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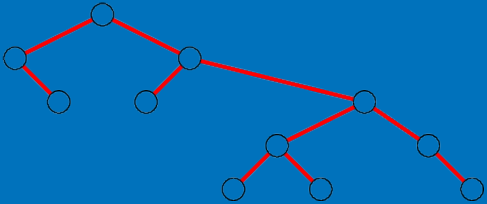
Experimental Combinatorics using Sage

Introduction



EAUMP summer school 2015

Overview

- Introduction to combinatorics
- Recursive generation of combinatorial objects
- Random generation
- Bijections and statistics

What does pure mathematics look like?

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L'équation de Boltzmann,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - v_*| \Big[f(v') f(v'_*) - f(v) f(v_*) \Big] \, dv_* \, d\sigma,$$

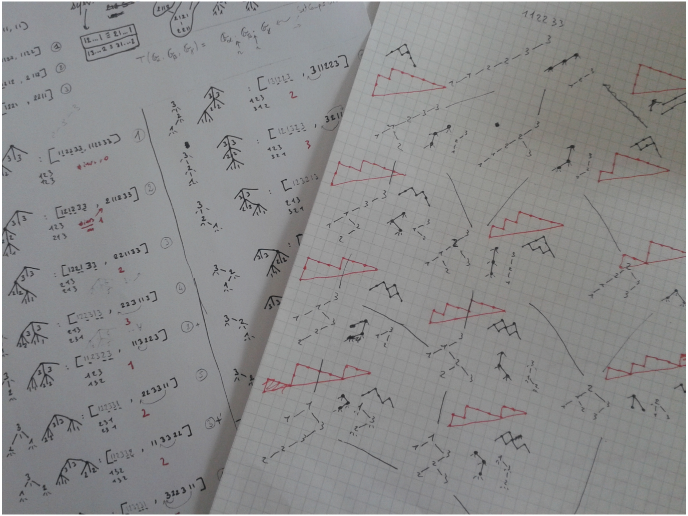
découverte aux alentours de 1870, modélise l'évolution d'un gaz raréfié, fait de milliards de milliards de particules, qui se cognent les unes contre les autres ; on représente la distribution statistique des positions et vitesses de ces particules par une fonction $f(t, x, v)$*, qui au temps* t *indique la densité de particules dont la position est (environ)* x *et dont la vitesse est (environ)* v*.*

Ludwig Boltzmann découvrit la notion statistique d'entropie, ou désordre, d'un gaz :

$$S = -\iint f \log f \, dx \, dv;$$

(from *Théorème Vivant* by Cédric Villani)

What does pure mathematics look like?



What does pure mathematics look like?

```
def inf_mperms(p1,p2):
    m = len([i for i in p1 if i==1])
    return perm_to_mperm(inf_perms(mperm_to_perm(p1),mperm_to_perm(p2)),m)

def is_last(perm,i):
    for b in perm[i+1:]:
        if b == perm[i]:
            return False
    return True

def mperm_to_tree(perm):
    values = list(set(perm))
    values.sort()
    values.reverse()
    m = len(perm) / len(values)
    tree = MDecreasingTree(m+1,None)
    for v in values:
        tree = tree.insert_from_mperm(perm,v)
    return tree


def mperm_to_tree2(perm, mfor0 = 1):
    if len(perm)==0:
        return MDecreasingTree(mfor0,None)
    n = max(perm)
    posr = [i for i in xrange(len(perm)) if perm[i]==n]
    m = len(posr)
    children = [[] for i in xrange(m+1)]
    right = {a for a in perm if a!=n}
    for i in xrange(m):
        pos = posr[i]
        for j in xrange(pos-1,-1,-1):
            a = perm[j]
            if a!=n:
                if is_last(perm,j):
                    if a in right:
                        children[i].append(a)
                        right.remove(a)
                elif a in right:
                    right.update([aa for aa in children[i] if aa < a])
                    children[i] = [b for b in children[i] if b >a]
    children[-1] = list(right)
    children_trees = [mperm_to_tree2([a for a in perm if a in c], mfor0 =m) for c in children]
    return MDecreasingTree(m+1,children_trees, label=n)
```

