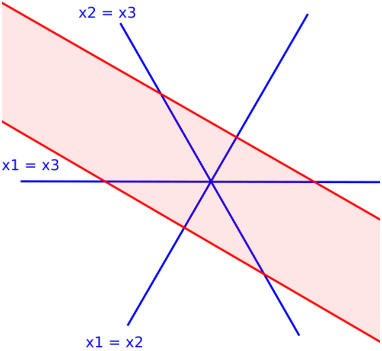
$$x_1 + x_2 \geq 3$$

$$x_1 + x_2 \leq 5$$



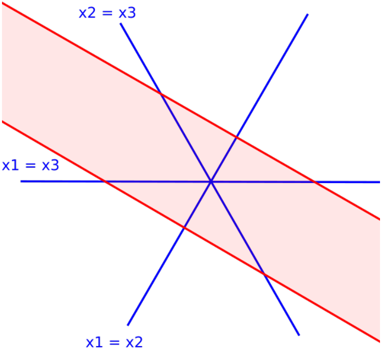
$$x_1 + x_2 \geq 3$$

$$x_3 \geq 1$$



$12|3 \quad x_1 + x_2 \geq 3$

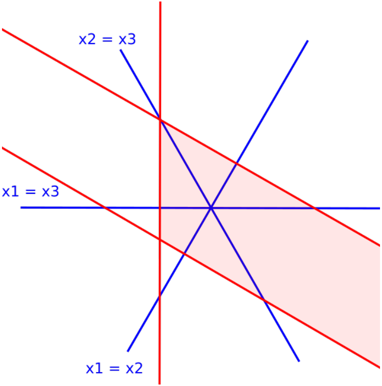
$3|12 \quad x_3 \geq 1$



$$J \subseteq [n] \rightarrow \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$

12|3 $\quad x_1 + x_2 \geq 3$

3|12 $\quad x_3 \geq 1$

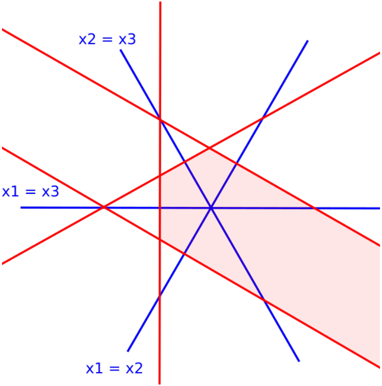


$$J \subseteq [n] \rightarrow \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$

$12|3 \quad x_1 + x_2 \geq 3$

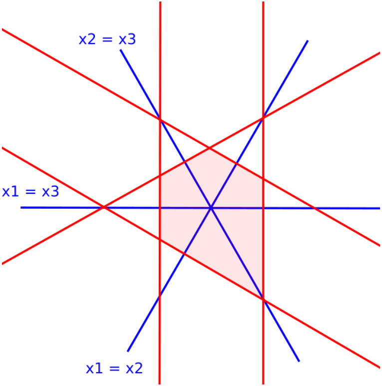
$2|13 \quad x_2 \geq 1$

$3|12 \quad x_3 \geq 1$



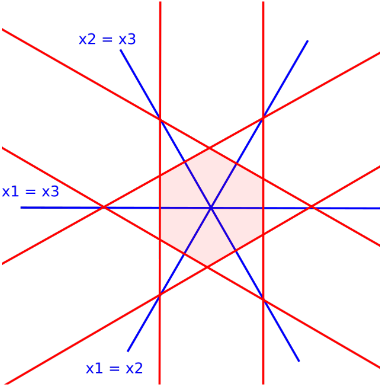
$$J \subseteq [n] \rightarrow \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$

$12 \mid 3$	$x_1 + x_2 \geq 3$
$2 \mid 13$	$x_2 \geq 1$
$23 \mid 1$	$x_2 + x_3 \geq 3$
$3 \mid 12$	$x_3 \geq 1$



$$J \subseteq [n] \rightarrow \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$

$12\|3$	$x_1 + x_2 \geq 3$
$2\|13$	$x_2 \geq 1$
$23\|1$	$x_2 + x_3 \geq 3$
$3\|12$	$x_3 \geq 1$
$13\|2$	$x_1 + x_3 \geq 3$



$$J \subseteq [n] \rightarrow \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$

12|3	$x_1 + x_2 \geq 3$
2|13	$x_2 \geq 1$
23|1	$x_2 + x_3 \geq 3$
3|12	$x_3 \geq 1$
13|2	$x_1 + x_3 \geq 3$
1|23	$x_1 \geq 1$

From the weak order to the Tamari lattice

We define a *surjection* from permutations to *binary trees* which gives us a new lattice.

