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Expérimentation manuelle et par ordinateur

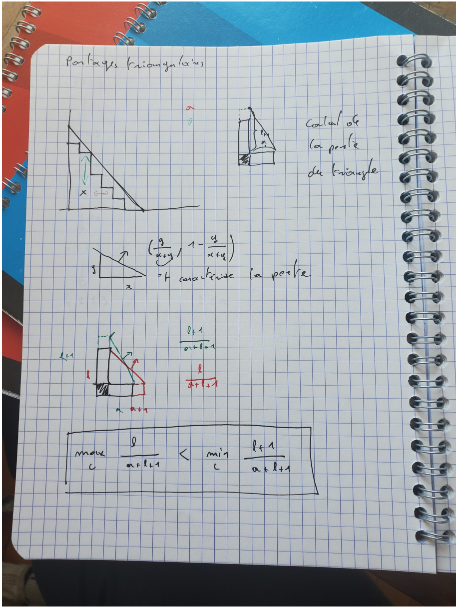
dans la recherche en combinatoire

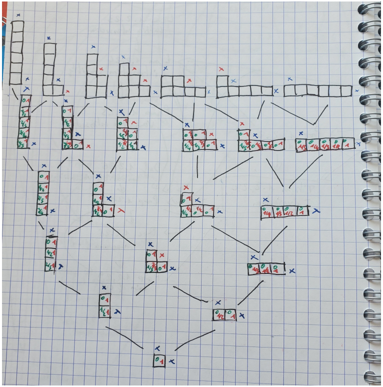
Un début de collaboration

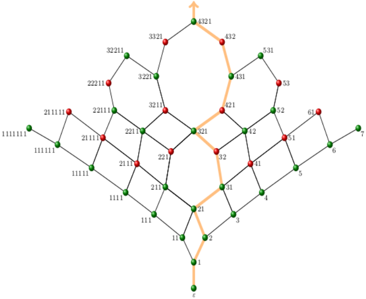
COMBINATORICS OF TRIANGULAR PARTITIONS

FRANÇOIS BERGERON AND MIKHAIL MAZIN

Abstract. The aim of this paper is to develop the combinatorics of constructions associated to what we call *triangular partitions*. As introduced in [9], these are the partitions whose cells are those lying below the line joining points $(r, 0)$ and $(0, s)$, for any given positive reals r and s. Classical notions such as Dyck paths and parking functions are naturally generalized by considering the set of partitions included in a given triangular partition. One of our striking results is that the restriction of the Young lattice to triangular partition has a planar Hasse diagram, with many nice properties. It follows that we may generalize the "first-return" recurrence, for the enumeration of classical Dyck paths, to the enumeration of all partitions contained in a fixed triangular one.


Partages triangulaires
Calcul de la pente du triangle
caractérise la pente
max
min




4321
3321
432
32211
531
3221
431
22211
53
3211
421
211111
22111
52
61
1111111
2211
321
42
7
21111
51
111111
221
32
6
2111
41
11111
211
31
5
1111
4
21
3
111
11
2
1
ε

Triangular Dyck paths

For a given triangular partition τ, we consider the set $\mathcal{D}_\tau := \{\alpha \mid \alpha \subseteq \tau\}$ of **τ-Dyck paths**. Observe that conjugation gives a bijection between $\mathcal{D}_\tau$ and $\mathcal{D}_{\tau'}$. We further write $\mathcal{D}_{(r,s)}$, when $\tau = \tau_{rs}$, and say that its elements are **$(r \times s)$-Dyck paths**. Observe that $\mathcal{D}_{(r,n)} = \mathcal{D}_{(kn,n)}$, for all $kn \leqslant r \leqslant kn+1$ and $k \in \mathbb{N}$, since the corresponding triangular partitions coincide. It is often convenient to consider that

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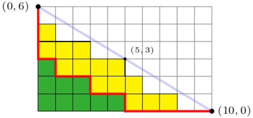


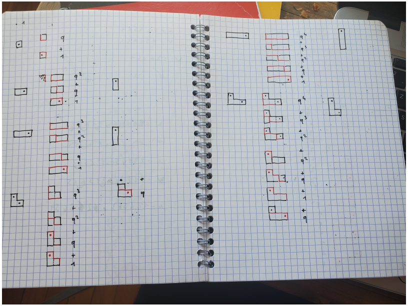
FIG. 4.1. The $\tau_{(10,6)}$-Dyck path 531000.

