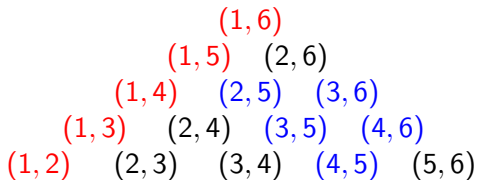


A Conjecture on descents, inversions and the weak order

Christophe Hohlweg – Viviane Pons

LACIM UQAM – LISN, Univ. Paris-Saclay

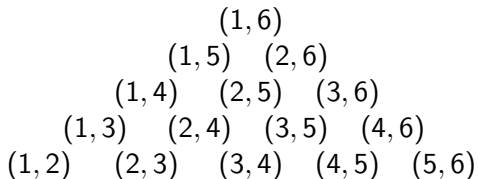


Let us start with a Coxeter group W and look at the (*left*) *inversion set* of an element $w \in W$

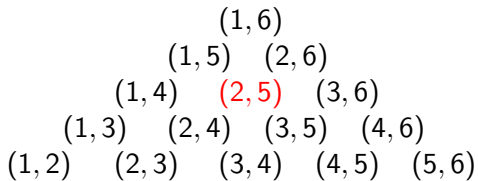
$$\text{invs}(w) = \{t \in T \mid \ell(tw) < \ell(t)\}.$$

where T is the set of *reflections* of W

Example in Type A

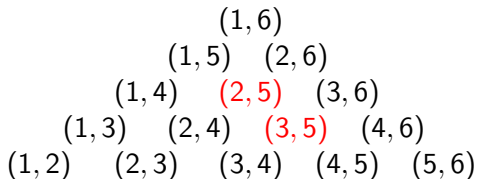
 $w = 526341$ 

Example in Type A

 $w = 526341$ 

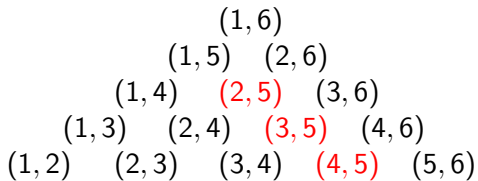
Example in Type A

$$w = 526341$$



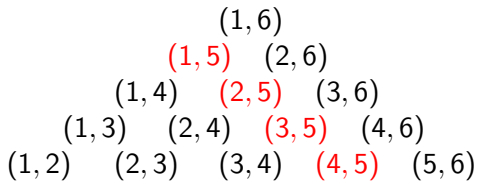
Example in Type A

$$w = 526341$$



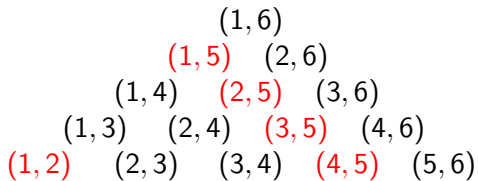
Example in Type A

$$w = 526341$$



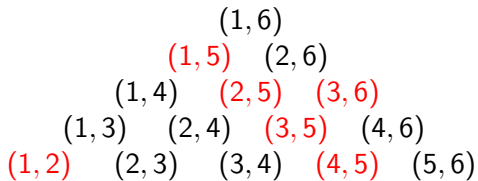
Example in Type A

$$w = 526341$$



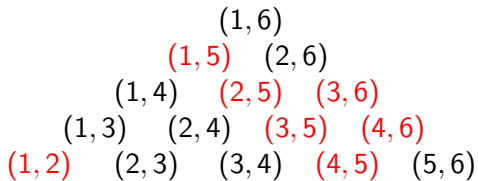
Example in Type A

$$w = 526341$$



Example in Type A

$$w = 526341$$



Example in Type A

$$w = 526341$$

$$\begin{array}{cccccc} & & & & & (1, 6) \\ & & & & & (1, 5) & (2, 6) \\ & & & & & (1, 4) & (2, 5) & (3, 6) \\ & & & & & (1, 3) & (2, 4) & (3, 5) & (4, 6) \\ & & & & & (1, 2) & (2, 3) & (3, 4) & (4, 5) & (5, 6) \end{array}$$

Example in Type A

$$w = 526341$$

$$\begin{array}{cccccc} & & & & & (1, 6) \\ & & & & & (1, 5) & (2, 6) \\ & & & & & (1, 4) & (2, 5) & (3, 6) \\ & & & & & (1, 3) & (2, 4) & (3, 5) & (4, 6) \\ & & & & & (1, 2) & (2, 3) & (3, 4) & (4, 5) & (5, 6) \end{array}$$

Example in Type A

$$w = 526341$$

$$\begin{array}{cccccc} & & & & & (1, 6) \\ & & & & & (1, 5) & (2, 6) \\ & & & & & (1, 4) & (2, 5) & (3, 6) \\ & & & & & (1, 3) & (2, 4) & (3, 5) & (4, 6) \\ & & & & & (1, 2) & (2, 3) & (3, 4) & (4, 5) & (5, 6) \end{array}$$