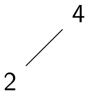
Binary search tree insertion

4

15324 $\rightarrow$

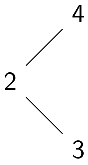
Binary search tree insertion

$15324 \rightarrow$



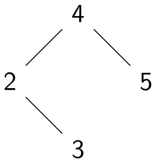
Binary search tree insertion

15324 $\rightarrow$



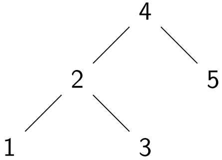
Binary search tree insertion

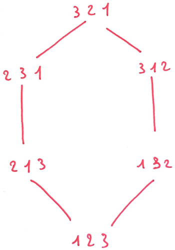
15324 $\rightarrow$

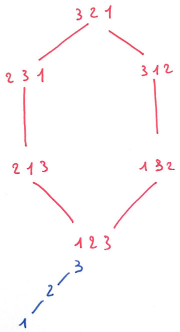


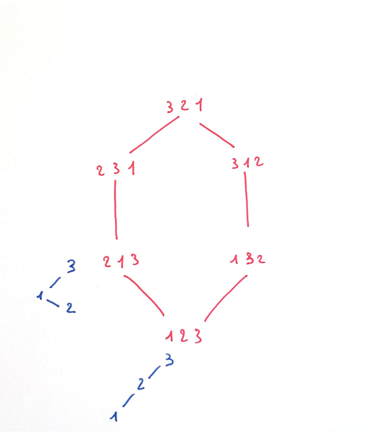
Binary search tree insertion

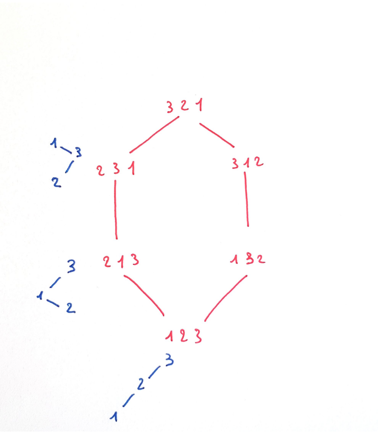
15324 →

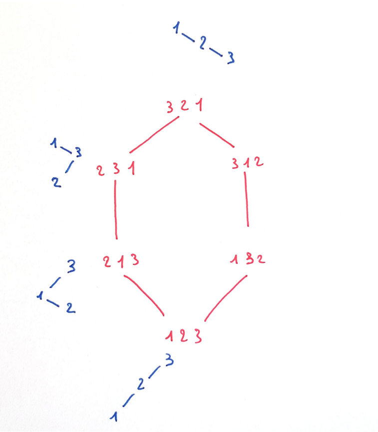



321
231
312
213
132
123

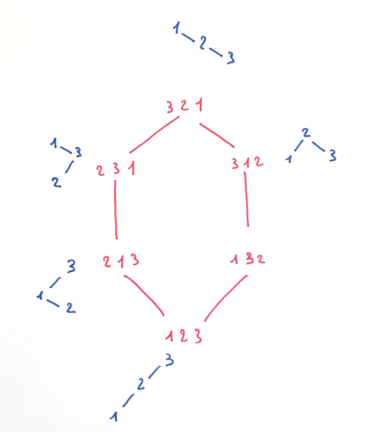

321
231
312
213
132
123
1
2
3

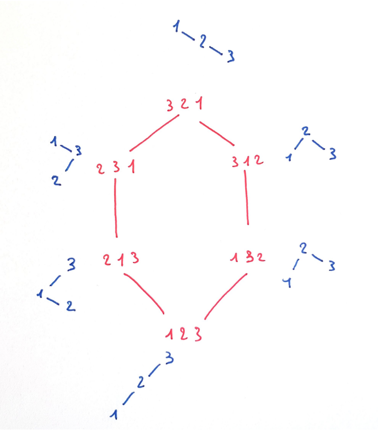

3 2 1
2 3 1
3 1 2
2 1 3
1 3 2
1 2 3
3
1
2
3
2
1

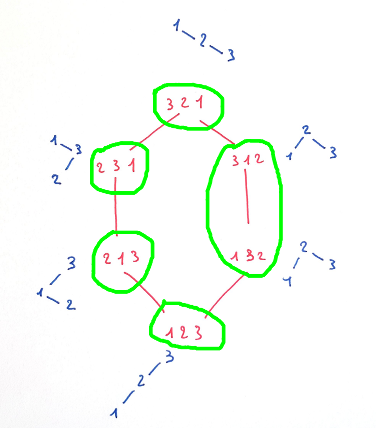

3 2 1
2 3 1
3 1 2
2 1 3
1 3 2
1 2 3

