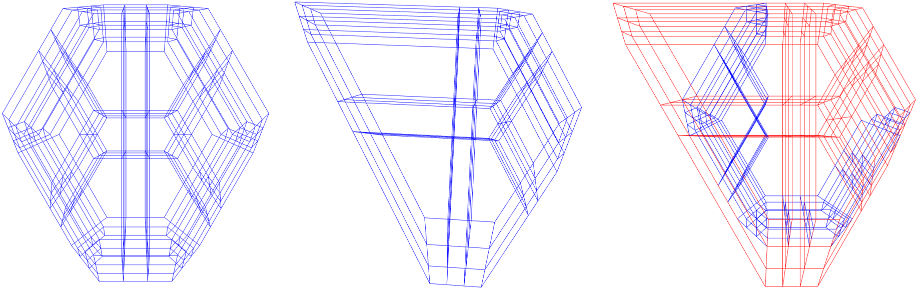
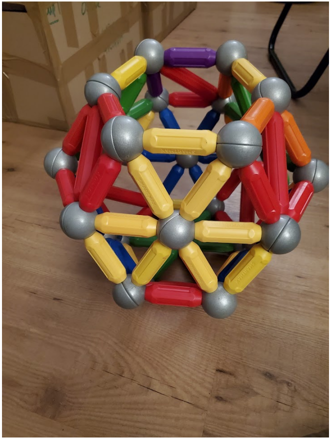
s-weak order and s-permutahedra

Cesar Ceballos – **Viviane Pons**
Univ. of Graz – **LISN, Univ. Paris-Saclay**



The Permutahedron



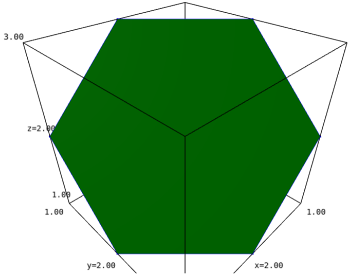
As a convex hull of permutations

Entrée [1]:
```
P = Polyhedron(list(Permutations(3)))
P
```

Out[1]: A 2-dimensional polyhedron in ZZ^3 defined as the convex hull of 6 vertices (use the .plot() method to plot)

Entrée [2]:
```
P.plot()
```

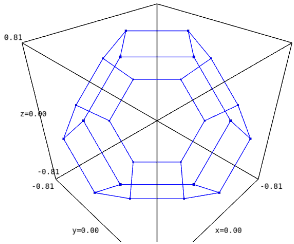
Out[2]:



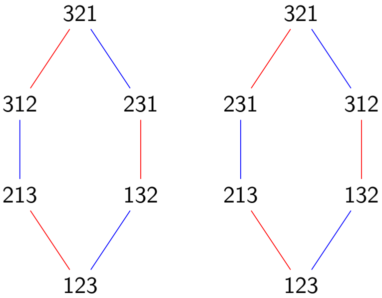
For size 4

Entrée [3]:
```
P = Polyhedron(list(Permutations(4)))
P
```
Out[3]: A 3-dimensional polyhedron in ZZ^4 defined as the convex hull of 24 vertices (use the .plot() method to plot)

Entrée [4]:
```
P.plot()
```
Out[4]:

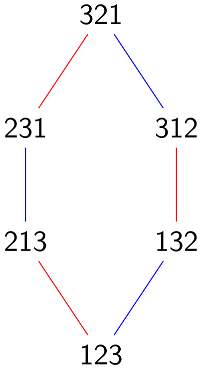


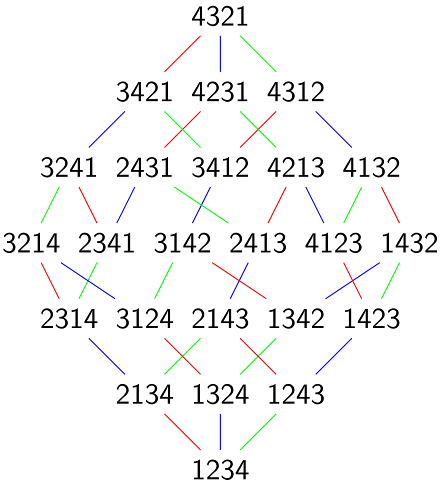
Weak order



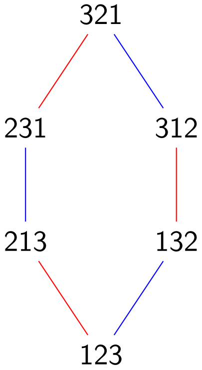
left weak order   right weak order

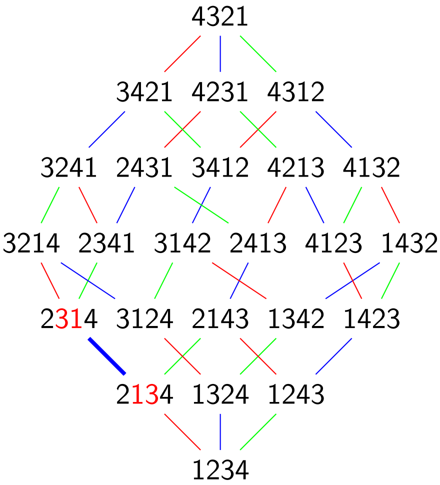
Right weak Order



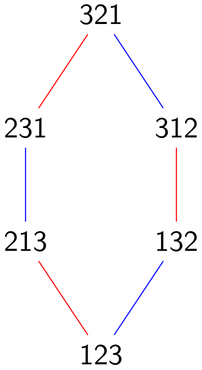


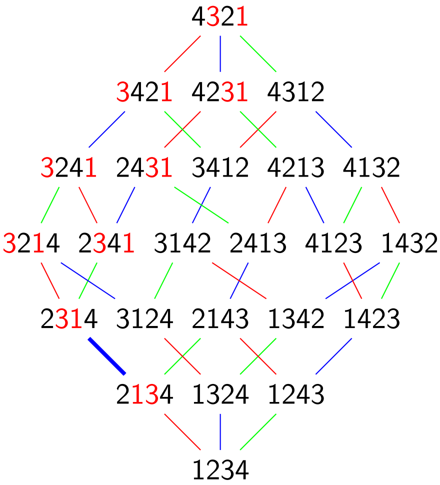
Right weak Order



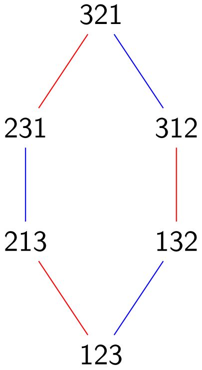


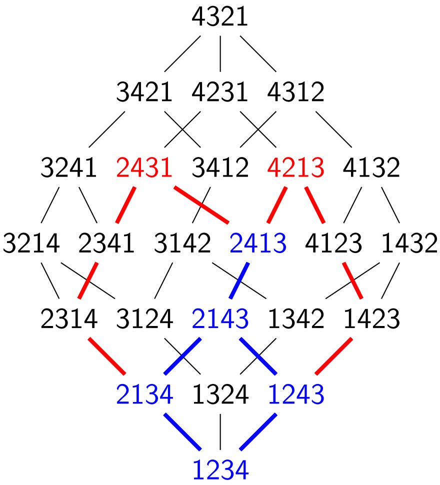
Right weak Order





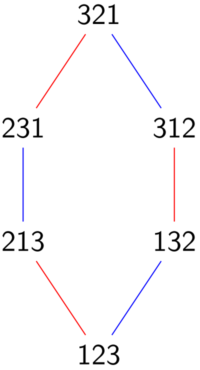
Right weak Order

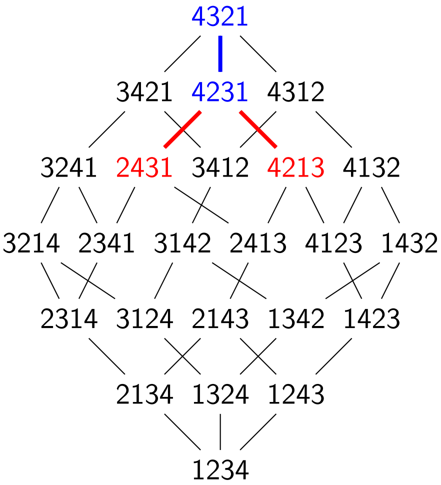




$2413 \wedge 4213 = 2413$ $\qquad$ $2413 \vee 4213 = 4231$

Right weak Order

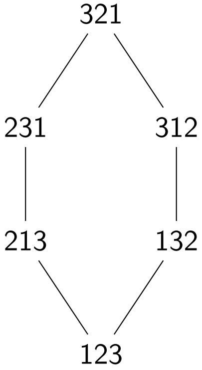


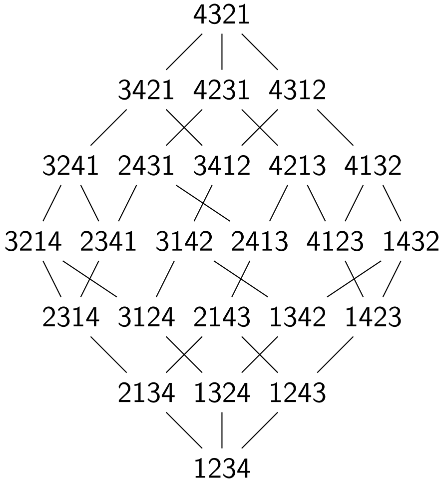


$$2413 \wedge 4213 = 2413$$

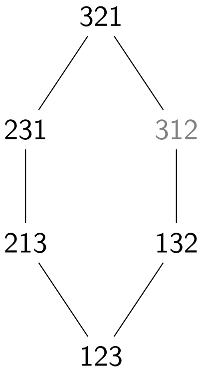
$$2413 \vee 4213 = 4231$$

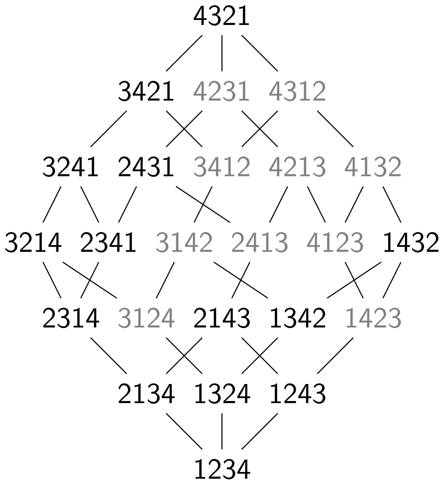
From the Weak Order to the Tamari lattice