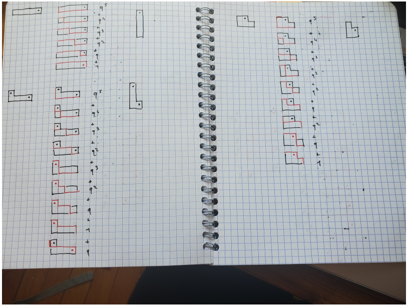
Proposition 5.1. *Denoting by $\partial = \partial_\tau$ the diagonal of a given triangular partition τ, then for the q-area enumerator of τ-Dyck, we have the recurrence*

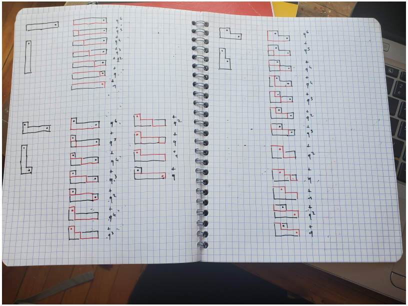
$$\mathcal{A}_\tau(q) = q^{|\partial|}\,\mathcal{A}_{\tau^\circ}(q) + \sum_{(\alpha,\beta)\in\Delta(\tau)} q^{|\alpha\cap\partial|}\,\mathcal{A}_{\alpha\setminus\partial}(q)\,\mathcal{A}_\beta(q), \tag{5.7}$$

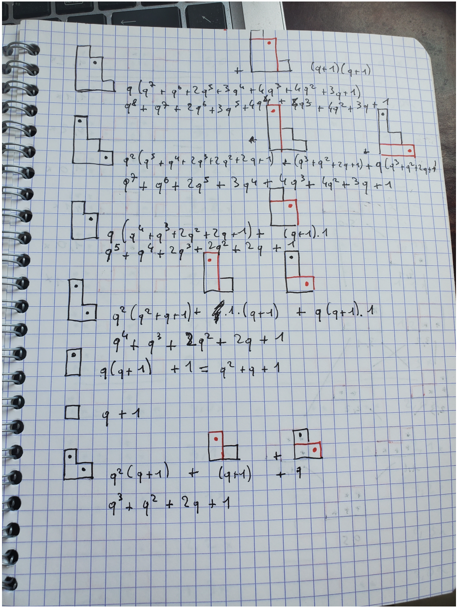
with initial condition $\mathcal{A}_\varepsilon(q) = 1$. In particular, setting $q = 1$, we have

$$\mathcal{A}_\tau = \mathcal{A}_{\tau^\circ} + \sum_{(\alpha,\beta)\in\Delta(\tau)} \mathcal{A}_{\alpha\setminus\partial}\,\mathcal{A}_\beta. \tag{5.8}$$








(q+1)(q+1)
q(q^7+q^6+2q^5+3q^4+4q^3+4q^2+3q+1)
q^2(q^5+q^4+2q^3+2q^2+2q+1)+(q^3+q^2+2q+1)+q(q^3+q^2+2q+1)
q^7+q^6+2q^5+3q^4+4q^3+4q^2+3q+1
q(q^4+q^3+2q^2+2q+1)+(q+1).1
q^5+q^4+2q^3+2q^2+2q+1
q^2(q^2+q+1)+1.(q+1)+q(q+1).1
q^4+q^3+2q^2+2q+1
q(q+1)+1=q^2+q+1
q+1
q^2(q+1)+(q+1)+q
q^3+q^2+2q+1

4.1. **Similar cells (Diagonal inversions).** Let τ be a given triangular partition. With the notations of Equation 1.2, for any $\alpha \subseteq \tau$ consider the set of cells c of α that have hooks in α that are "similar" to τ:

$$\mathrm{Sim}_\tau(\alpha) := \{c \in \alpha \mid t'(c,\alpha) \leqslant t_\tau < t''(c,\alpha)\},$$