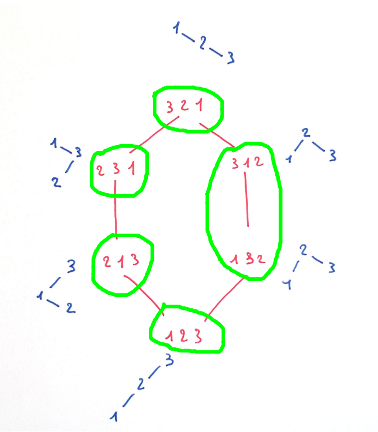


3 2 1
2 3 1
3 1 2
2 1 3
1 3 2
1 2 3

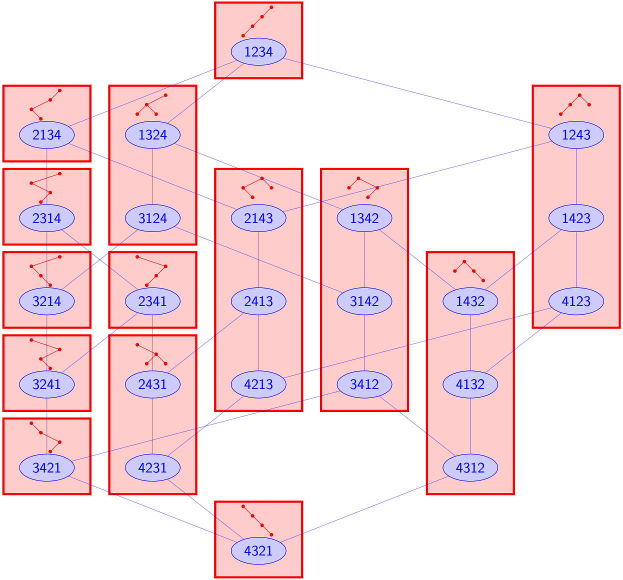

321
231
312
213
132
123


321
231
312
213
132
123


321
231
312
213
132
123



The **Tamari lattice** is a lattice on binary trees. It is a **quotient lattice** of the weak order.


1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

More about the Tamari lattice

Binary trees are counted by the Catalan numbers

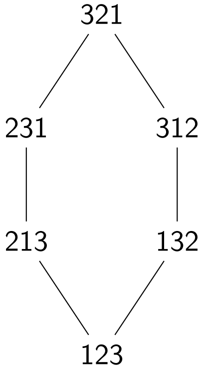
$$\frac{1}{n+1}\binom{2n}{n}$$

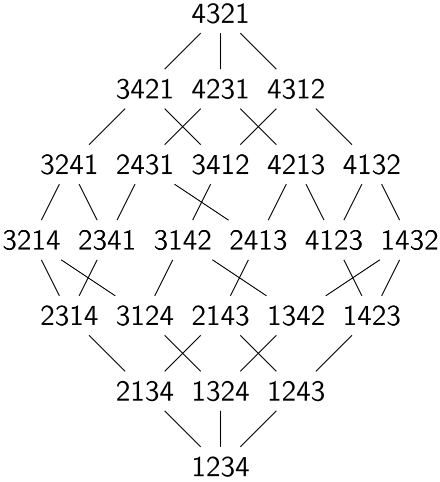
.

The Tamari lattice can be defined on **many** families of combinatorial objects, such as

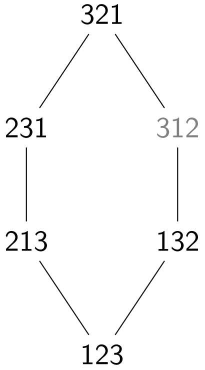
- Triangulations of regular polygons
- Dyck paths
- Ordered forests
- certain pattern avoiding permutations (312 avoiding and 231 avoiding)
- ways to parenthesize an expression (original definition of Tamari)
- ...