What does pure mathematics look like?

```
AUTHORS:

- Florent Hivert (2010-2011): initial implementation.

REFERENCES:

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   *Hopf algebra of the planar binary trees*,
   Advances in Mathematics, volume 139, issue 2,
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.. [HNT05] Florent Hivert, Jean-Christophe Novelli, and Jean-Yves Thibon.
   *The algebra of binary search trees*,
   :arxiv:`math/0401089v2`.

.. [CP12] Gregory Chatel, Viviane Pons.
   *Counting smaller trees in the Tamari order*,
   :arxiv:`1212.0751v1`.
"""
#*****************************************************************************
#       Copyright (C) 2010 Florent Hivert <Florent.Hivert@univ-rouen.fr>,
#
#  Distributed under the terms of the GNU General Public License (GPL)
#  as published by the Free Software Foundation; either version 2 of
#  the License, or (at your option) any later version.
#                  http://www.gnu.org/licenses/
#*****************************************************************************
from sage.structure.list_clone import ClonableArray
from sage.combinat.abstract_tree import (AbstractClonableTree,
                                         AbstractLabelledClonableTree)
from sage.combinat.ordered_tree import LabelledOrderedTrees
from sage.rings.integer import Integer
from sage.misc.classcall_metaclass import ClasscallMetaclass
from sage.misc.lazy_attribute import lazy_attribute, lazy_class_attribute
from sage.combinat.combinatorial_map import combinatorial_map

class BinaryTree(AbstractClonableTree, ClonableArray):
    """
    Binary trees.

    Binary trees here mean ordered (a.k.a. plane) finite binary
    trees, where "ordered" means that the children of each node are
    ordered.

    Binary trees contain nodes and leaves, where each node has two
```

What is combinatorics?

The study of finite and enumerated sets of **discrete structures**.

Example 1

12 21

123 132 213 231 312 321

1234 1243 1324 1342 1423 1432 2134 2143 2314 2341 2413 2431 3124 3142 3214 3241 3412 3421 4123 4132 4213 4231 4312 4321

Example 1

Size 2

12 21

$\longrightarrow$ **2**

Size 3

123 132 213 231 312 321

$\longrightarrow$ **6**

Size 4

1234 1243 1324 1342 1423 1432 2134 2143 2314 2341 2413 2431 3124 3142 3214 3241 3412 3421 4123 4132 4213 4231 4312 4321

$\longrightarrow$ **24**

Example 1

Size 2

12 21

$\longrightarrow$ **2**

Size 3

123 132 213 231 312 321

$\longrightarrow$ **6**

Size 4

1234 1243 1324 1342 1423 1432 2134 2143 2314 2341 2413 2431 3124 3142 3214 3241 3412 3421 4123 4132 4213 4231 4312 4321

$\longrightarrow$ **24**

Size 5

12345 12354 12435 12453 12534 12543 13245 ...

$\longrightarrow$ **??**

Example 1

Size 2

12 21

$\longrightarrow$ **2**

Size 3

123 132 213 231 312 321

$\longrightarrow$ **6**

Size 4

1234 1243 1324 1342 1423 1432 2134 2143 2314 2341 2413 2431 3124 3142 3214 3241 3412 3421 4123 4132 4213 4231 4312 4321

$\longrightarrow$ **24**

Size 5

12345 12354 12435 12453 12534 12543 13245 ...

$\longrightarrow$ **120**

Example 1: Permutations of size n

All possible lists of the numbers $1, \dots, n$.

Number of permutations of size n

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Number of permutations of size n

$$n! = n(n-1)(n-2)\ldots 1$$

.

Example 2: n chooses k

I want k numbers among $1, 2, \dots n$ (the order doesn't count).

Example : 4 chooses 2

1,2 1,3 1,4 2,3 2,4 3,4

Formula

??

