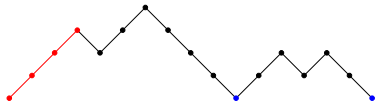
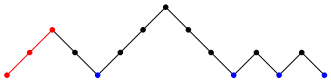
Mini project: Dyck Paths

The initial rise of a Dyck Path is the number of initial up steps, the number of touch points is the number of time the path touch the initial line after the beginning. For example:

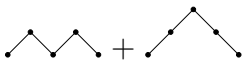


Initial Rise: 3, Touch points: 2



Initial Rise: 2, Touch points: 4

1. Compute the initial rise and touch points of all Dyck Paths of size 3 and 4
2. If A is a Dyck path with initial rise k and B is a Dyck path with initial rise ℓ. What is the initial rise of $A \times B$? (To answer this question, first compute many examples and then try to express the general rule).
3. Let P_n be a polynomial defined this way: for each Dyck path d of size n, I sum y to the power of the initial rise of d. As an example, the Dyck paths of size 2 are given by

   They have respective initial rise of 1 and 2 and so

   $$P_2 = y + y^2.$$

   Check that $P_3 = y^3 + 2y^2 + 2y$ and $P_4 = y^4 + 3y^3 + 5y^2 + 5y$.
4. If A is a Dyck path with k touch points and B is a Dyck path with ℓ touch points, how many touch points does $A \times B$ have? (Compute examples and find the general rule)
5. Now we define the polynomial P'_n this way: for each Dyck path d of size n, I sum y to the power of the number of touch points of d. As an example, $P'_2 = y^2 + y$ because there is one Dyck path of size 2 with 2 touch points and one with one touch point. Compute P'_3 and P'_4, what do you notice?
6. Can you find a map between Dyck paths such that the image of a Dyck path d with initial rise k is a Dyck path d' with k touch points?
7. Look at the statistics on findstat.org for Dyck path, compute their polynomials (the way we did for initial rises and touch points) and see the ones that give similar polynomials.