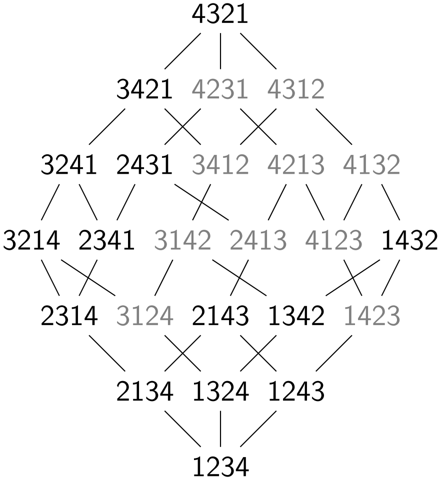
312 - avoiding permutations



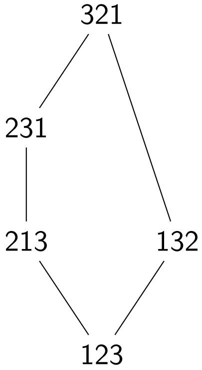


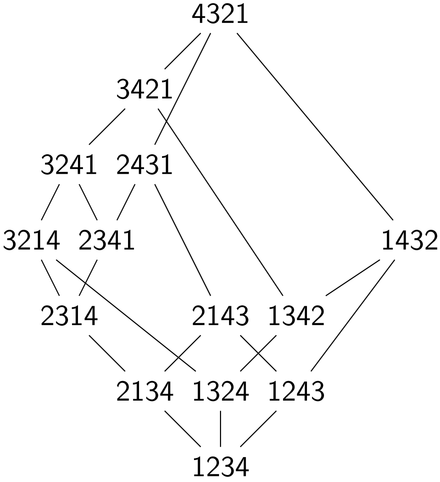
312 - avoiding permutations



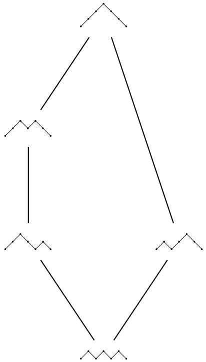


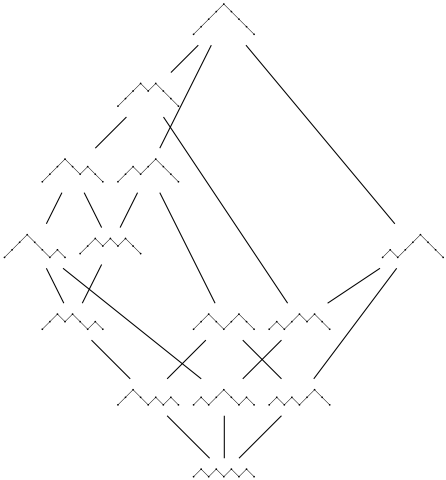
312 - avoiding permutations





Dyck paths





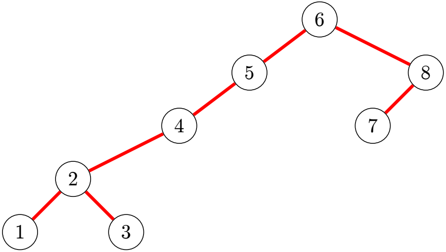
The Associahedron – Stasheff polytope

Different constructions

- Loday 2004
- Billera-Filliman-Sturmfels 1990
- Gelfand-Kapranov-Zelevinsky 1994
- Chapoton-Fomin-Zelevinsky 2002
- Hohlweg-Lange 2007
- Ceballos-Santos-Ziegler 2011
- Hohlweg-Lange-Thomas 2012

Loday's Associahedron

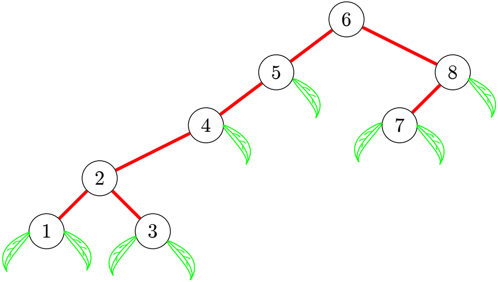
$i \rightarrow$ (# of left leaves) $\times$ (# of right leaves)



$$(\ , \ , \ , \ , \ , \quad , \ , \ )$$

Loday's Associahedron

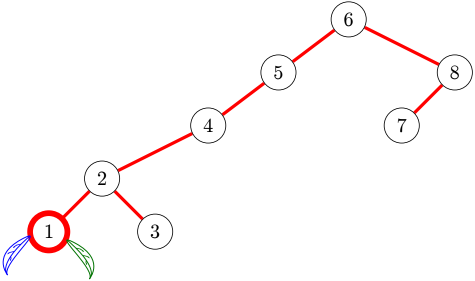
$i \rightarrow$ (# of left leaves) $\times$ (# of right leaves)



$(\; , \; , \; , \; , \; , \quad , \; , \;)$

Loday's Associahedron

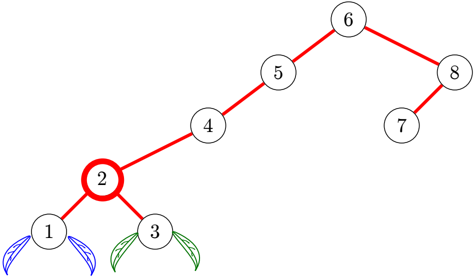
$i \rightarrow$ (# of left leaves) $\times$ (# of right leaves)



$$(1, \ , \ , \ , \ , \ \ , \ , \ )$$

Loday's Associahedron

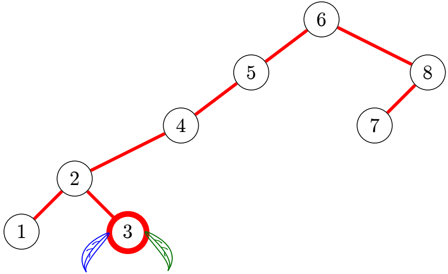
$i \rightarrow (\#\text{ of left leaves}) \times (\#\text{ of right leaves})$