Example 2: n chooses k

I want k numbers among $1, 2, \ldots n$ (the order doesn't count).

Example : 4 chooses 2
1,2 1,3 1,4 2,3 2,4 3,4

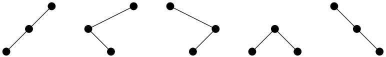
Formula

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

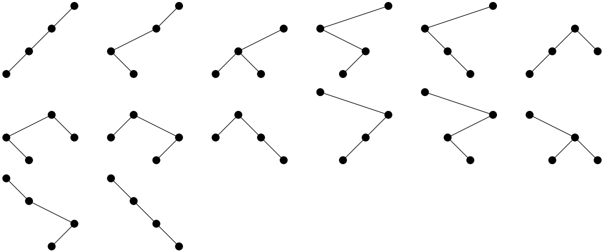
Example 3

Size 2

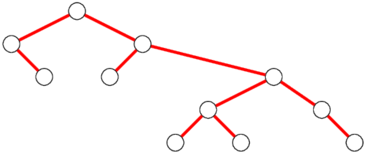
Size 3

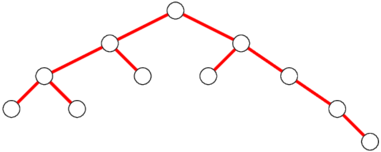


Size 4



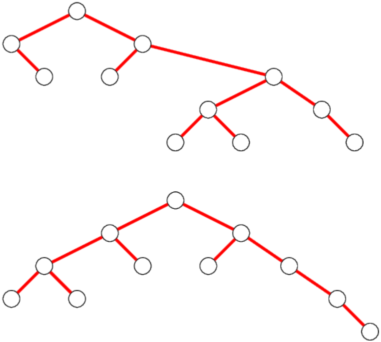
Example 3: binary trees





Questions:

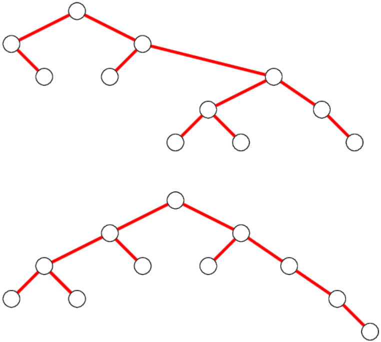
Example 3: binary trees



Questions:

- How many binary trees with 11 nodes?

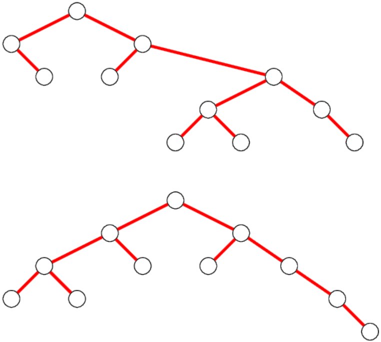
Example 3: binary trees



Questions:

- How many binary trees with 11 nodes? 58786

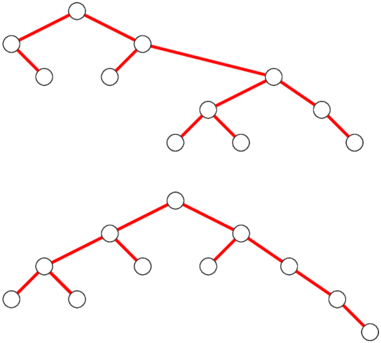
Example 3: binary trees



Questions:

- How many binary trees with 11 nodes? 58786
- What does a "random" binary tree look like?

Example 3: binary trees



Questions:

- How many binary trees with 11 nodes? 58786
- What does a "random" binary tree look like?
- Are there other combinatorial objects somehow linked to binary trees?

Motivations?

- "Real world" motivation: physics models, Bioinformatics
- Algorithmic motivations: random generation, algorithms complexity
- Algebraic: algebraic geometry, representation theory, group theory

Methods?

Computer exploration, mathematics experimentation.