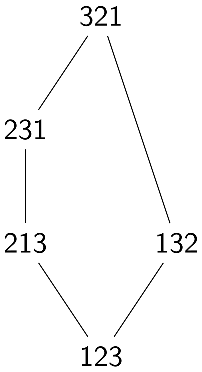


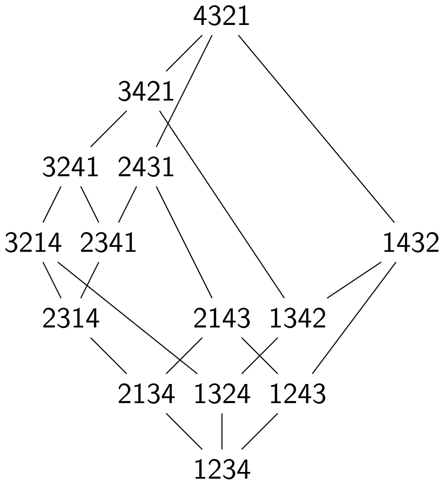
From the Weak Order to the Tamari lattice



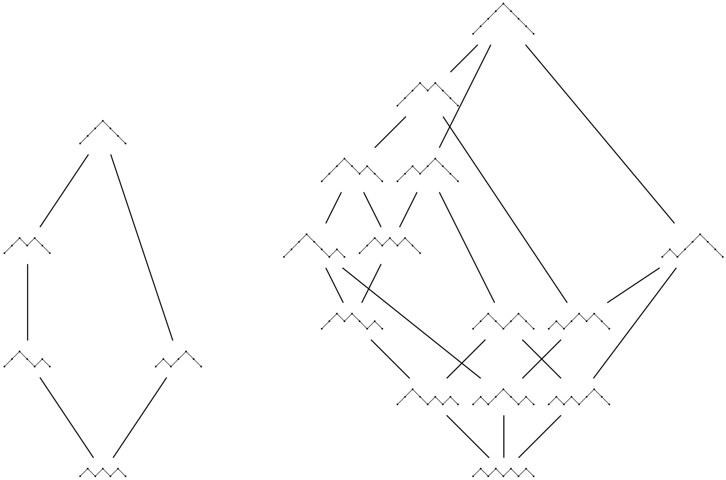


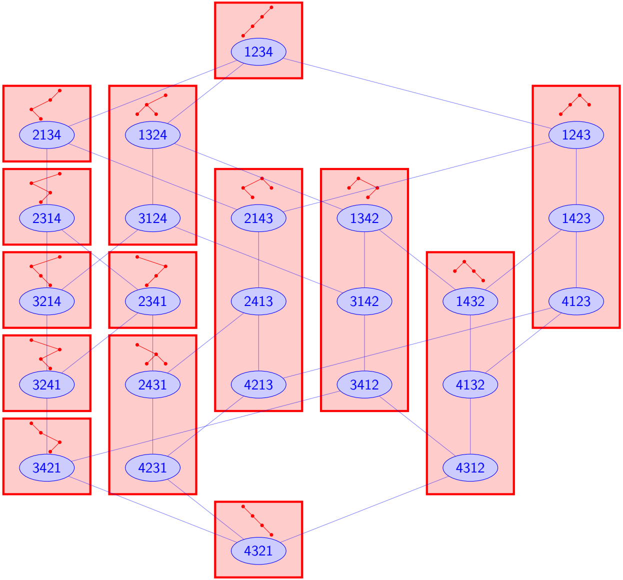
From the Weak Order to the Tamari lattice



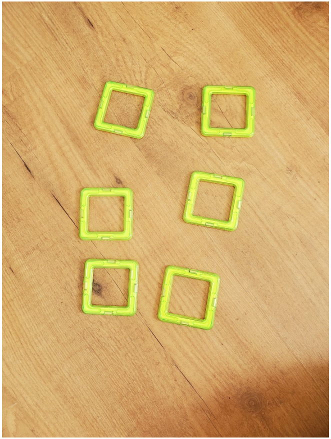


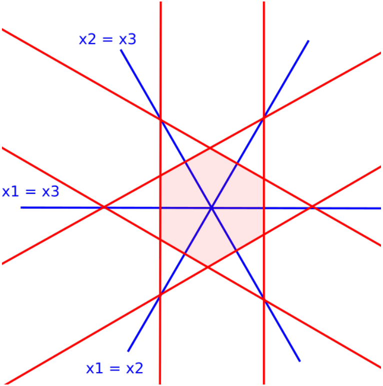
From the Weak Order to the Tamari lattice