$$(1, 4, \ , \ , \ , \quad , \ , \ )$$

Loday's Associahedron

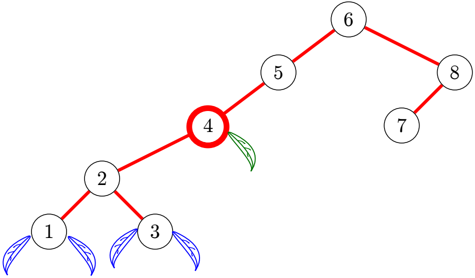
$i \to$ (# of left leaves) $\times$ (# of right leaves)



$$(1, 4, 1, \ , \ , \ \ , \ , \ )$$

Loday's Associahedron

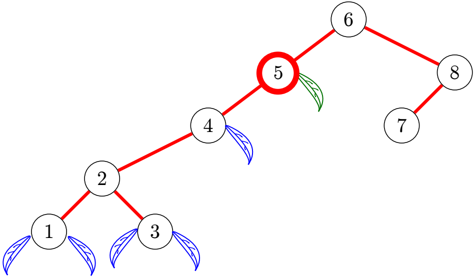
$i \rightarrow$ (# of left leaves) $\times$ (# of right leaves)



$$(1, 4, 1, 4, \ , \quad , \ , \ )$$

Loday's Associahedron

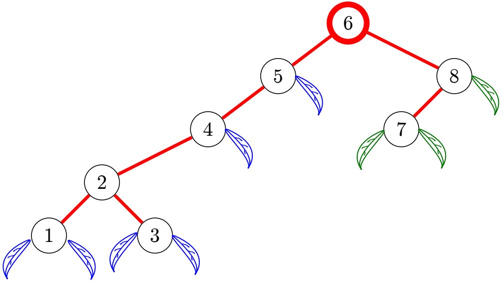
$i \rightarrow$ (# of left leaves) $\times$ (# of right leaves)



$$(1, 4, 1, 4, 5, \quad , \quad , \quad)$$

Loday's Associahedron

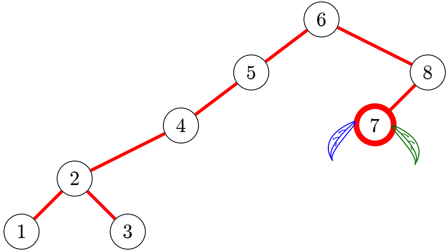
$i \rightarrow$ (# of left leaves) $\times$ (# of right leaves)



$$(1, 4, 1, 4, 5, 18, \quad , \quad)$$

Loday's Associahedron

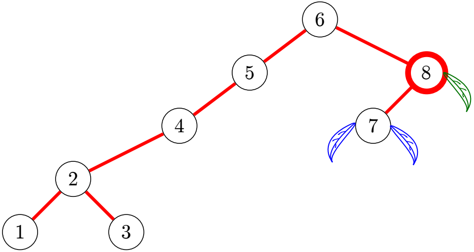
$i \rightarrow$ (# of left leaves) $\times$ (# of right leaves)



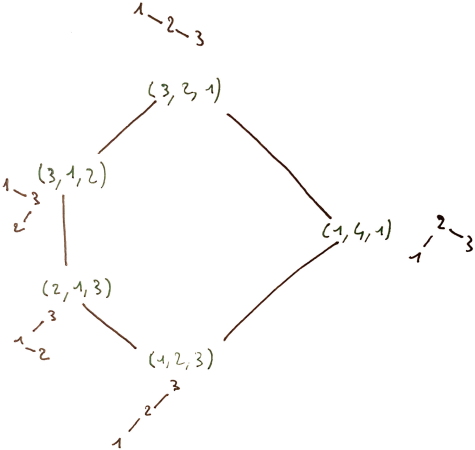
$$(1, 4, 1, 4, 5, 18, 1, \quad)$$

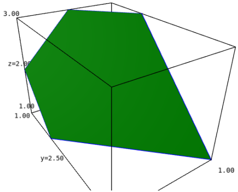
Loday's Associahedron

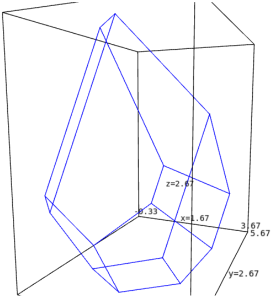
$i \rightarrow$ (# of left leaves) $\times$ (# of right leaves)



