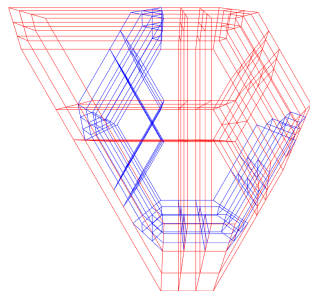
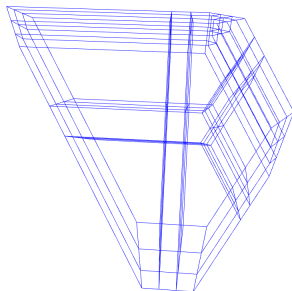
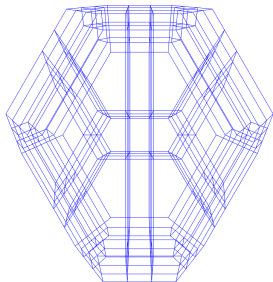


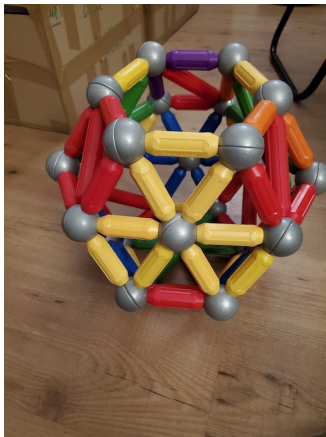
# $s$ -weak order and $s$ -permutahedra

Cesar Ceballos – Viviane Pons

Univ. of Graz – LISN, Univ. Paris-Saclay



# The Permutahedron



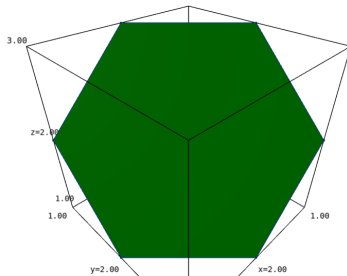
## As a convex hull of permutations

```
Entrée [1]: P = Polyhedron(List(Permutations(3)))
P
```

```
Out[1]: A 2-dimensional polyhedron in ZZ^3 defined as the convex hull of 6 vertices (use the .plot() method to plot)
```

```
Entrée [2]: P.plot()
```

```
Out[2]:
```



①

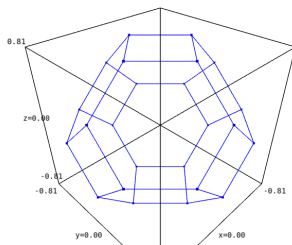
## For size 4

Entrée [3]: `P = Polyhedron(list(Permutations(4)))`  
P

Out[3]: A 3-dimensional polyhedron in  $\mathbb{Z}^4$  defined as the convex hull of 24 vertices (use the `.plot()` method to plot)

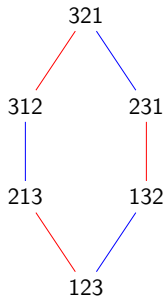
Entrée [4]: `P.plot()`

Out[4]:

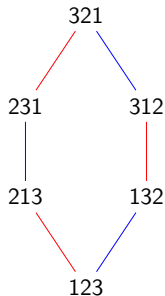


①

Weak order

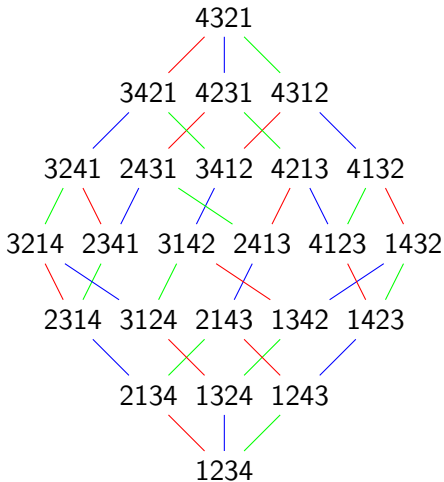
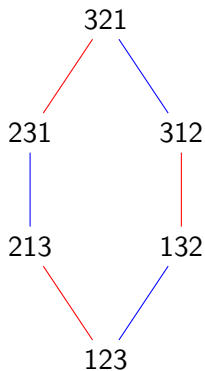