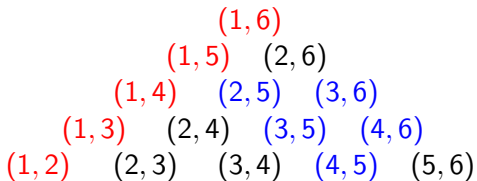
Example in type A : a bipartition of an element w

$w = 526341$

$u = 234561$

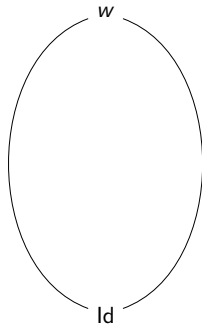
$v = 152634$

$$\text{invs}(w) = \text{invs}(u) \sqcup \text{invs}(v)$$



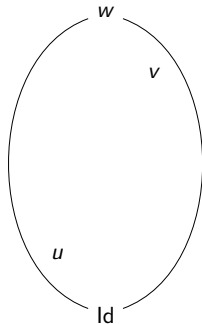
Bipartitions and the (right) weak order

(right) Weak order: order elements of W by inclusion of (left) inversion sets.



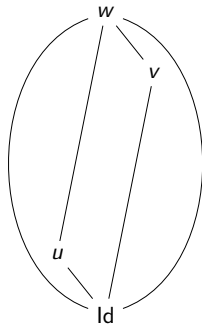
Bipartitions and the (right) weak order

(right) Weak order: order elements of W by inclusion of (left) inversion sets.



Bipartitions and the (right) weak order

(right) Weak order: order elements of W by inclusion of (left) inversion sets.

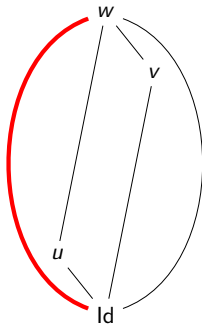


$$w = u \vee v$$

$$d = u \wedge v$$

Bipartitions and the (right) weak order

(right) Weak order: order elements of W by inclusion of (left) inversion sets.



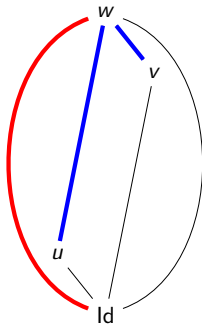
$$w = u \vee v$$

$$d = u \wedge v$$

$$d(id, w)$$

Bipartitions and the (right) weak order

(right) Weak order: order elements of W by inclusion of (left) inversion sets.



$$w = u \vee v$$

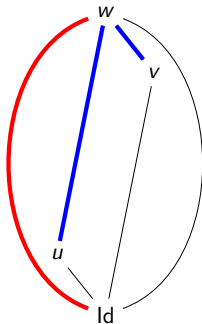
$$d = u \wedge v$$

$$d(id, w)$$

$$d(u, v)$$

Bipartitions and the (right) weak order

(right) Weak order: order elements of W by inclusion of (left) inversion sets.



$$w = u \vee v$$

$$d = u \wedge v$$

$$d(id, w)$$

$$d(u, v)$$

u, v bipartition of $w \Leftrightarrow$

$$d(u, v) = d(id, w)$$

Some numbers

	A_2	A_3	A_4	A_5
Elements which admit bipartition	1	11	81	554
Partition-indecomposable elements	5	13	39	166

Inversion partitions in the literature

Appear in a geometric problem

Eigenvalue problem and a new product in cohomology of flag varieties,
Belkale and Kumar, 2006

Inversion partitions in the literature

Appear in a geometric problem

Eigenvalue problem and a new product in cohomology of flag varieties,
Belkale and Kumar, 2006

Also

Decomposing sets of inversions, Katthän, 2013

Decomposing inversion sets of permutations and applications to faces of
the Littlewood-Richardson cone, Dewji, Dimitrov, McCabe, Roth, Wehlau
and Wilson, 2017

Our motivation

A conjecture formulated by Ressayre on the number of descents

(right) descents of elements

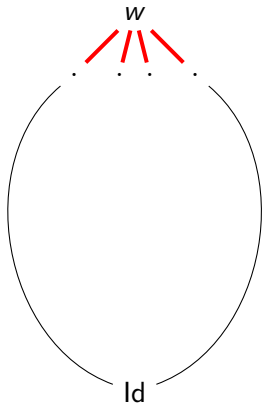
Let $w \in W$, a (*right*) descent s is a simple reflection such that $\ell(ws) = \ell(w) - 1$. We write $d_R(w)$ the number of right descents.