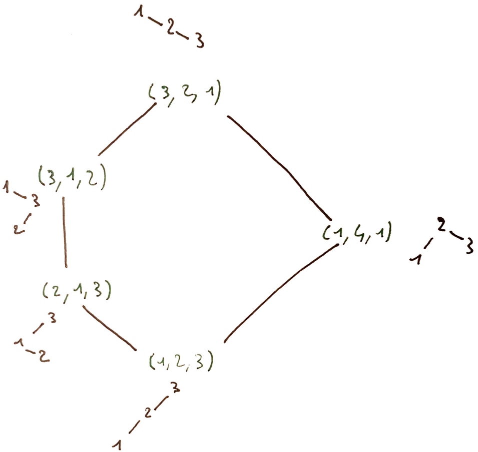
$(1, 4, 1, 4, 5, 18, 1, 8)$


(3, 2, 1)
(3, 1, 2)
(2, 1, 3)
(1, 2, 3)

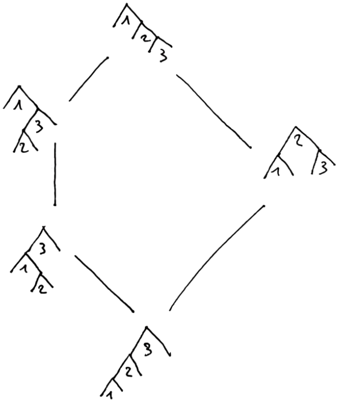

3.00
z=2.00
1.00
1.00
y=2.50
1.00


z=2.67
-0.33
x=1.67
3.67
5.67
y=2.67

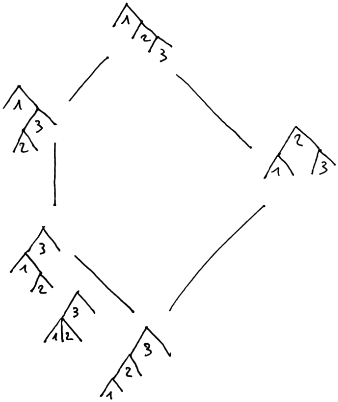
Schröder trees



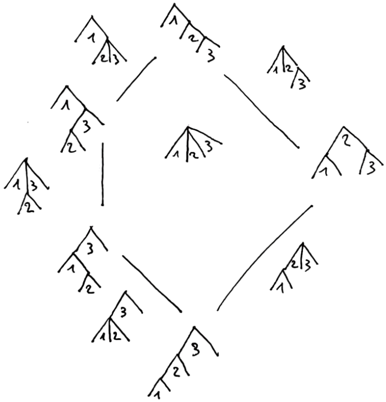
Schröder trees



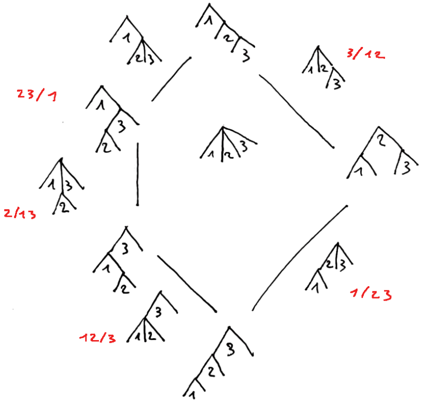
Schröder trees

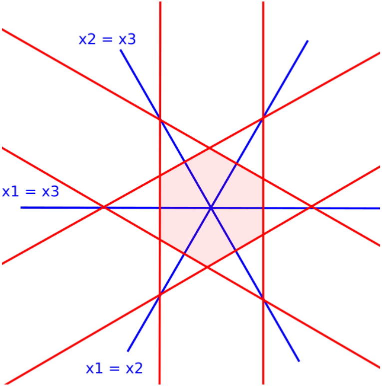


Schröder trees

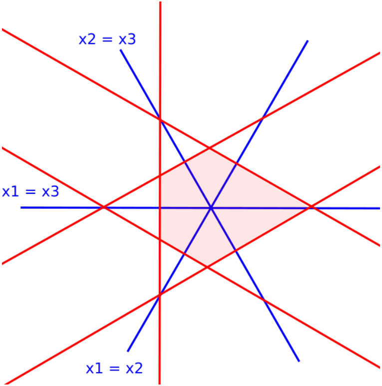


Schröder trees

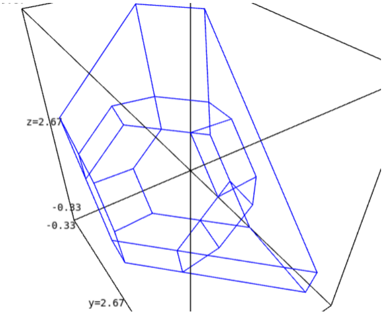




12|3	$x_1 + x_2 \geq 3$
2|13	$x_2 \geq 1$
23|1	$x_2 + x_3 \geq 3$
3|12	$x_3 \geq 1$
13|2	$x_1 + x_3 \geq 3$
1|23	$x_1 \geq 1$

