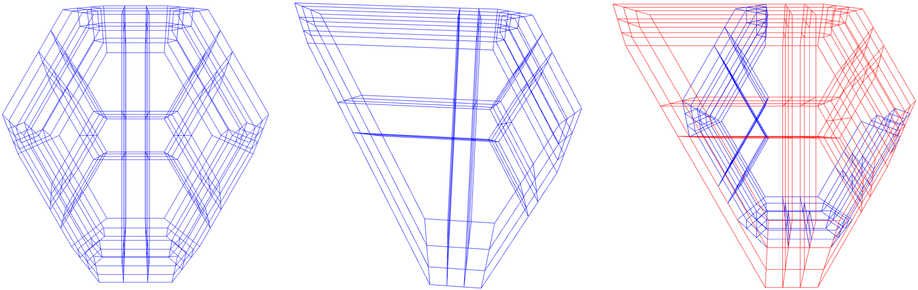
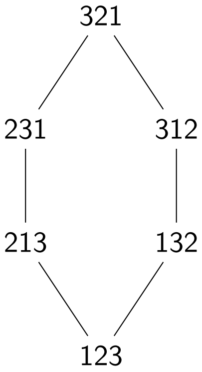
s-weak order and s-permutahedra

Viviane Pons, LISN, Univ. Paris-Saclay
Cesar Ceballos, Univ. of Vienna



Weak Order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213 132
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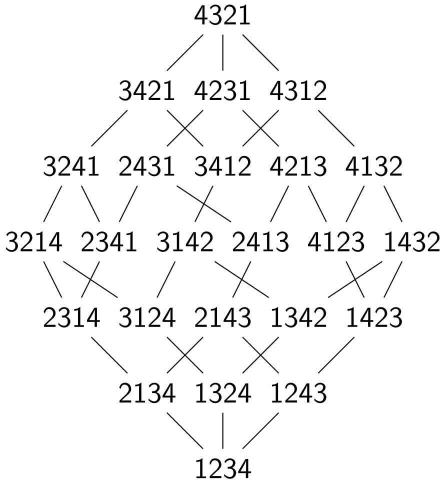
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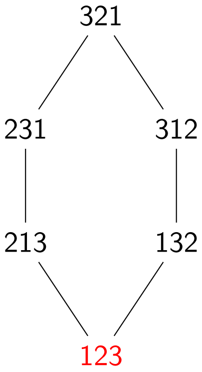
3241 2431 3412 4213 4132
3214 2341 3142 2413 4123 1432
2314 3124 2143 1342 1423
2134 1324 1243
1234