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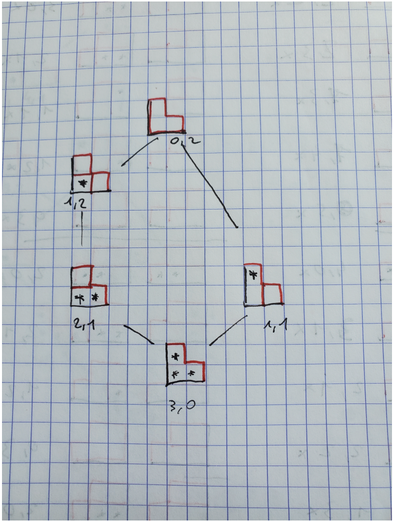
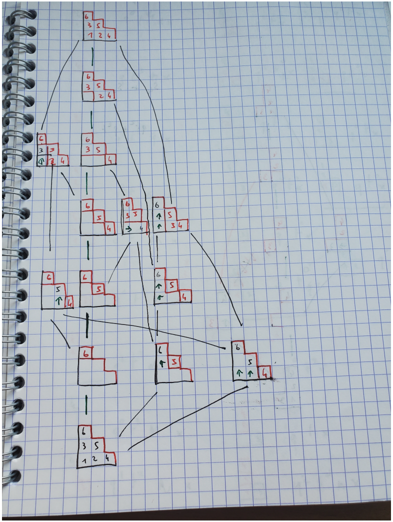
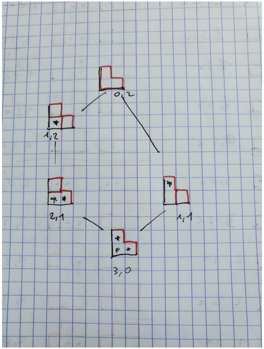
FIG. 4.4. Sim-cells for subpartitions of 32. The first 6 subpartitions are similar to 32.