1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

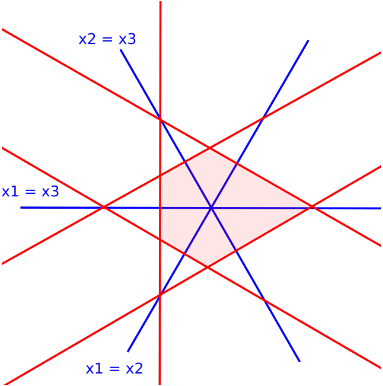
From the permuathedron to the associahedron : removing faces





$$J \subseteq [n] \to \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$

$12\|3$	$x_1 + x_2 \geq 3$
$2\|13$	$x_2 \geq 1$
$23\|1$	$x_2 + x_3 \geq 3$
$3\|12$	$x_3 \geq 1$
$13\|2$	$x_1 + x_3 \geq 3$
$1\|23$	$x_1 \geq 1$

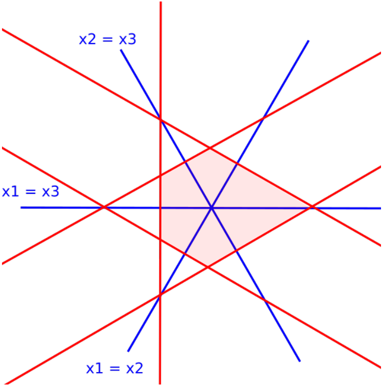


$$J \subseteq [n] \rightarrow \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$

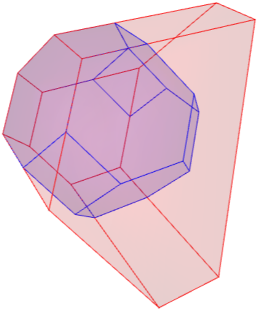
J interval

$12\|3$	$x_1 + x_2 \geq 3$
$2\|13$	$x_2 \geq 1$
$23\|1$	$x_2 + x_3 \geq 3$
$3\|12$	$x_3 \geq 1$
$13\|2$	
$1\|23$	$x_1 \geq 1$

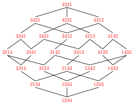
$$J \subseteq [n] \rightarrow \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$

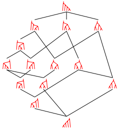


J interval



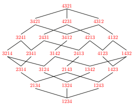
Weak order



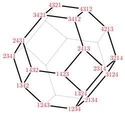


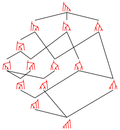
Tamari lattice

Weak order

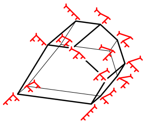


Permutahedron



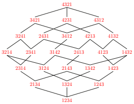


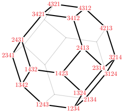
Tamari lattice

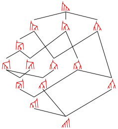


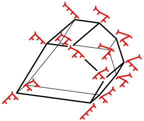
Associahedron

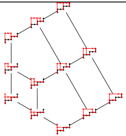
Weak order	Permutahedron		





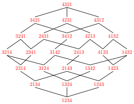




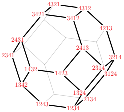


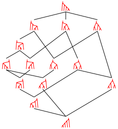
Tamari lattice	Associahedron	ν-Tamari Préville-Ratelle, Viennot	

Weak order

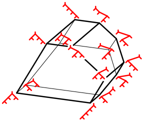


Permutahedron

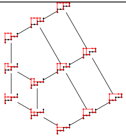




Tamari lattice

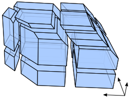


Associahedron



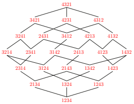
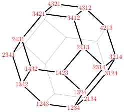
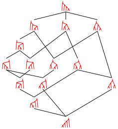
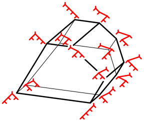
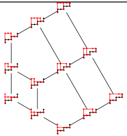
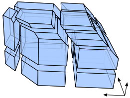
ν-Tamari

Préville-Ratelle, Viennot

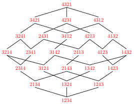


ν-Associahedron

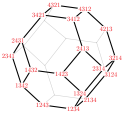
Ceballos, Padrol, Sarmiento

Weak order	Permutahedron	?	?
			
			
Tamari lattice	Associahedron	ν-Tamari	ν-Associahedron
		Préville-Ratelle, Viennot	Ceballos, Padrol, Sarmiento

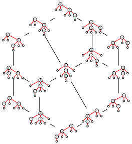
Weak order



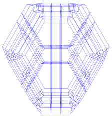
Permutahedron

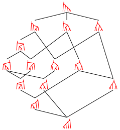


s-Weak order

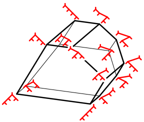


s-Permutahedron

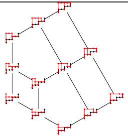




Tamari lattice

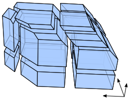


Associahedron



ν-Tamari

Préville-Ratelle, Viennot



ν-Associahedron

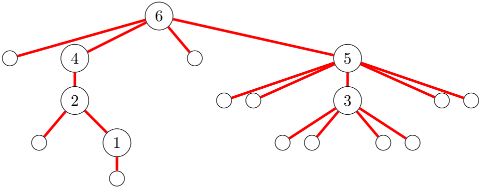
Ceballos, Padrol, Sarmiento

s-weak order

s-decreasing trees

- s is a sequence of non-negative integers
- The node i has $s(i) + 1$ children
- Nodes labels decreasing from root to leaves

$$s = (0, 1, 3, 0, 4, 3)$$



How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

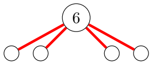
Number of s-decreasing trees:

How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

(1+3)

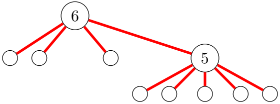


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$(1+3)\times(1+3+4)$

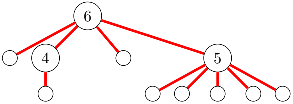


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$(1+3)\times(1+3+4)\times(1+3+4+0)$

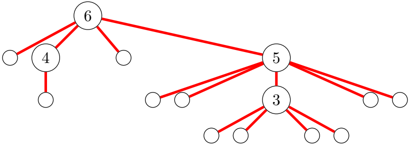


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$(1+3)\times(1+3+4)\times(1+3+4+0)\times(1+3+4+0+3)$

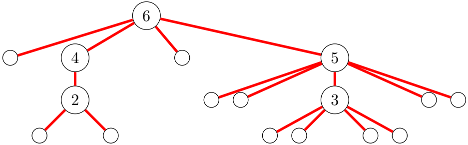


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$$(1+3)\times(1+3+4)\times(1+3+4+0)\times(1+3+4+0+3)\times(1+3+4+0+3+1)$$

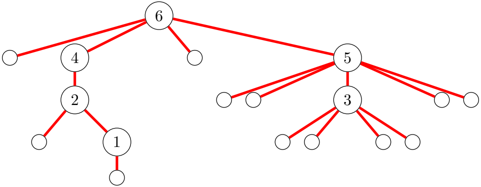


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$$(1{+}3){\times}(1{+}3{+}4){\times}(1{+}3{+}4{+}0){\times}(1{+}3{+}4{+}0{+}3){\times}(1{+}3{+}4{+}0{+}3{+}1)$$

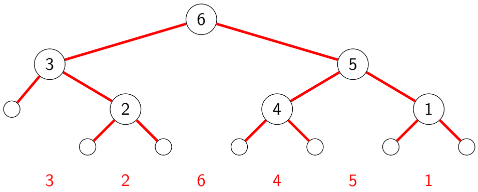


Permutations

$$s = (1,1,1,1,1,1)$$

Number of s-decreasing trees:

$$(1+1)\times(1+1+1)\times(1+1+1+1)\times(1+1+1+1+1)\times(1+1+1+1+1+1)$$

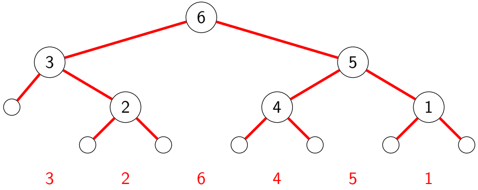


Permutations

$$s = (1, 1, 1, 1, 1, 1)$$

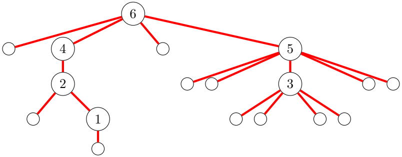
Number of s-decreasing trees: 6!

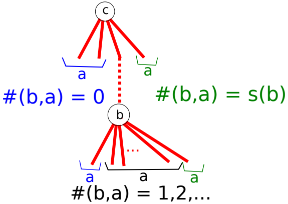
$$(1+1)\times(1+1+1)\times(1+1+1+1)\times(1+1+1+1+1)\times(1+1+1+1+1+1)$$



Tree-inversions

For all $b > a$, we define $0 \leq \#(b,a) \leq s(b)$.





$\#(6,5) = 3$ $\#(6,4) = 1$ $\#(6,3) = 3$ $\#(6,2) = 1$ $\#(6,1) = 1$

$\#(5,4) = 0$ $\#(5,3) = 2$ $\#(5,2) = 0$ $\#(5,1) = 0$

$\#(4,3) = 0$ $\#(4,2) = 0$ $\#(4,1) = 0$

$\#(3,2) = 0$ $\#(3,1) = 0$

$\#(2,1) = 1$

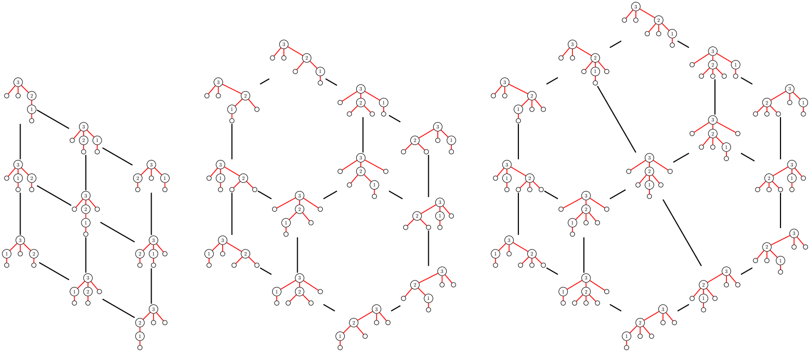
The s-weak order

R, T, s-decreasing trees:

$$R \preccurlyeq T \Leftrightarrow \forall b > a, \#_R(b,a) \leq \#_T(b,a)$$

Theorem (Ceballos, P.)