Weak Order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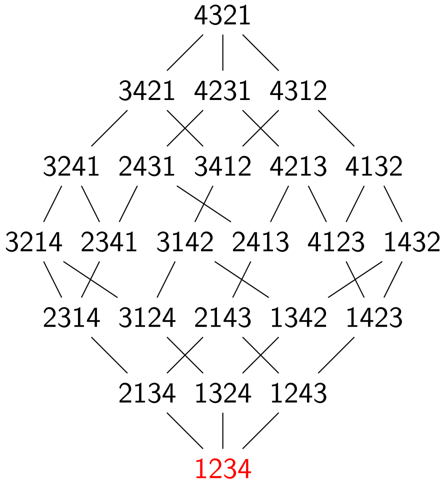
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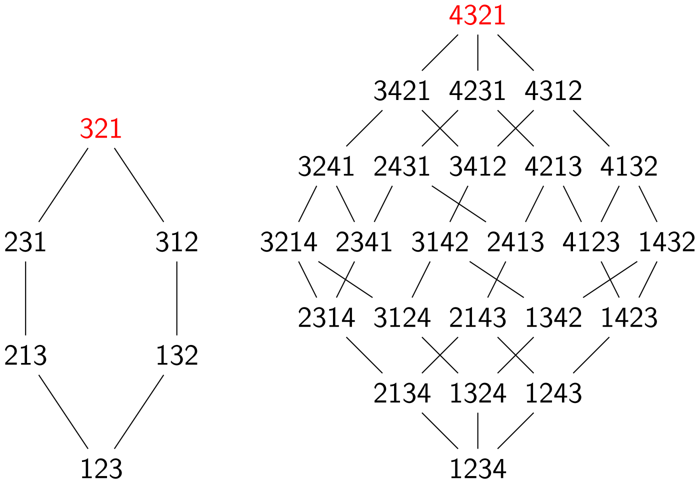
Weak Order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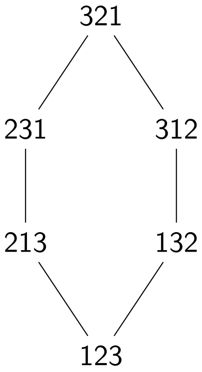
Weak Order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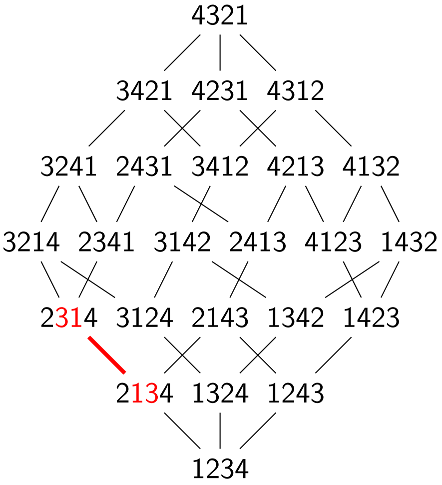
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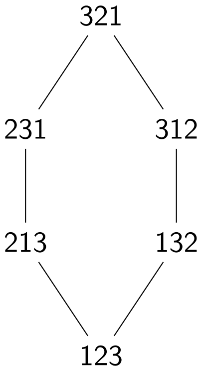
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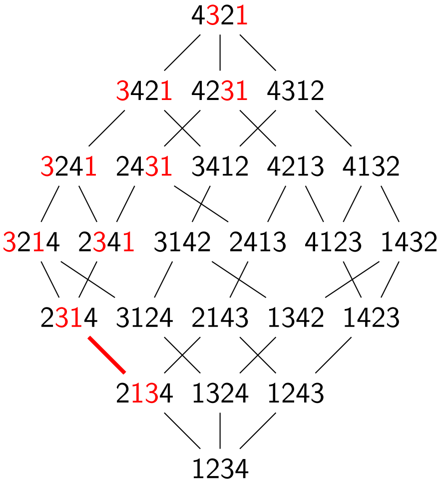
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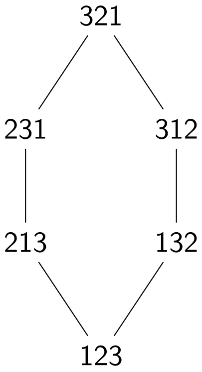
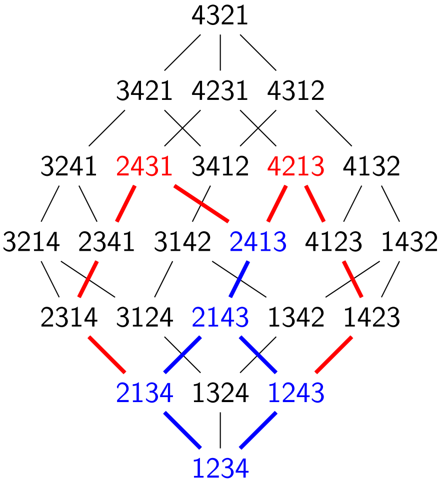
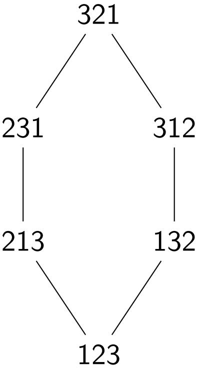
Weak Order