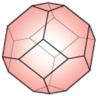


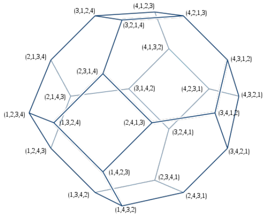
$12\|3$	$x_1 + x_2 \geq 3$
$2\|13$	$x_2 \geq 1$
$23\|1$	$x_2 + x_3 \geq 3$
$3\|12$	$x_3 \geq 1$
$13\|2$	
$1\|23$	$x_1 \geq 1$


z=2.67
-0.33
-0.33
y=2.67

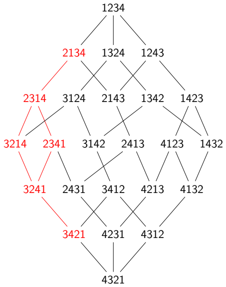
The Malvenuto–Reutenauer Hopf algebra

Geometry





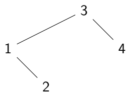
Combinatorics



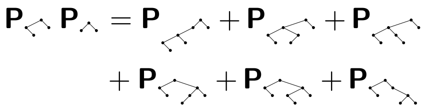
Algebra

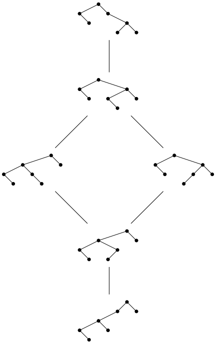
$$\mathbf{F}_{21} \,.\, \mathbf{F}_{12} = \mathbf{F}_{21 \sqcup\!\sqcup 12}$$
$$= \mathbf{F}_{2134} + \mathbf{F}_{2314} + \mathbf{F}_{2341} + \mathbf{F}_{3214} + \mathbf{F}_{3241} + \mathbf{F}_{3421}$$

The Loday–Ronco Hopf algebra

$\mathbf{P}$ 

$= \mathbf{F}_{2143} + \mathbf{F}_{2413} + \mathbf{F}_{4213}$

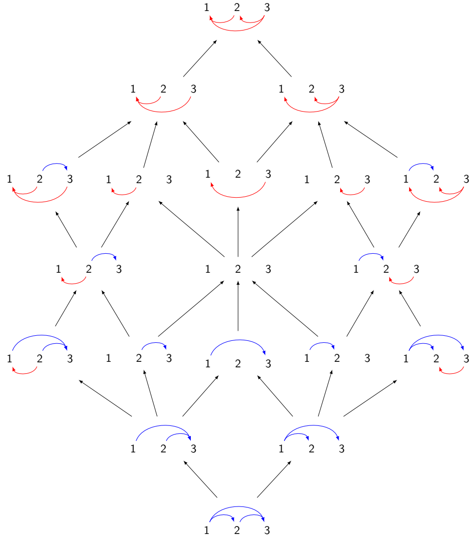




And also...

- A Hopf algebra on the faces of the permutahedron and the associahedron (Chapoton, 2000)
- A Hopf algebra on the intervals (Pilaud, P. 2020)
- A Hopf algebra on Permutrees (and Shröder permutrees) (Pilaud, P. 2018)

The lattice and Hopf algebra on integer poset (Pilaud, P.)



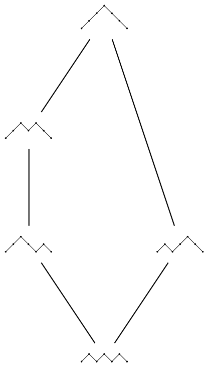
Tamari Intervals

2005 **Chapoton** proves a nice formula counting intervals of the Tamari lattice (also counting triangular maps!)

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$$

F. Chapoton. Sur le nombre d'intervalles dans les treillis de Tamari. *Sém. Lothar. Combin.*, 2005.

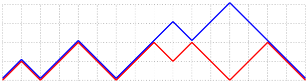
Tamari Intervals

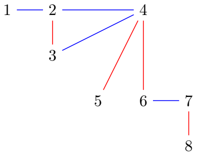


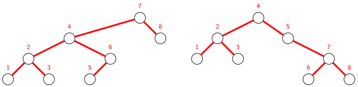
$$\frac{2}{3\times 4}\binom{4\times 3+1}{3-1} = \frac{2}{3\times 4}\binom{13}{2} = \frac{2\times 13\times 12}{4\times 3\times 2} = 13$$

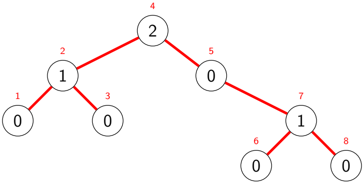
Many combinatorial interpretations

- Tamari interval-posets (Châtel-P. 2015)
- Rooted triangulations (Tute 62)
- proofs in certain sequent calculus (Zeilberger 2019)
- closed flows on ordered forest (Châtel-Chapoton-Pons 2014)
- Tamari diagrams (Combes 2019)
- Grafting trees (P. 2019)
- Uniquely sorted permutations avoiding 132 (Defant 2020)