left weak order

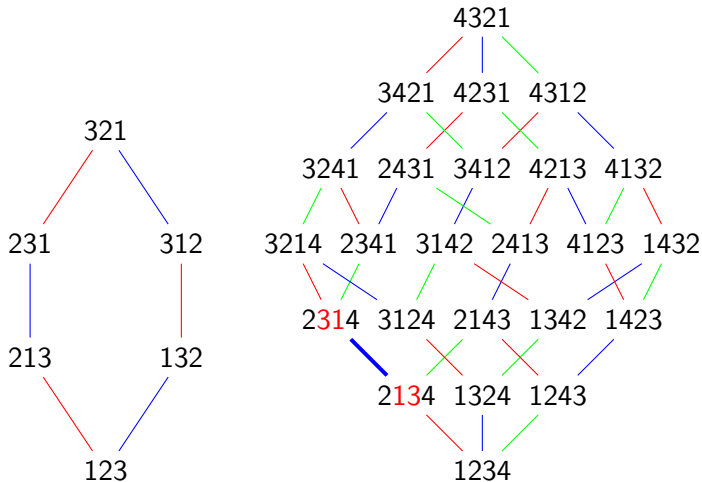


right weak order

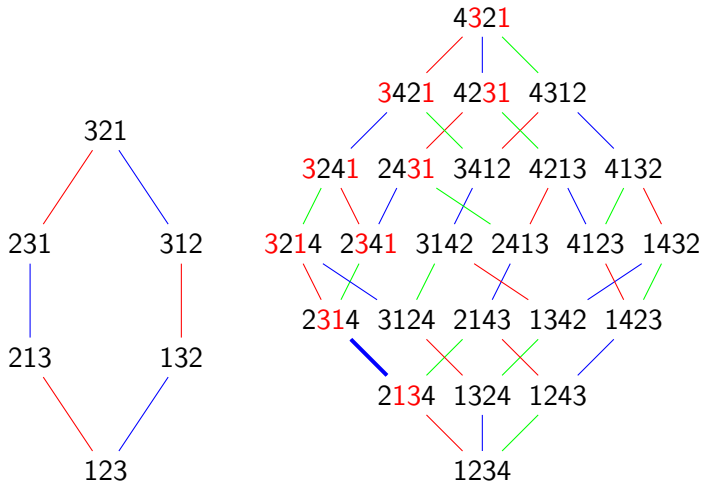
## Right weak Order



## Right weak Order

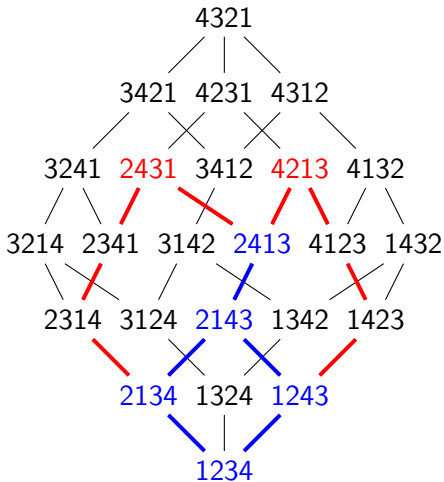
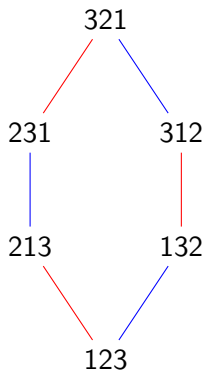


## Right weak Order





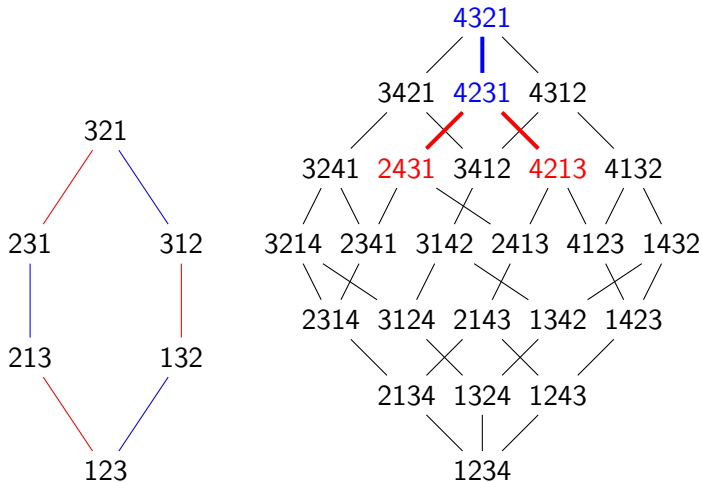
## Right weak Order



$$2413 \wedge 4213 = 2413$$

$$2413 \vee 4213 = 4231$$

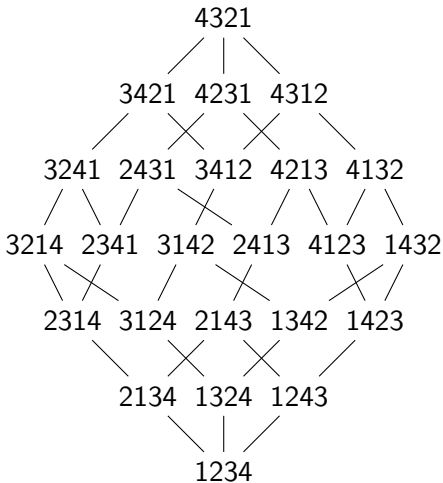
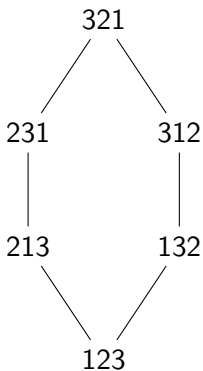
## Right weak Order