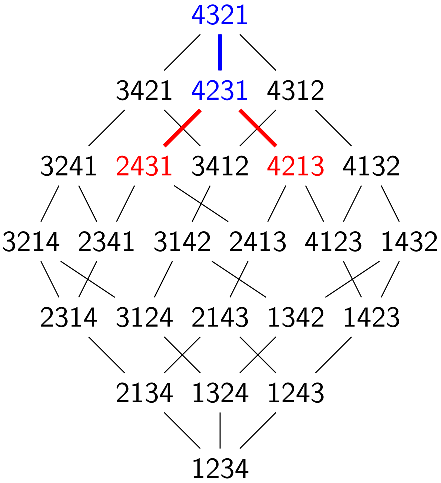




$2413 \wedge 4213 = 2413$ $2413 \vee 4213 = 4231$

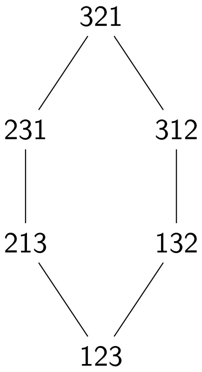
Weak Order

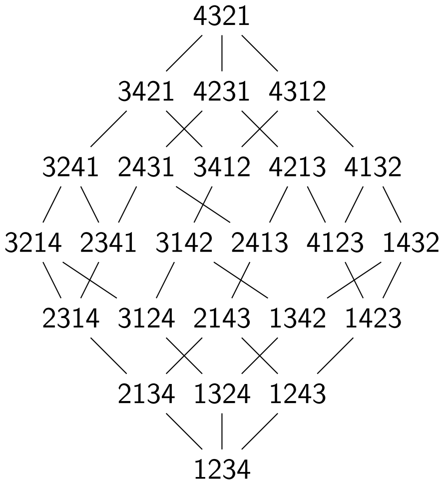




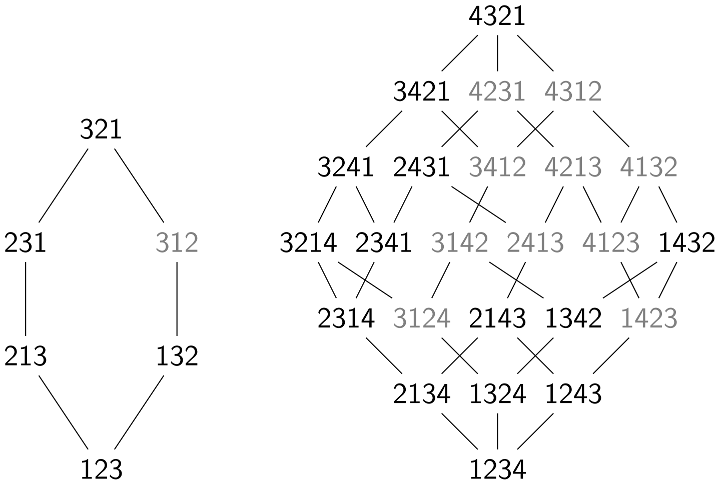
$2413 \wedge 4213 = 2413$ $\qquad$ $2413 \vee 4213 = 4231$

From the Weak Order to the Tamari lattice

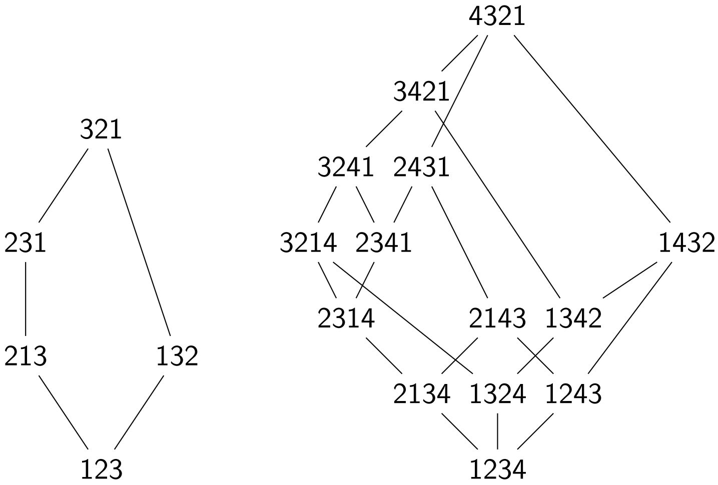




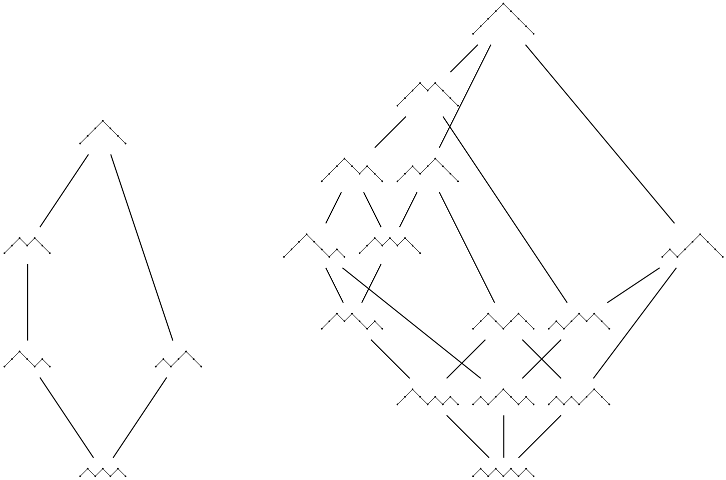
From the Weak Order to the Tamari lattice



From the Weak Order to the Tamari lattice

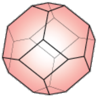


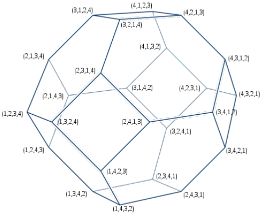
From the Weak Order to the Tamari lattice



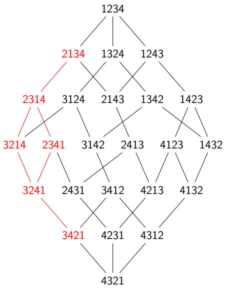
Context

Geometry



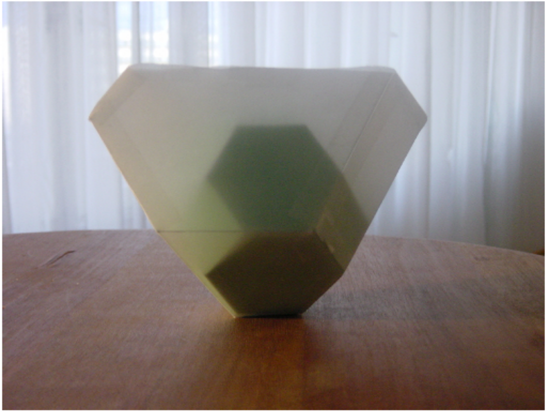


Combinatorics

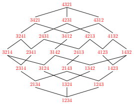


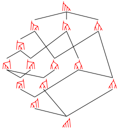
Algebra

$$\mathbf{F}_{21} . \mathbf{F}_{12} = \mathbf{F}_{21 \sqcup\!\sqcup 12}$$
$$= \mathbf{F}_{2134} + \mathbf{F}_{2314} + \mathbf{F}_{2341} + \mathbf{F}_{3214}\ \mathbf{F}_{3241} + \mathbf{F}_{3421}$$