Example in type A

$$526341 \times (1, 2) = 256341$$

$$d_R(5.26.34.1) = 3$$

On the weak order



descents of $w =$ coatoms of w

The Conjecture

If u, v is a bipartition of w , then

$$d_R(w) = d_R(u) + d_R(v)$$

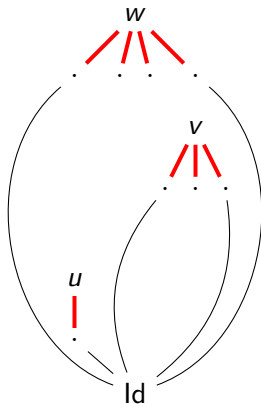
Example in type A

$$w = 5.26.34.1 \quad d_R(w) = 3$$

$$u = 23456.1 \quad d_R(u) = 1$$

$$v = 15.26.34 \quad d_R(v) = 2$$

On the weak order



The Conjecture

True for Weil groups (through algebraic geometry, case by case)

Intersection multiplicity one for the belkale-kumar product, Ressayre and Francone, 2023

Direct (elementary) proof in type A and B

Hohlweg and P., 2024

Still open for other Coxeter groups

Idea of the proof on the example

Recursive decomposition

“left words” “right words”

$$w = 5.26.34.1$$

$$u = 23456.1$$

$$v = 15.26.34$$

Idea of the proof on the example

Recursive decomposition

“left words” “right words”

$$w = 5.26.34.1$$

$$u = 23456.1$$

$$v = 15.26.34$$

Idea of the proof on the example

Recursive decomposition

“left words” “right words”

$$w = 5.26.34.1$$

$$u = 23456.1$$

$$v = 15.26.34$$

Idea of the proof on the example

Recursive decomposition

	“left words”	“right words”
$w = 5.26.34.1$	5.2	6.34.1
$u = 23456.1$	25	346.1
$v = 15.26.34$	5.2	16.34

Idea of the proof on the example

Recursive decomposition

	“left words”	“right words”
$w = 5.26.34.1$	2.1	4.23.1
$u = 23456.1$	12	234.1
$v = 15.26.34$	2.1	14.23

Idea of the proof on the example

Recursive decomposition

	“left words”	“right words”	“decreased right words”
$w = 5.26.34.1$	2.1	4.23.1	23.1
$u = 23456.1$	12	234.1	
$v = 15.26.34$	2.1	14.23	

Idea of the proof on the example

Recursive decomposition

	“left words”	“right words”	“decreased right words”
$w = 5.26.34.1$	2.1	4.23.1	23.1
$u = 23456.1$	12	234.1	23.1
$v = 15.26.34$	2.1	14.23	123

Also in type B

$$\begin{array}{l}
 w = 1.\bar{4}\bar{3}.\bar{6}\bar{2}.\bar{5}|5.26.34.\bar{1} \\
 u = \bar{4}\bar{3}.\bar{6}\bar{2}.\bar{5}\bar{1}|15.26.34 \\
 v = 1.\bar{6}\bar{5}\bar{4}\bar{3}\bar{2}|23456.\bar{1}
 \end{array}$$

“left words” “right words”

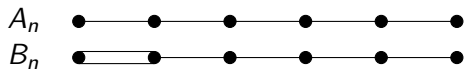
Also in type B

	“left words”	“right words”
$w = 1.\bar{4}\bar{3}.\bar{6}\bar{2}.\bar{5} 5.26.34.\bar{1}$	$\bar{2}.\bar{5} 5.2$	$6.34.\bar{1}$
$u = \bar{4}\bar{3}.\bar{6}\bar{2}.\bar{5}\bar{1} 15.26.34$	$\bar{2}.\bar{5} 5.2$	$\bar{1}6.34$
$v = 1.\bar{6}\bar{5}\bar{4}\bar{3}\bar{2} 23456.\bar{1}$	$\bar{5}\bar{2} 25$	$346.\bar{1}$

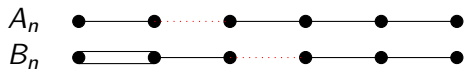
Also in type B

	“left words”	“right words”
$w = 1.\bar{4}\bar{3}.\bar{6}\bar{2}.\bar{5} 5.26.34.\bar{1}$	$\bar{1}.\bar{2} 2.1$	4.23.1
$u = \bar{4}\bar{3}.\bar{6}\bar{2}.\bar{5}\bar{1} 15.26.34$	$\bar{1}.\bar{2} 2.1$	14.23
$v = 1.\bar{6}\bar{5}\bar{4}\bar{3}\bar{2} 23456.\bar{1}$	$\bar{2}\bar{1} 12$	234.1

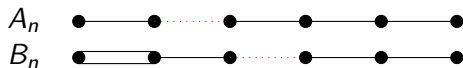
Coxeterize?



Coxeterize?



Coxeterize?



Goal: find a group generic recursive decomposition of elements that “keeps” the descents

Thank you!

A conjecture on descents, inversions and the weak order, Hohlweg and P.,
arXiv:2412.09227, 2024