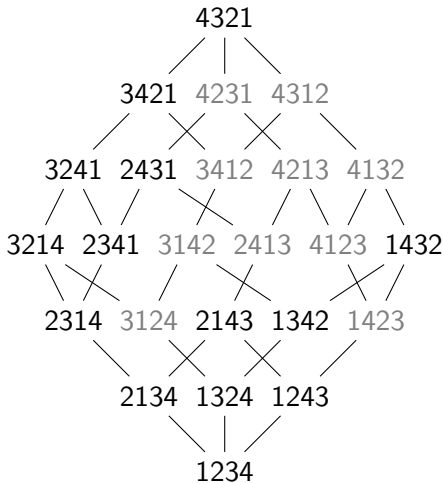
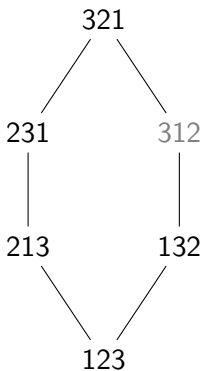
$$2413 \wedge 4213 = 2413$$

$$2413 \vee 4213 = 4231$$

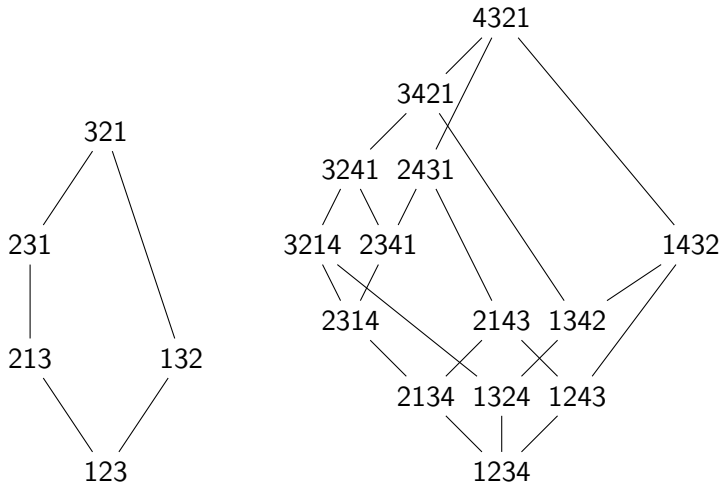
## From the Weak Order to the Tamari lattice



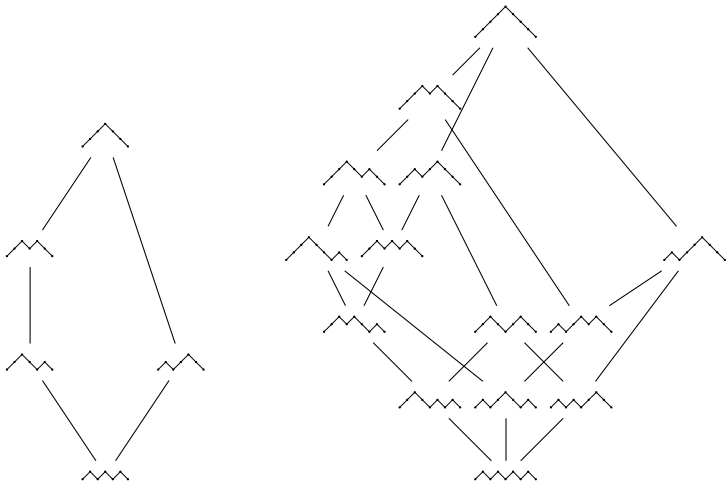
## From the Weak Order to the Tamari lattice