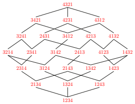
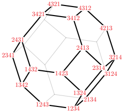
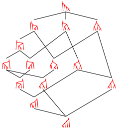
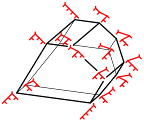


Weak order

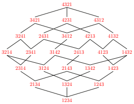


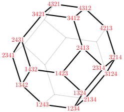


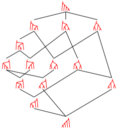
Tamari lattice

Weak order	Permutahedron		
			
			
Tamari lattice	Associahedron		

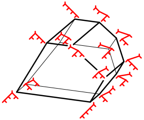
Weak order	Permutahedron		



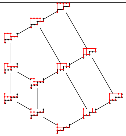




Tamari lattice



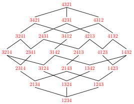
Associahedron



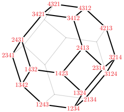
ν-Tamari

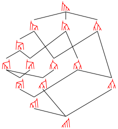
Préville-Ratelle, Viennot

Weak order

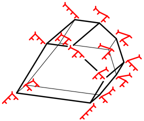


Permutahedron

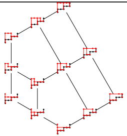




Tamari lattice

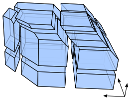


Associahedron



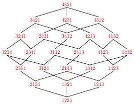
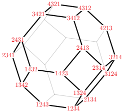
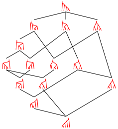
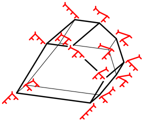
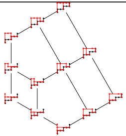
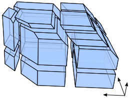
ν-Tamari

Préville-Ratelle, Viennot

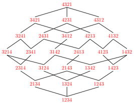


ν-Associahedron

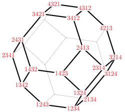
Ceballos, Padrol, Sarmiento

Weak order	Permutahedron	?	?
			
			
Tamari lattice	Associahedron	ν-Tamari	ν-Associahedron
		Préville-Ratelle, Viennot	Ceballos, Padrol, Sarmiento

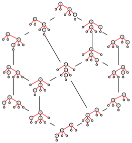
Weak order



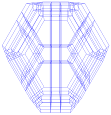
Permutahedron

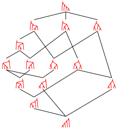


s-Weak order

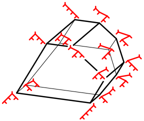


s-Permutahedron

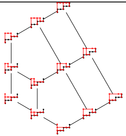




Tamari lattice

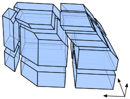


Associahedron



ν-Tamari

Préville-Ratelle, Viennot



ν-Associahedron

Ceballos, Padrol, Sarmiento

s-decreasing trees

Let s be a sequence of n non-negative integers. An s-decreasing tree is a planar tree labeled with $1 \dots n$ such that each node i has $s(i) + 1$ children and labels are decreasing from root to leaves.

$$s = (0, 1, 3, 0, 4, 3)$$

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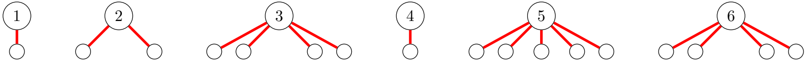

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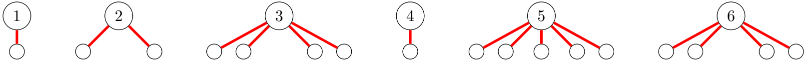
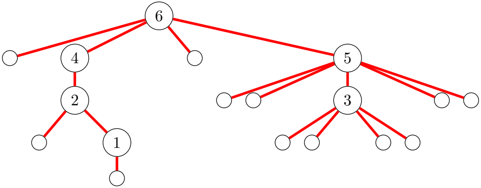




How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

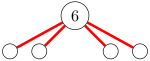
Number of s-decreasing trees:

How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$(1+3)$

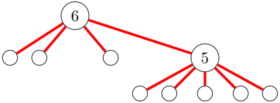


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$(1+3)\times(1+3+4)$

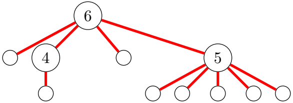


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$(1+3)\times(1+3+4)\times(1+3+4+0)$