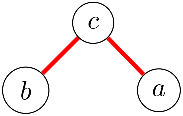
The s-weak order is always a lattice.



$(0, 0, 2)$ $(0, 1, 2)$ $(0, 2, 2)$

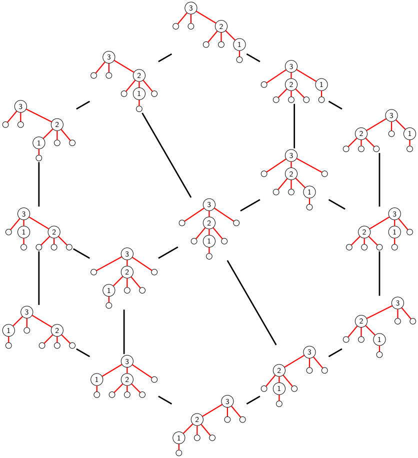
s-Tamari lattice

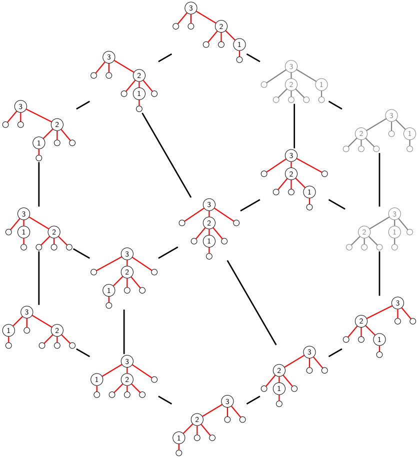
Select trees which avoid "pattern 231": $a < b < c$

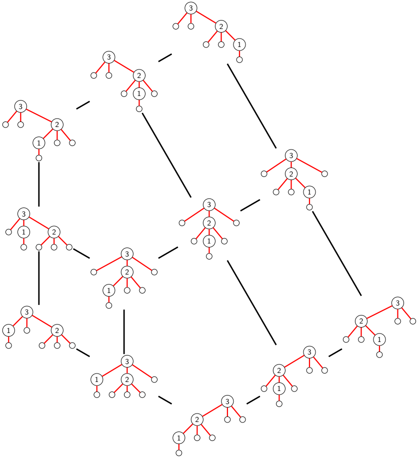


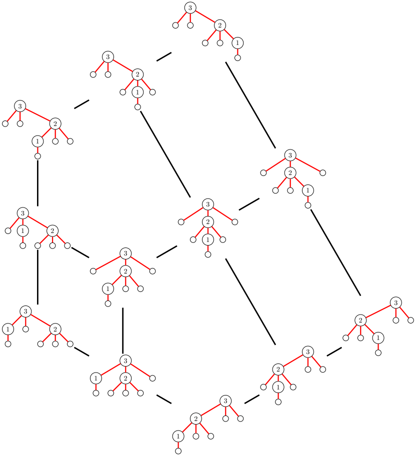
Theorem (Ceballos, P.)

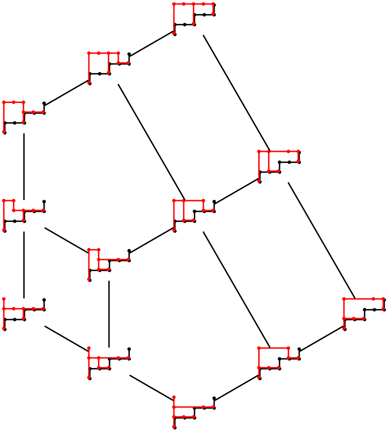
The set of 231-avoiding s-decreasing trees form a sublattice, the s-Tamari lattice, isomorphic to the ν-Tamari lattice.

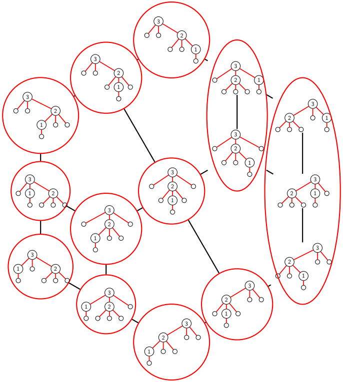



3
2
1





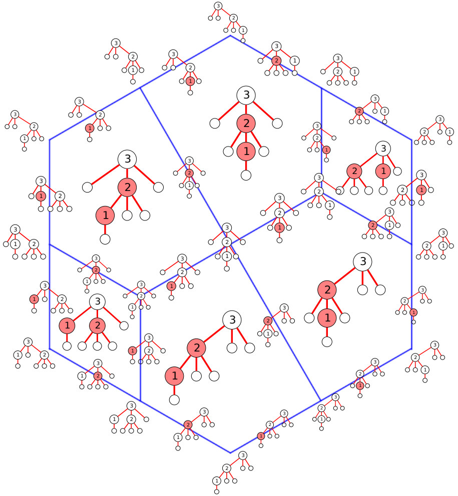




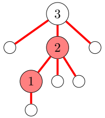
Theorem (Ceballos, P.)

The s-Tamari lattice is a quotient lattice of the s-weak order if s does not contain any zeros.