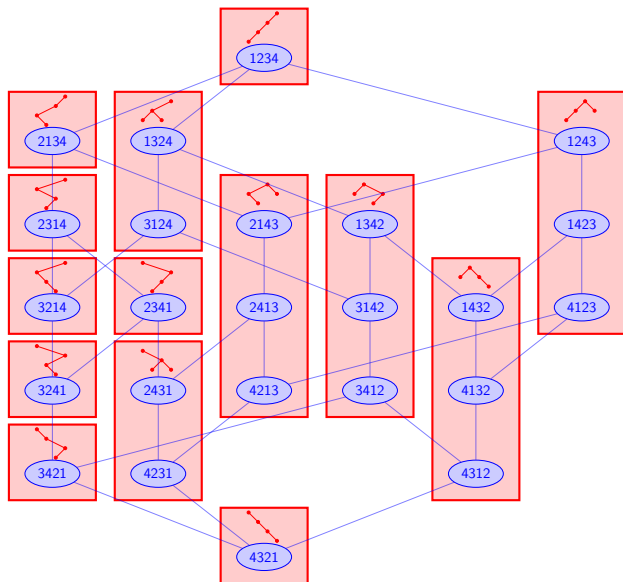


## From the Weak Order to the Tamari lattice

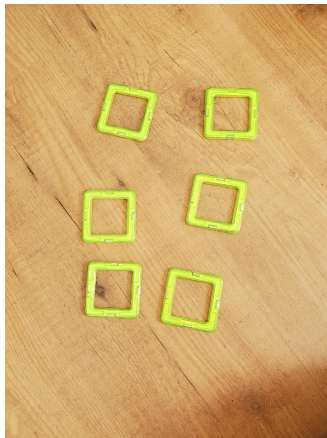


## From the Weak Order to the Tamari lattice

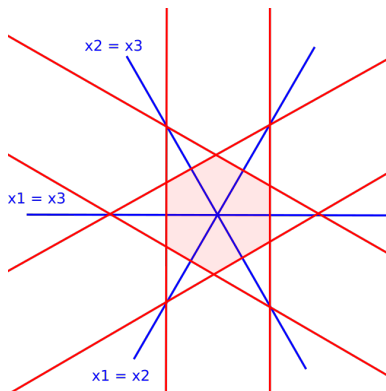




# From the permutahedron to the associahedron : removing faces



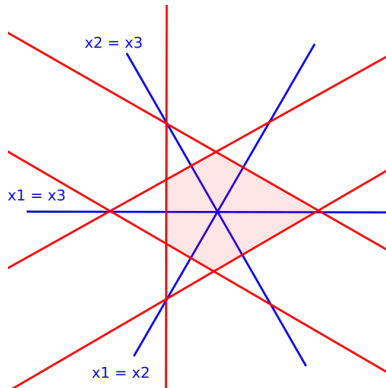




$$J \subseteq [n] \rightarrow \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$

$$\begin{array}{l} 12|3 \quad x_1 + x_2 \geq 3 \\ 2|13 \quad x_2 \geq 1 \\ 23|1 \quad x_2 + x_3 \geq 3 \\ 3|12 \quad x_3 \geq 1 \\ 13|2 \quad x_1 + x_3 \geq 3 \\ 1|23 \quad x_1 \geq 1 \end{array}$$

$$J \subseteq [n] \rightarrow \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$



$J$  interval

$$12|3 \quad x_1 + x_2 \geq 3$$

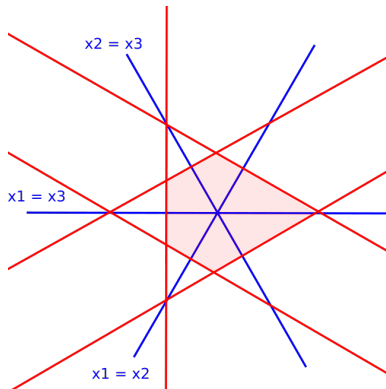
$$2|13 \quad x_2 \geq 1$$

$$23|1 \quad x_2 + x_3 \geq 3$$

$$3|12 \quad x_3 \geq 1$$

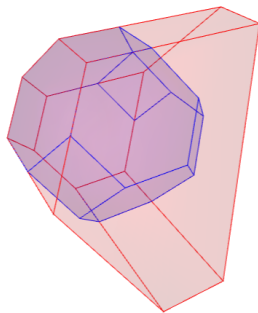
$$13|2$$

$$1|23 \quad x_1 \geq 1$$



$$J \subseteq [n] \rightarrow \sum_{j \in J} x_j \geq \binom{|J| + 1}{2}$$

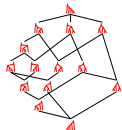
$J$  interval



Weak order



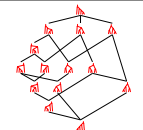
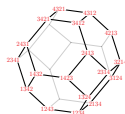
Tamari lattice



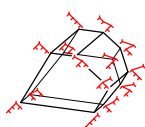
Weak order



Permutahedron

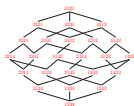


Tamari lattice

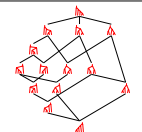
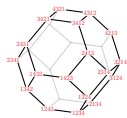


Associahedron

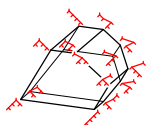
Weak order



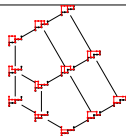
Permutahedron



Tamari lattice



Associahedron

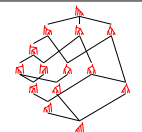
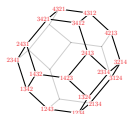


$\nu$ -Tamari  
Préville-Ratelle, Viennot

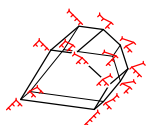
Weak order



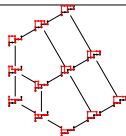
Permutahedron



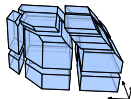
Tamari lattice



Associahedron



$\nu$ -Tamari  
Préville-Ratelle, Viennot

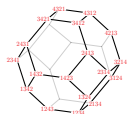


$\nu$ -Associahedron  
Ceballos, Padrol, Sarmiento

Weak order

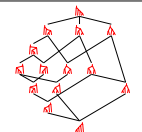


Permutahedron

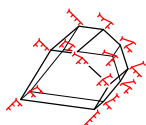


?

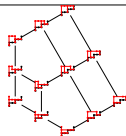
?