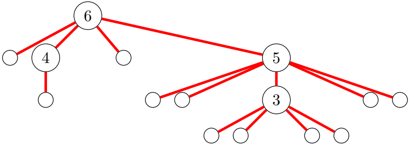


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$(1+3)\times(1+3+4)\times(1+3+4+0)\times(1+3+4+0+3)$

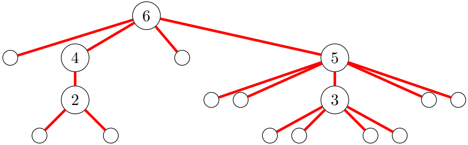


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$$(1+3)\times(1+3+4)\times(1+3+4+0)\times(1+3+4+0+3)\times(1+3+4+0+3+1)$$

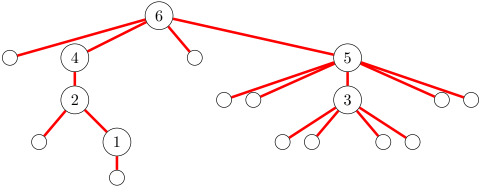


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$$(1+3)\times(1+3+4)\times(1+3+4+0)\times(1+3+4+0+3)\times(1+3+4+0+3+1)$$

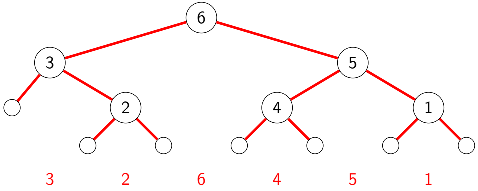


Permutations

$$s = (1, 1, 1, 1, 1, 1)$$

Number of s-decreasing trees:

$$(1+1)\times(1+1+1)\times(1+1+1+1)\times(1+1+1+1+1)\times(1+1+1+1+1+1)$$

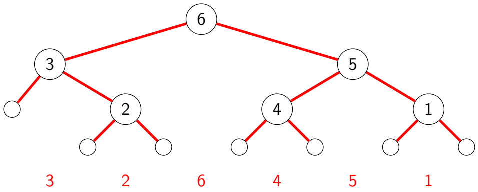


Permutations

$$s = (1, 1, 1, 1, 1, 1)$$

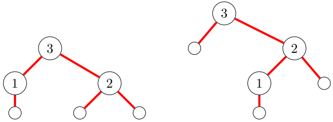
Number of s-decreasing trees: 6!

$$(1+1)\times(1+1+1)\times(1+1+1+1)\times(1+1+1+1+1)\times(1+1+1+1+1+1)$$

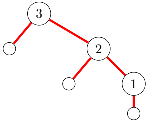


Tree-inversions

For all $b > a$, we define $0 \leq \#(b, a) \leq s(b)$.



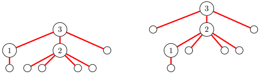
$\#(2, 1) = 0$



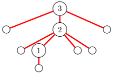
$\#(2, 1) = 1$

Tree-inversions

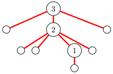
For all $b > a$, we define $0 \leq \#(b, a) \leq s(b)$.



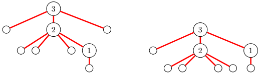
$\#(2, 1) = 0$



$\#(2, 1) = 1$



$\#(2, 1) = 2$



$\#(2, 1) = 3$

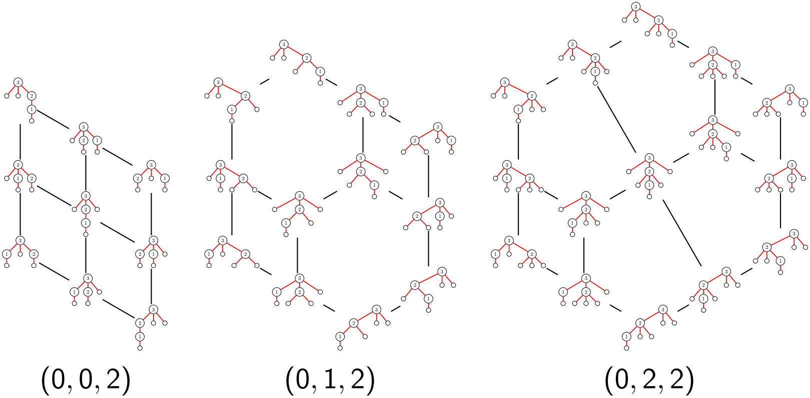
The s-weak order

R, T, s-decreasing trees:

$$R \preccurlyeq T \Leftrightarrow \forall b > a, \#_R(b, a) \leq \#_T(b, a)$$

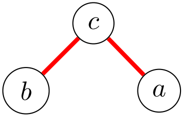
Theorem

The s-weak order is always a lattice.