Treillis de Tamari



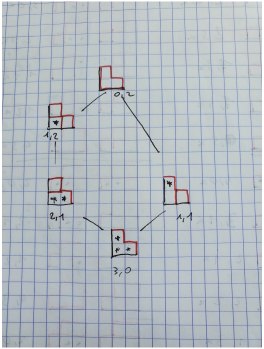


Symétries et Treillis de Tamari



$$f = s_{1,1} + s_3$$

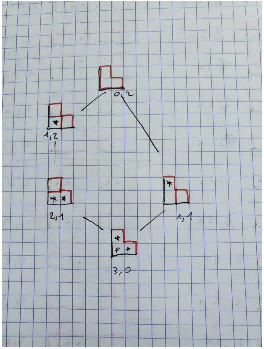
Symétries et Treillis de Tamari



$$f = s_{1,1} + s_3$$

$$f(q,t) = q^3 + q^2 t + q t^2 + t^3 + qt$$
$$f(1,1) = 5$$

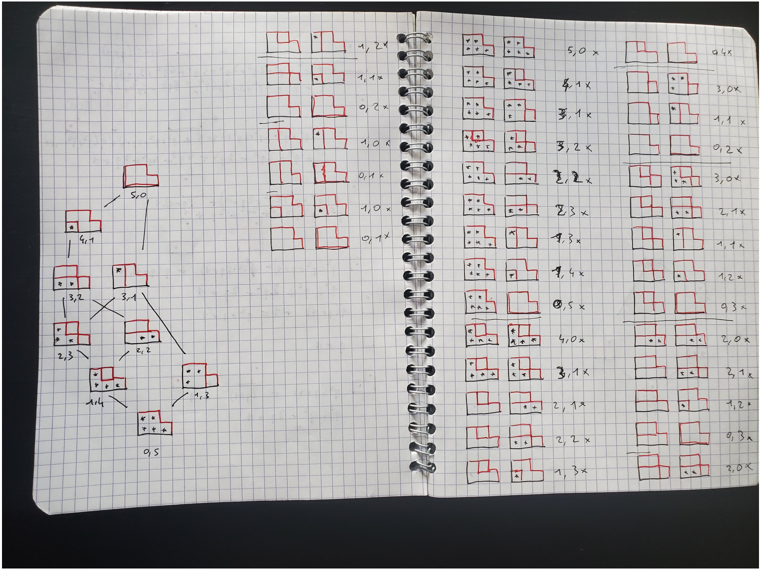
Symétries et Treillis de Tamari

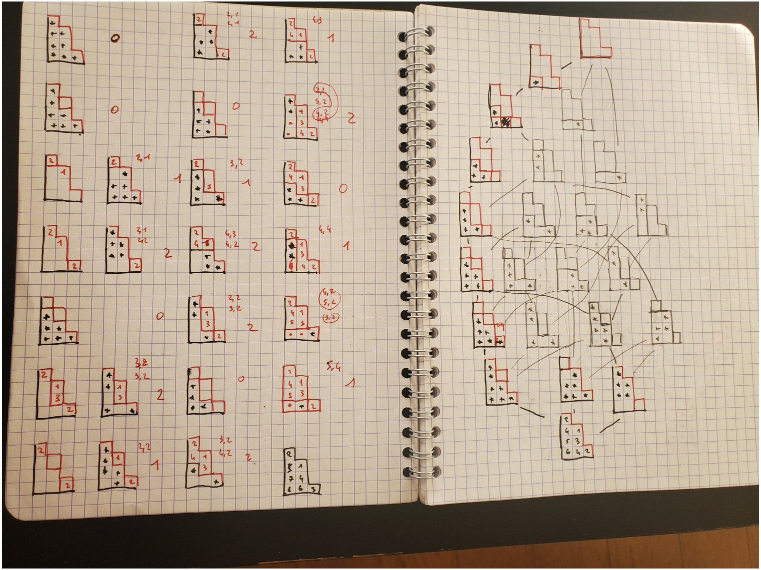


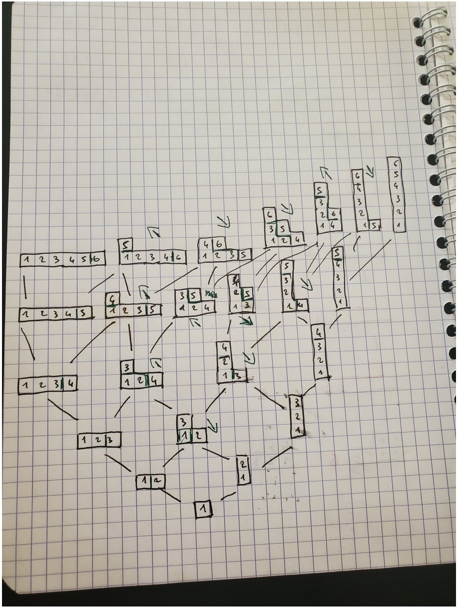
$$f = s_{1,1} + s_3$$

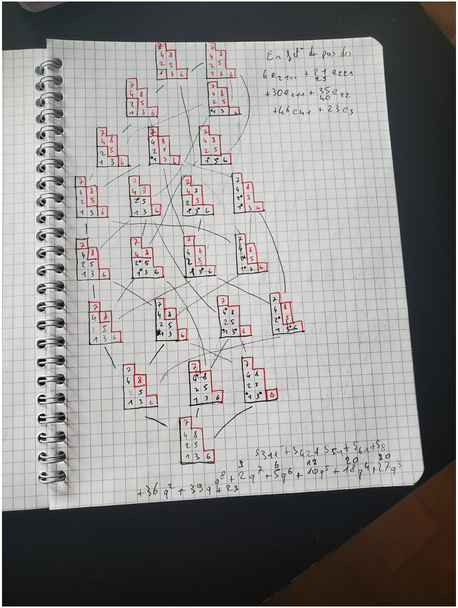
$$f(q,t) = q^3 + q^2t + qt^2 + t^3 + qt$$
$$f(1,1) = 5$$

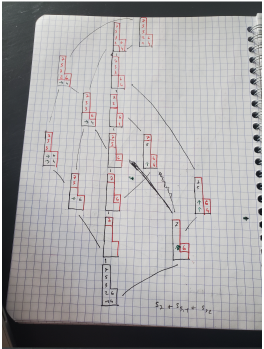
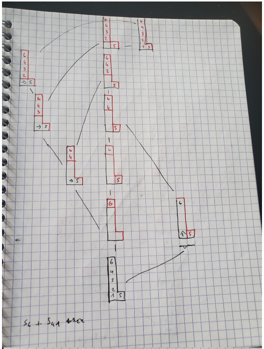
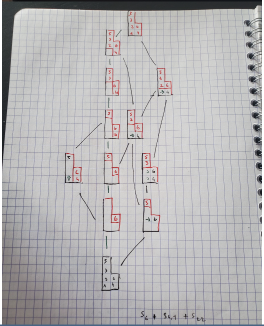
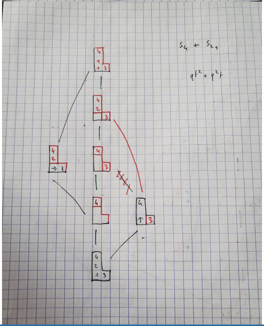
$$\begin{aligned} f(q,t,r) &= q^3 + q^2t + qt^2 + t^3 + q^2r + qtr \\ &= +t^2r + qr^2 + tr^2 + r^3 + qt + qr + tr \end{aligned}$$
$$f(1,1,1) = 13$$