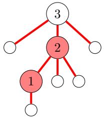
Geometry: the *s*-Permutahedron

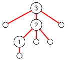


Faces : pure intervals

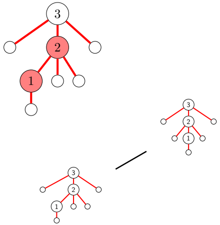


Faces : pure intervals

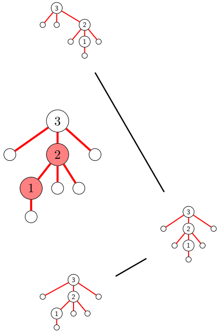




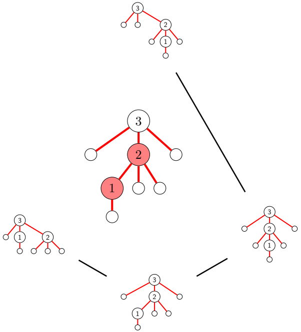
Faces : pure intervals



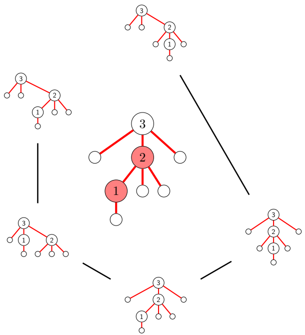
Faces : pure intervals



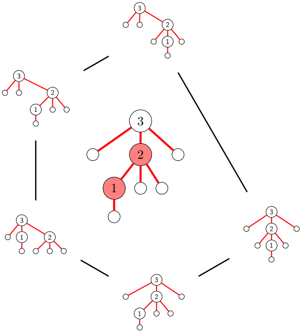
Faces : pure intervals



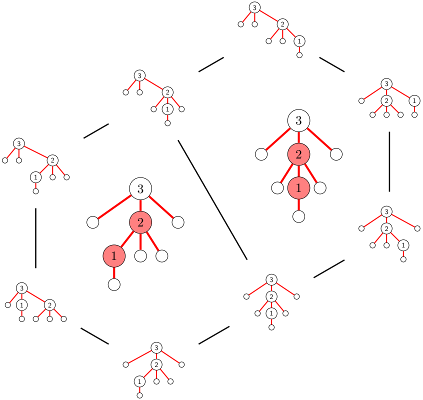
Faces : pure intervals



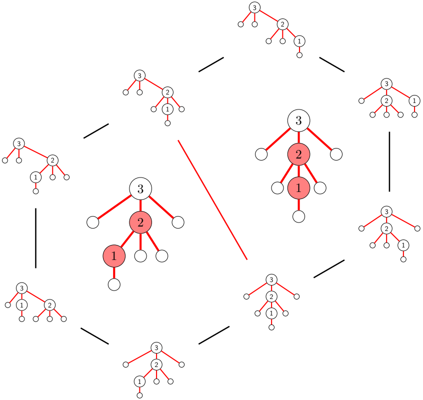
Faces : pure intervals



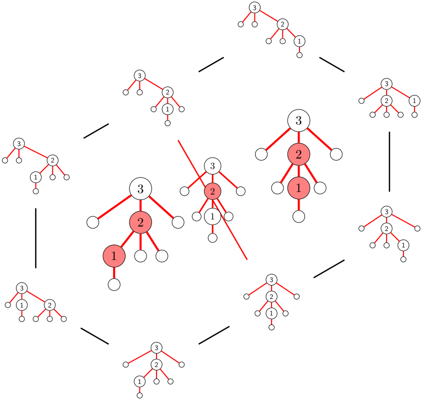
Faces : pure intervals



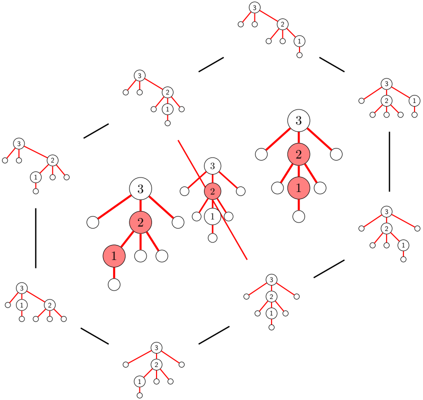
Faces : pure intervals



Faces : pure intervals



Faces : pure intervals



Theorem (Ceballos, P.)

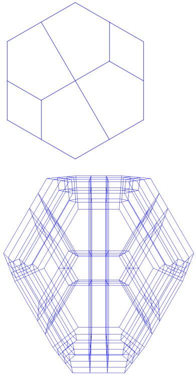
The intersection of two pure intervals is a pure interval.

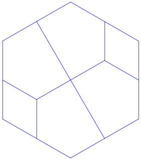
Conjecture (Ceballos, P.)

The s-Permutahedron is a polytopal complex

Conjecture (Ceballos, P.)

It can be realized as a polytopal subdivision of the classical permutahedron.



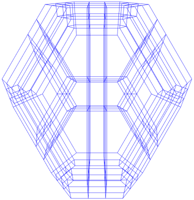


Conjecture (Ceballos, P.)