(0, 0, 2)
(0, 1, 2)
(0, 2, 2)

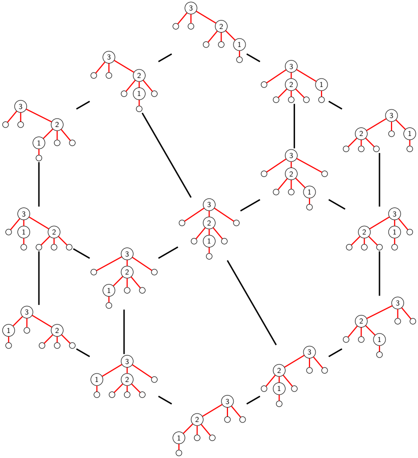
s-Tamari lattice

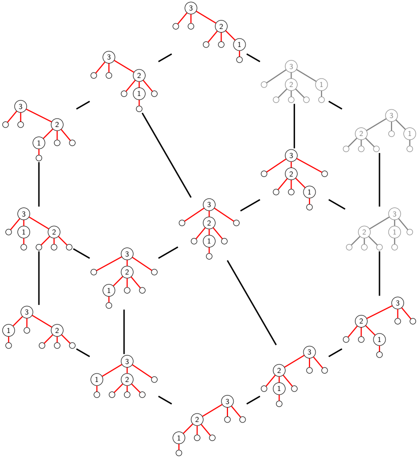
Select trees which avoid "pattern 231": $a < b < c$

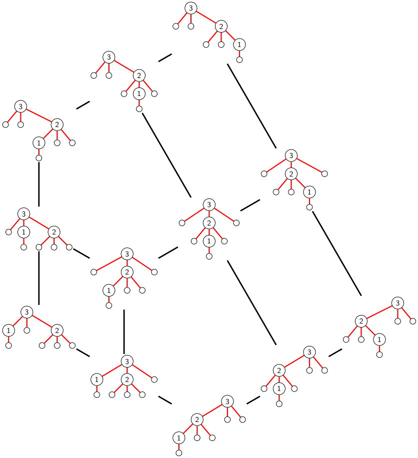


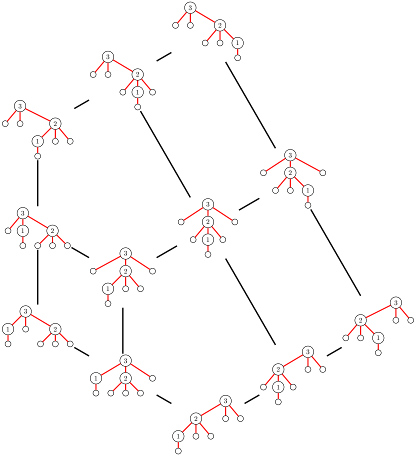
Theorem

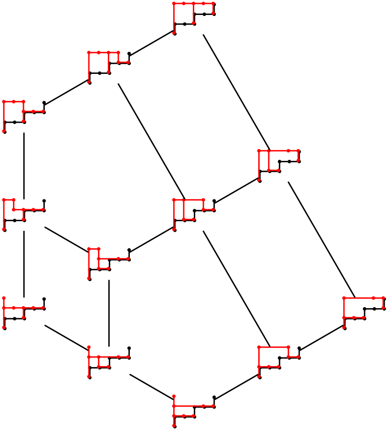
The set of 231-avoiding s-decreasing trees form a sublattice, the s-Tamari lattice, isomorphic to the ν-Tamari lattice.