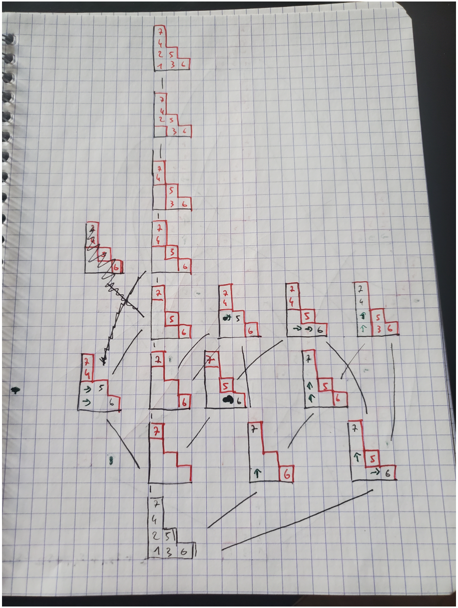

5,0
4,1
3,2
3,1
2,3
2,2
1,4
1,3
0,5

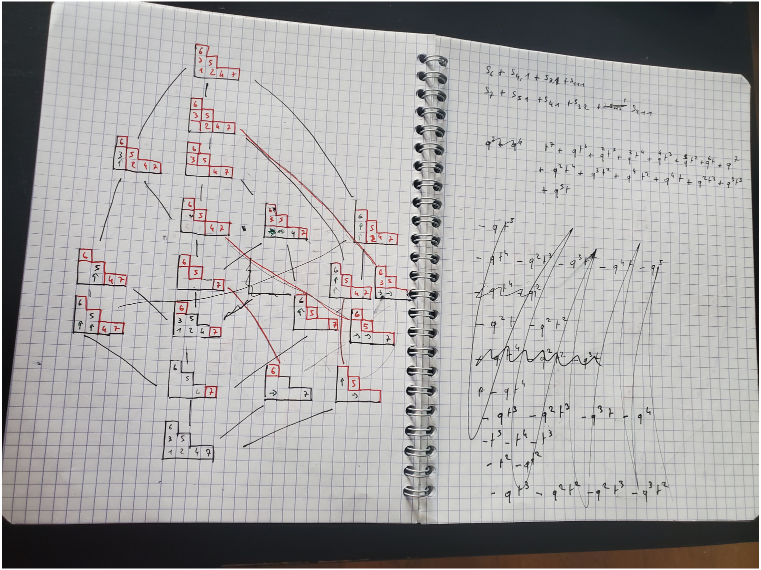


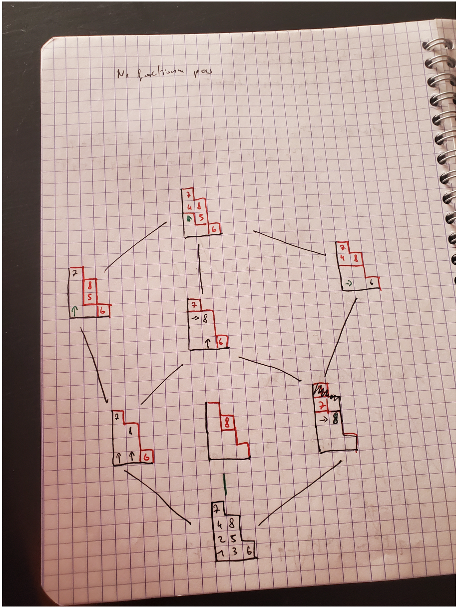


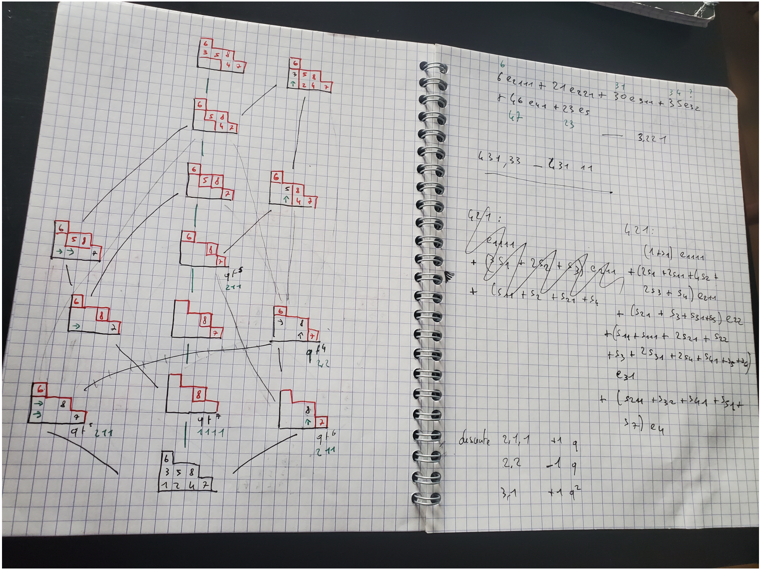

En fct du pas des





S4 + S21
qt2 + q2t






Ne fonctionne pas


discute 2,1,1 +1 q
2,2 −1 q
3,1 +1 q²