The s-Permutahedron is a polytopal complex

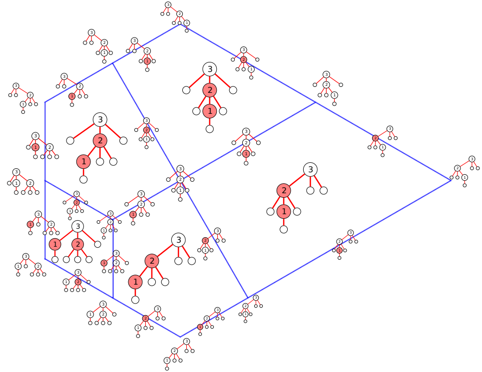
Conjecture (Ceballos, P.)

It can be realized as a polytopal subdivision of the classical permutahedron.



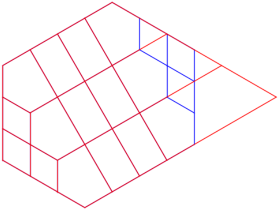
Solved for s without 0 (González D'Leon, Morales, Philippe, Tamayo and Yip)

The s-Associahedron



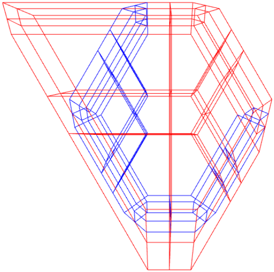
Theorem (Ceballos, P.)

The s-associahedron is isomorphic to the ν-associahedron for $\nu = NE^{s(n)} \dots NE^{s(1)}$.



Conjecture (Ceballos, P.)

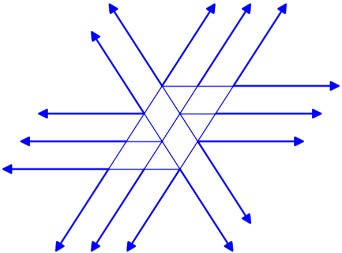
A realization of the s-associahedron can be obtained by "removing" some facets of the s-permutahedron.

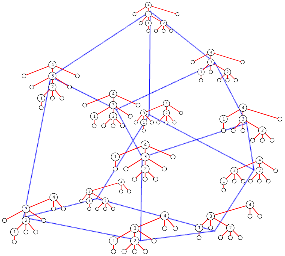


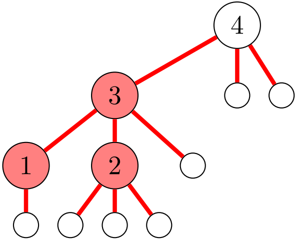
Work in progress

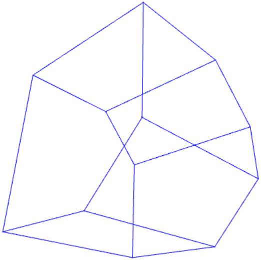
Proof of Polytopal Complex conjecture for all s, using:

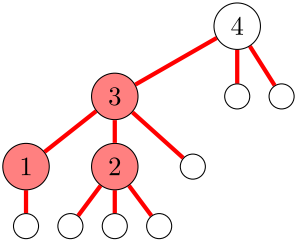
- The s-braid arrangement
- The ascentopes

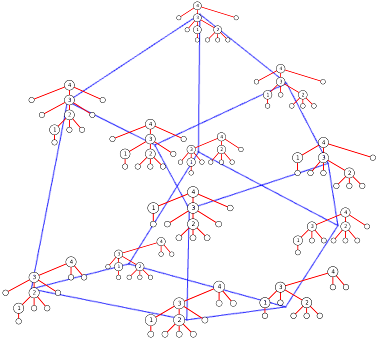




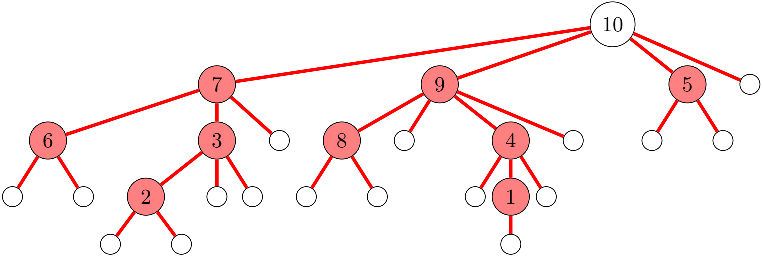

4
3
1
2







- 12 vertices
- 18 edges
- 8 facets



f vector =
$(1, 2178, 9801, 19008, 20790, 14082, 6099, 1680, 282, 26, 1)$

Work in progress (Philippe, Pilaud)

Realization of the lattice quotients of the s-weak order using the s-braid arrangement.

References

- Ceballos, P. The s-weak order and s-permutahedra. *FPSAC 2019*
- Ceballos, P. The s-weak order and s-permutahedra I: combinatorics and lattice structure. 2022+.arXiv:2212.11556
- Ceballos, P. The s-weak order and s-permutahedra II: The combinatorial complex of pure intervals. 2023+. arXiv:2309.14261
- Lacina. Poset topology of s weak order via SB-labelings. *J. Combinatorics* 2022.
- González D'León, Morales, Philippe, Jiménez, Yip. Realizing the s-permutahedron via flow polytopes. 2023+. arXiv:2307.03474