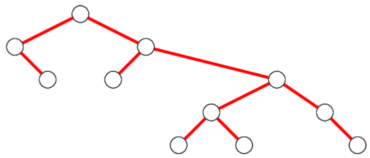
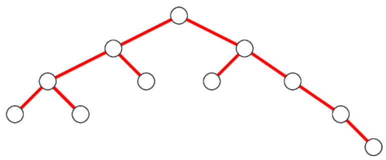
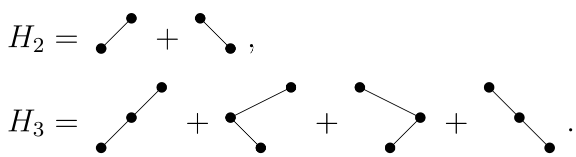
1. Construct the two following binary trees in Sage





2. Let H_n be the sum over binary trees of size n with depth n. So for example, we have



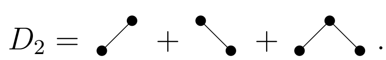
   (a) Compute H_4 by hand, you should find a sum over 8 elements.

   (b) Find a recursive formula for H_n depending on sums H_k, with $k < n$.

   (c) Let $h_n = H_n(1)$ be the number of elements in the sum H_n. What is the recursive formula for h_n? Use it to prove that $h_n = 2^n$.

   (d) Use the recursive formula for H_n to generate all trees of size n with n.

3. Now, we define D_n to be the sum over binary trees of depth n without any restriction on the size. As an example, we have



   (a) Compute D_3 by hand, you should find 21 elements.

   (b) Express D_n in terms of D_k with $k < n$. **Help**: if B is a tree of size n, it can be exactly one of these cases:

      - B is a product of two trees of depth $n - 1$,
      - B is a product of one tree of depth $n - 1$ on the left and one tree of depth $k < n - 1$ on the right,
      - B is a product of one tree of depth $k < n - 1$ on the left and one tree of depth $n - 1$ on the right.

   (c) Use this formula to generate all binary trees of a given depth.

   (d) Compute the number of trees of depth n for all n from 0 to 4. You should find these numbers: 1, 1, 3, 21, 651.