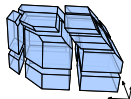
Tamari lattice



Associahedron



$\nu$ -Tamari  
Préville-Ratelle, Viennot



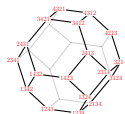
$\nu$ -Associahedron  
Ceballos, Padrol, Sarmiento



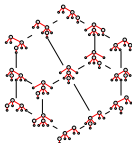
Weak order



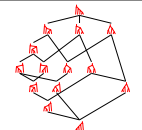
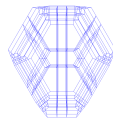
Permutahedron



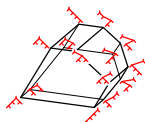
$s$ -Weak order



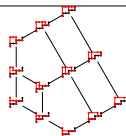
$s$ -Permutahedron



Tamari lattice



Associahedron



$\nu$ -Tamari  
Préville-Ratelle, Viennot



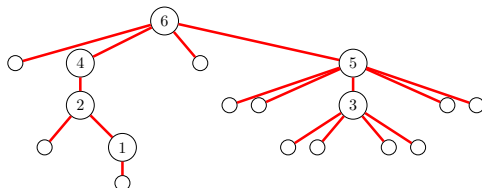
$\nu$ -Associahedron  
Ceballos, Padrol, Sarmiento

# $s$ -weak order

## $s$ -decreasing trees

- ▶  $s$  is a sequence of non-negative integers
- ▶ The node  $i$  has  $s(i) + 1$  children
- ▶ Nodes labels decreasing from root to leaves

$$s = (0, 1, 3, 0, 4, 3)$$



How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of  $s$ -decreasing trees:

How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of  $s$ -decreasing trees:

$(1+3)$

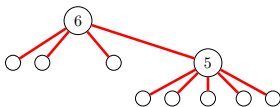


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of  $s$ -decreasing trees:

$$(1+3) \times (1+3+4)$$

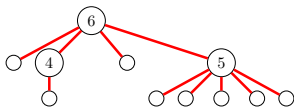


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of  $s$ -decreasing trees:

$$(1+3) \times (1+3+4) \times (1+3+4+0)$$

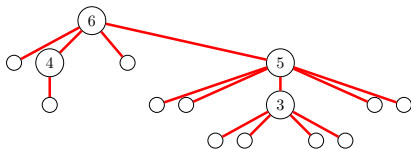


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of  $s$ -decreasing trees:

$$(1+3) \times (1+3+4) \times (1+3+4+0) \times (1+3+4+0+3)$$

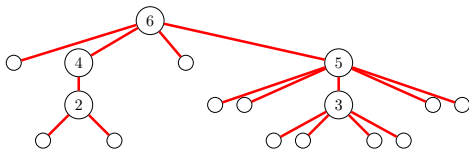


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of  $s$ -decreasing trees:

$$(1+3) \times (1+3+4) \times (1+3+4+0) \times (1+3+4+0+3) \times (1+3+4+0+3+1)$$



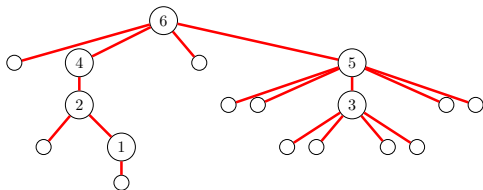


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of  $s$ -decreasing trees:

$$(1+3) \times (1+3+4) \times (1+3+4+0) \times (1+3+4+0+3) \times (1+3+4+0+3+1)$$

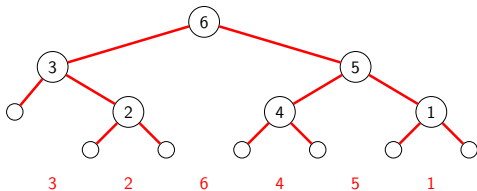


## Permutations

$$s = (1, 1, 1, 1, 1, 1)$$

Number of  $s$ -decreasing trees:

$$(1+1) \times (1+1+1) \times (1+1+1+1) \times (1+1+1+1+1) \times (1+1+1+1+1+1)$$

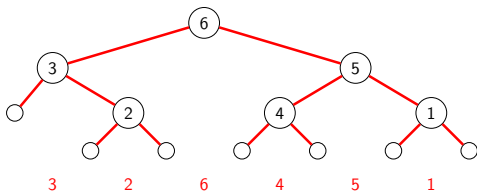


## Permutations

$$s = (1, 1, 1, 1, 1, 1)$$

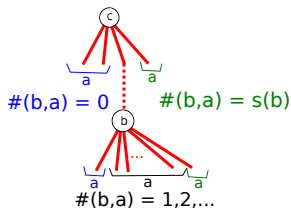
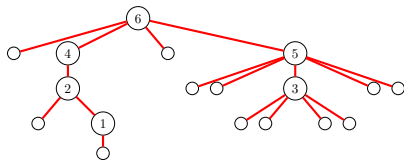
Number of  $s$ -decreasing trees:  $6!$

$$(1+1) \times (1+1+1) \times (1+1+1+1) \times (1+1+1+1+1) \times (1+1+1+1+1+1)$$



## Tree-inversions

For all  $b > a$ , we define  $0 \leq \#(b, a) \leq s(b)$ .



$$\begin{array}{cccccc}
 \#(6, 5) = 3 & \#(6, 4) = 1 & \#(6, 3) = 3 & \#(6, 2) = 1 & \#(6, 1) = 1 \\
 \#(5, 4) = 0 & \#(5, 3) = 2 & \#(5, 2) = 0 & \#(5, 1) = 0 \\
 & \#(4, 3) = 0 & \#(4, 2) = 0 & \#(4, 1) = 0 \\
 & & \#(3, 2) = 0 & \#(3, 1) = 0 \\
 & & & \#(2, 1) = 1
 \end{array}$$