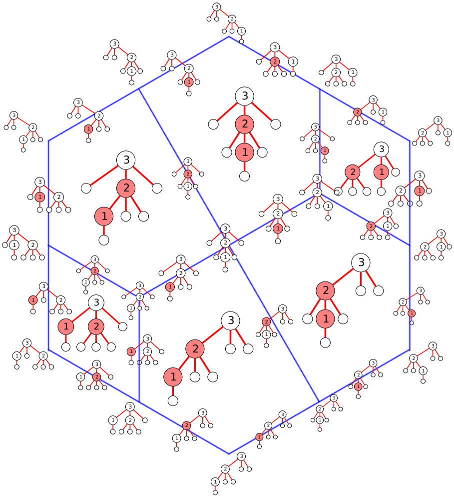


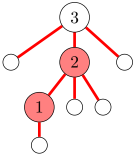






Geometry: the *s*-Permutahedron




3
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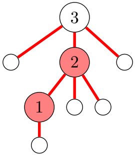
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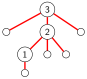
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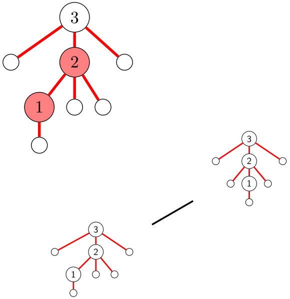
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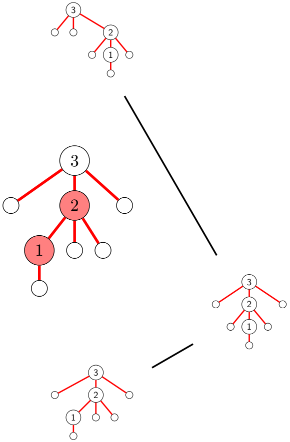
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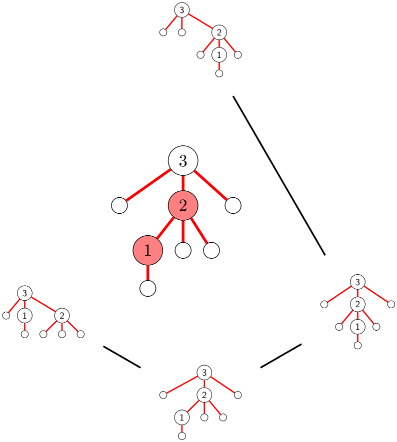
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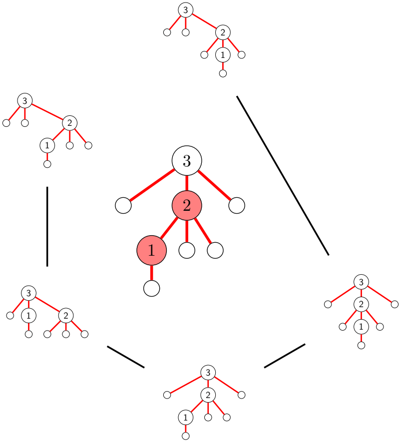
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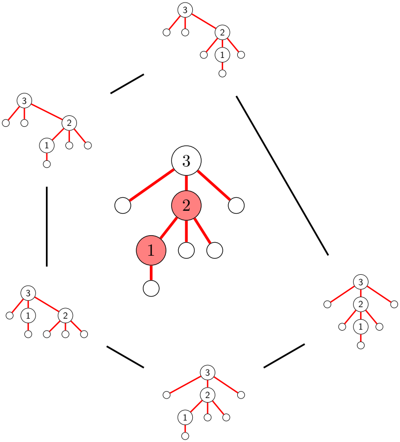
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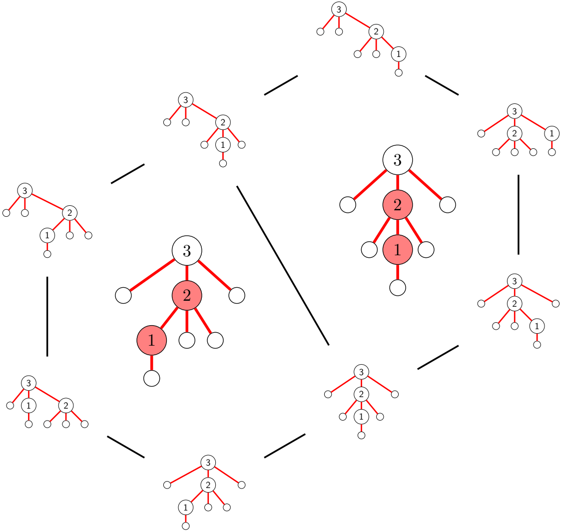
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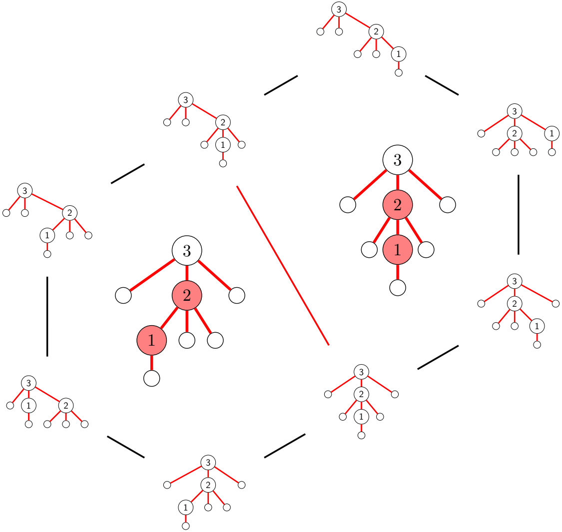
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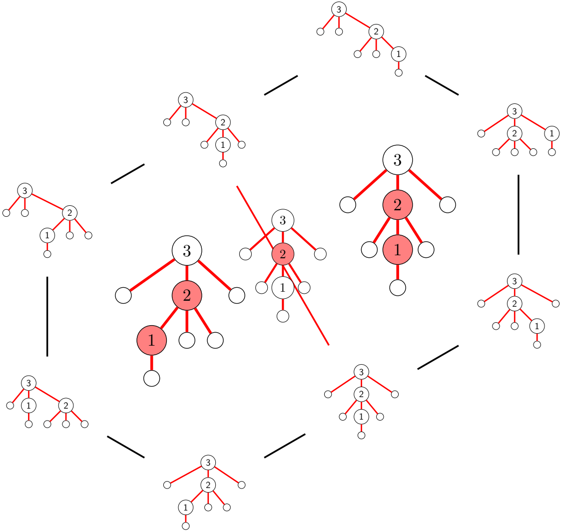
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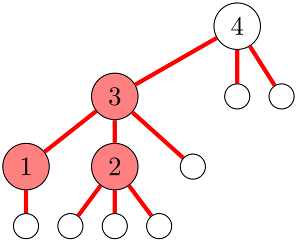
Polytopal complex? Ascentopes

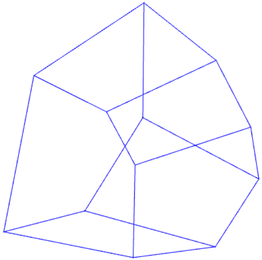
Can each face be realized as a polytope?