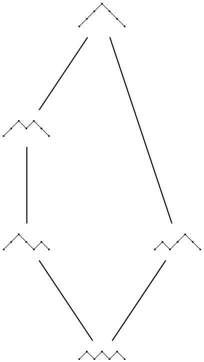






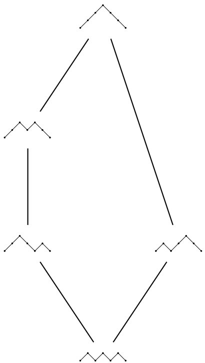


Dimensions of co-invariant spaces



$$f = s_{1,1} + s_3$$

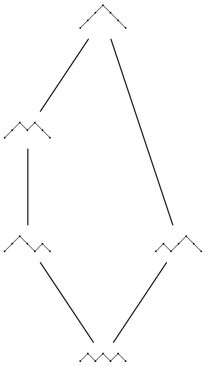
Dimensions of co-invariant spaces



$$f = s_{1,1} + s_3$$

$$f(q,t) = q^3 + q^2 t + q t^2 + t^3 + qt$$
$$f(1,1) = 5$$

Dimensions of co-invariant spaces



$$f = s_{1,1} + s_3$$

$$f(q,t) = q^3 + q^2t + qt^2 + t^3 + qt$$
$$f(1,1) = 5$$

$$f(q,t,r) = q^3 + q^2t + qt^2 + t^3 + q^2r + qtr$$
$$= +t^2r + qr^2 + tr^2 + r^3 + qt + qr + tr$$
$$f(1,1,1) = 13$$

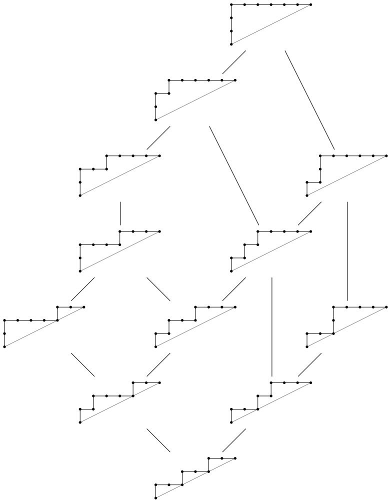
One question among many

What about multichains of the Tamari lattice?

Generalizations

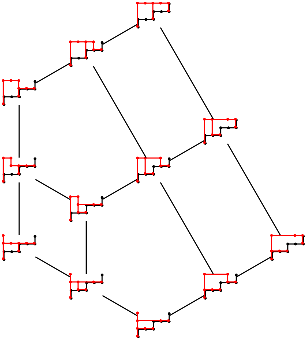
The Tamari lattice has many generalizations

The m-Tamari lattices



- Nice formula for the intervals (Bergeron–Préville-Ratelle 2012, Bousquet-Mélou–Fusy–Préville-Ratelle 2011)
- co-invariant spaces (Bergeron–Préville-Ratelle 2012)
- Hopf algebra (Novelli–Thibon 2014)
- Polytopal complex realization (Ceballos–Padrol–Sarmiento 2019)

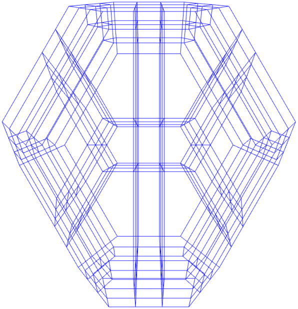
The ν-Tamari lattices

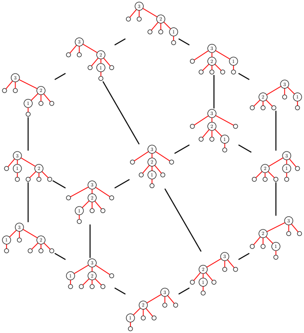


- Definition by Viennot and Préville-Ratelle, 2015
- Geometric realization: Ceballos–Padrol–Sarmiento, 2019
- Enumeration of intervals: Fang–Préville-Ratelle, 2017

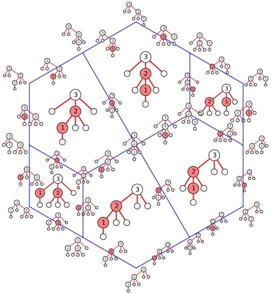
The *s*-weak order and the *s*-Permutahedron

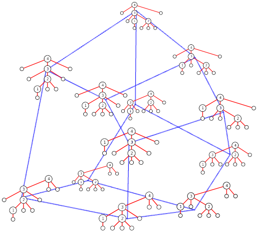
Ceballos–P., 2019





Polytopal complex and Ascentopes





One question among many

- What is the triangular Tamari lattice?

Thanks!