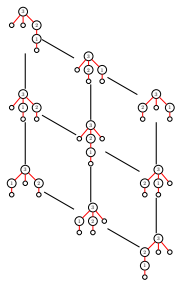
## The $s$ -weak order

$R, T$ ,  $s$ -decreasing trees:

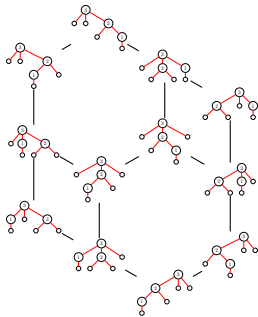
$$R \preceq T \Leftrightarrow \forall b > a, \#_R(b, a) \leq \#_T(b, a)$$

## Theorem (Ceballos, P.)

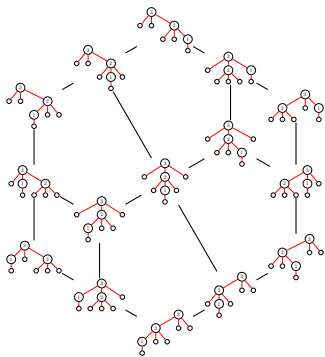
*The  $s$ -weak order is always a lattice.*



$(0, 0, 2)$



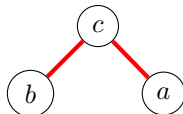
$(0, 1, 2)$



$(0, 2, 2)$

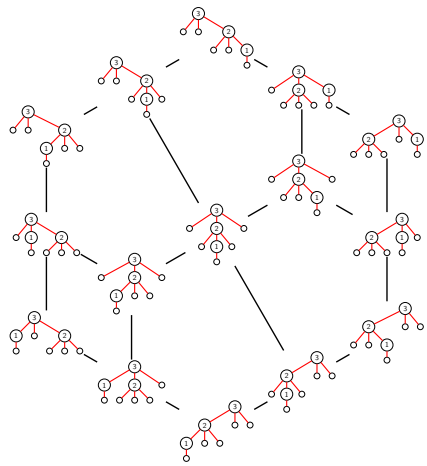
## $s$ -Tamari lattice

Select trees which avoid “pattern 231”:  $a < b < c$

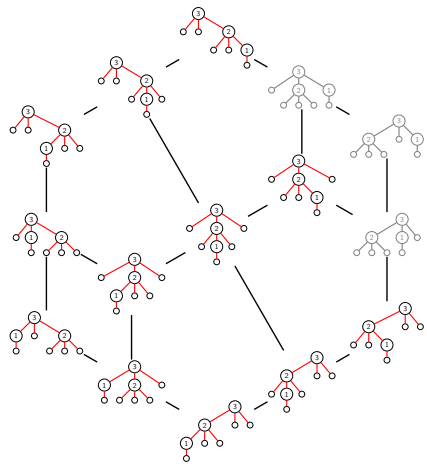


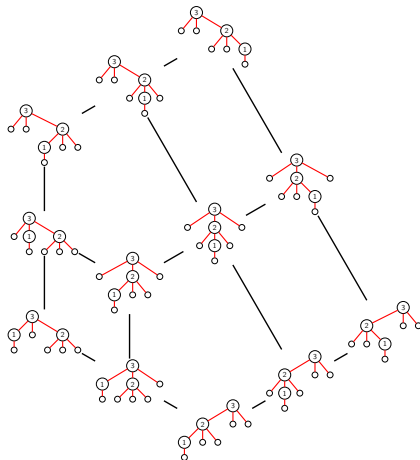
### Theorem (Ceballos, P.)

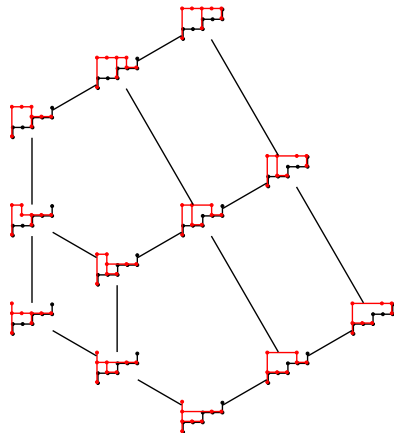
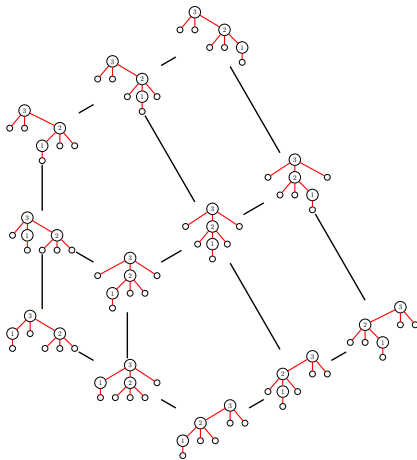
*The set of 231-avoiding  $s$ -decreasing trees form a sublattice, the  $s$ -Tamari lattice, isomorphic to the  $\nu$ -Tamari lattice.*