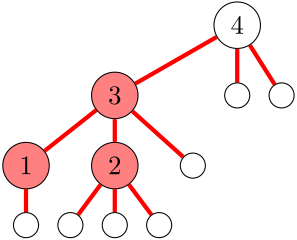
Polytopal complex? Ascentopes

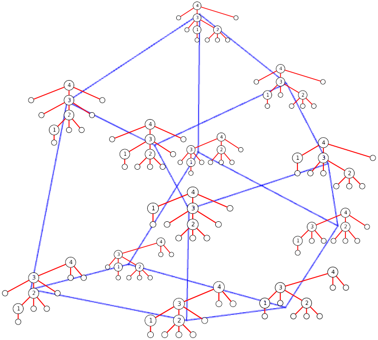
Can each face be realized as a polytope?

To each facet, we associate a **generalized permutahedron**: we define a injection between maximal faces included in a given facet and the facets of the permutahedron.

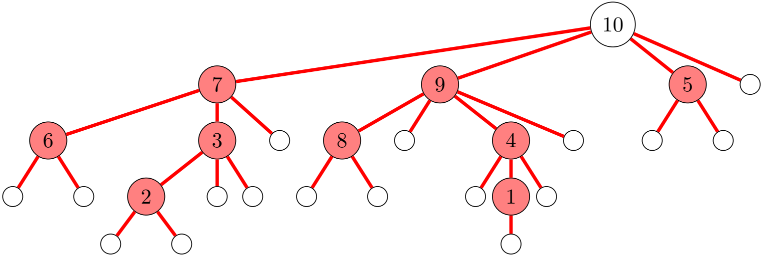

4
3
1
2







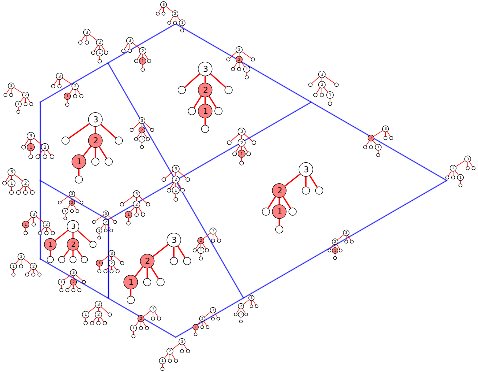
- 12 vertices
- 18 edges
- 8 facets



f vector $=$

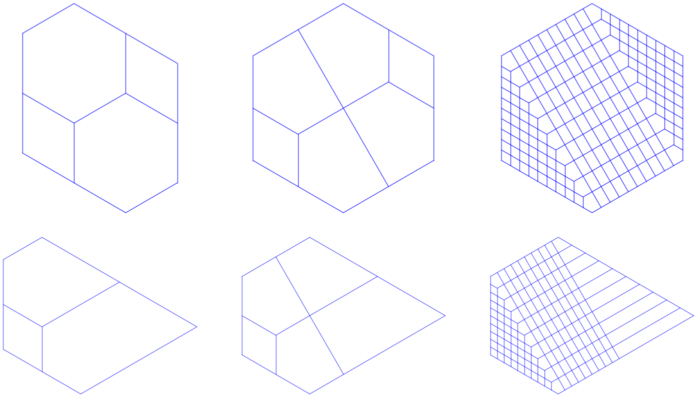
$$(1, 2178, 9801, 19008, 20790, 14082, 6099, 1680, 282, 26, 1)$$

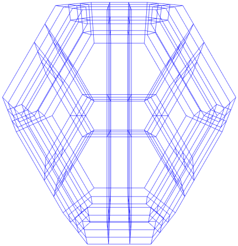
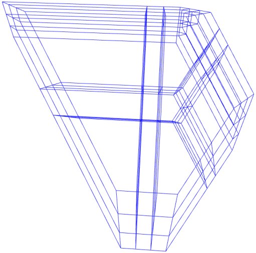
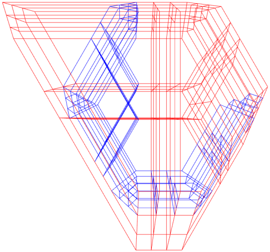
The s-Associahedron



Theorem: The s-associahedron is isomorphic to the ν-associahedron for $\nu = NE^{s(n)} \dots NE^{s(1)}$.

Polytopal subdivision



Conjecture 1

The s-permutohedron can be realized as a polytopal subdivision of the permutohedron.

Conjecture 2

One can obtain a realization of the s-associahedron by removing some facets of the s-permutohedron realization.