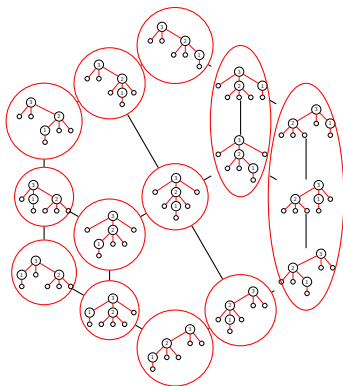








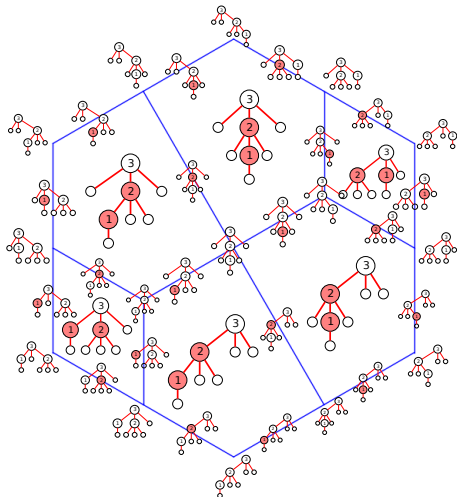




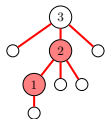
### Theorem (Ceballos, P.)

*The  $s$ -Tamari lattice is a quotient lattice of the  $s$ -weak order if  $s$  does not contain any zeros.*

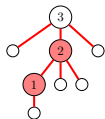
# Geometry: the $s$ -Permutahedron



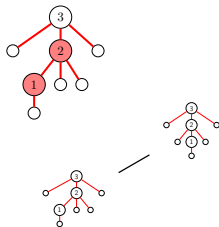
## Faces : pure intervals



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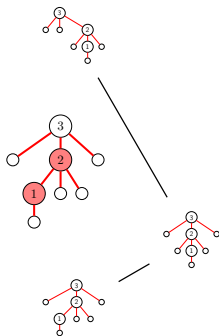


## Faces : pure intervals

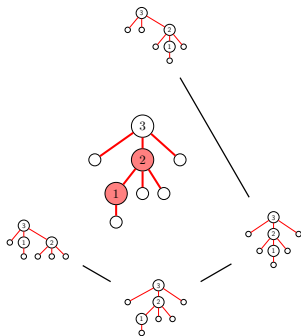




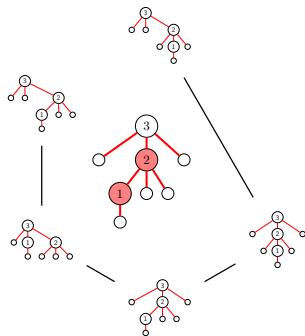
## Faces : pure intervals



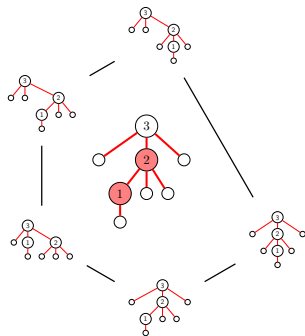
## Faces : pure intervals



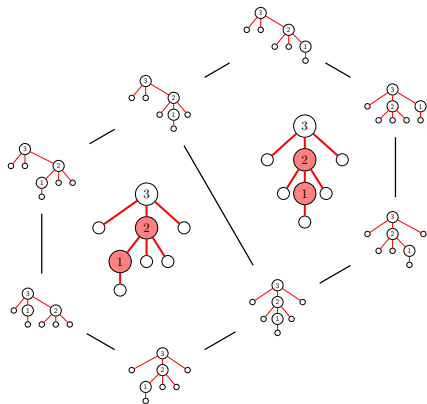
## Faces : pure intervals



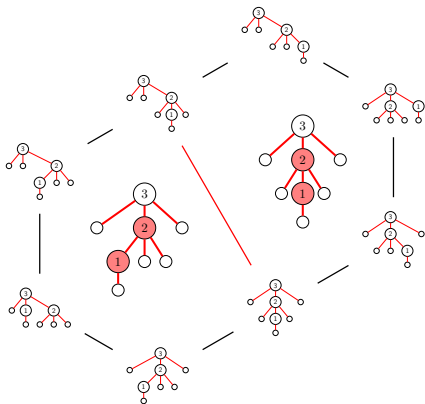
## Faces : pure intervals



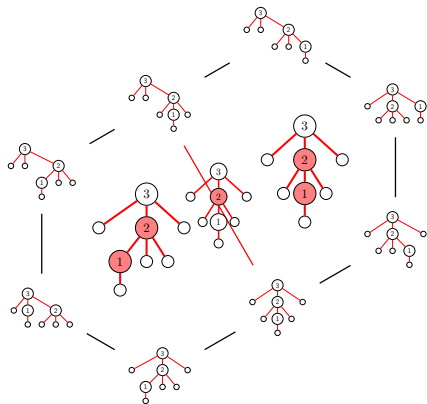
## Faces : pure intervals



## Faces : pure intervals



## Faces : pure intervals





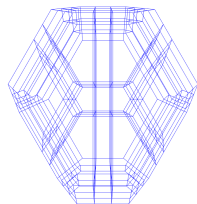
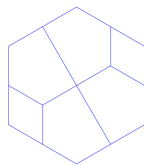


## Conjecture (Ceballos, P.)

*The  $s$ -Permutahedron is a polytopal complex*

## Conjecture (Ceballos, P.)

*It can be realized as a polytopal subdivision of the classical permutahedron.*

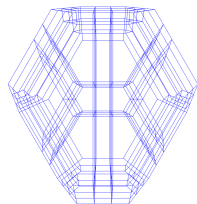
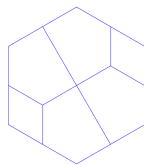


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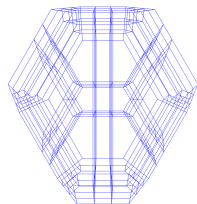
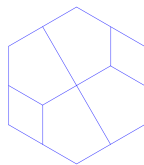
**2023 Solved for  $s$  without 0 via flow polytopes** (González D'Leon, Morales, Philippe, Tamayo and Yip)

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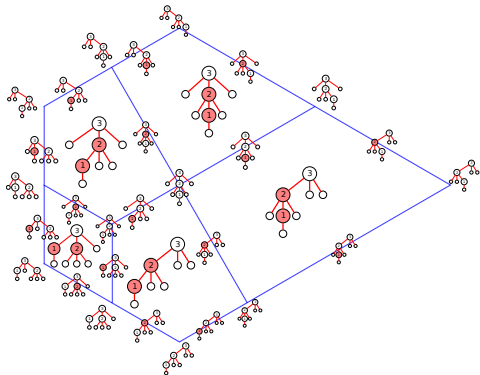
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**2023 Solved for  $s$  without 0 via flow polytopes** (González D'Leon, Morales, Philippe, Tamayo and Yip)

**2024 Solved for all  $s$**  (Philippe, Pilaud)

## The $s$ -Associahedron

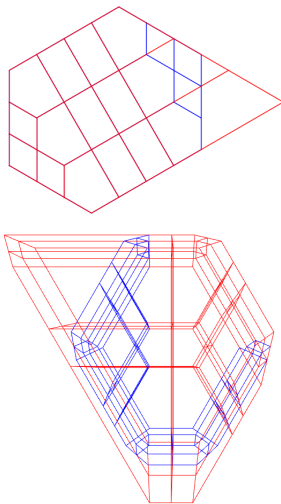


### Theorem (Ceballos, P.)

*The  $s$ -associahedron is isomorphic to the  $\nu$ -associahedron for  $\nu = NE^{s(n)} \dots NE^{s(1)}$ .*

## Conjecture (Ceballos, P.)

*A realization of the  $s$ -associahedron can be obtained by “removing” some facets of the  $s$ -permutahedron.*

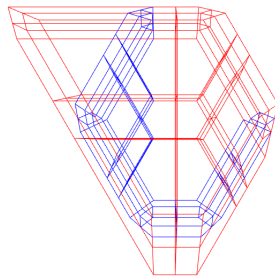
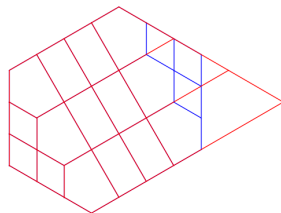


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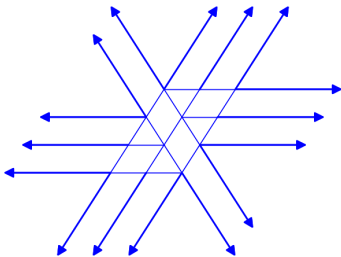
**2024 Solved for all quotients**

(Philippe, Pilaud)

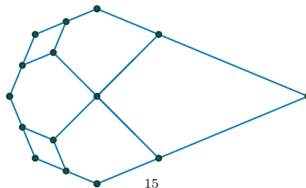
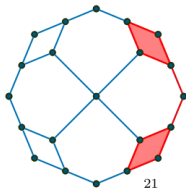


## How?

The  $s$ -foam (previously called  $s$ -braid arrangement)



What's next? Other types?





## References

- ▶ Ceballos, P. The  $s$ -weak order and  $s$ -permutahedra. *FPSAC 2019*
- ▶ Ceballos, P. The  $s$ -weak order and  $s$ -permutahedra I: combinatorics and lattice structure. 2022+. *to appear in SIDMA* arXiv:2212.11556
- ▶ Ceballos, P. The  $s$ -weak order and  $s$ -permutahedra II: The combinatorial complex of pure intervals. 2023+. *to appear in EJC* arXiv:2309.14261
- ▶ Lacina. Poset topology of  $s$  weak order via SB-labelings. *J. Combinatorics* 2022.
- ▶ González D'León, Morales, Philippe, Tamayo Jiménez, Yip. Realizing the  $s$ -permutahedron via flow polytopes. 2023+. arXiv:2307.03474
- ▶ Philippe, Pilaud, Geometric realizations of the  $s$ -weak order and its lattice quotients, 2024+ arXiv:2405.02092