A lattice on decreasing trees : the metasylvester lattice

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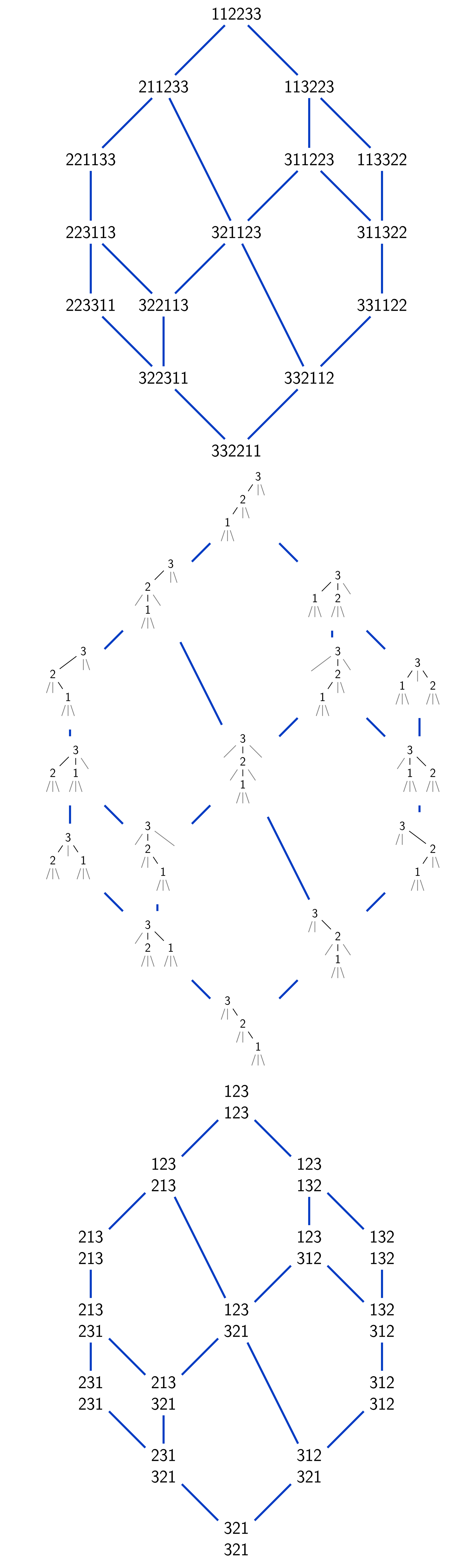
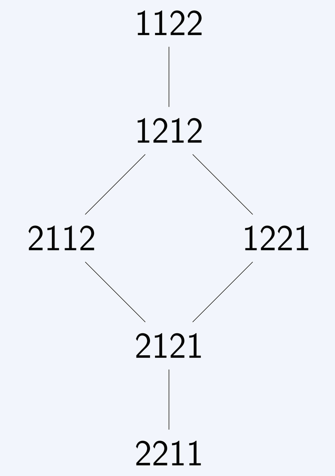
The generalization of the well known Tamari lattice to the family of m-Tamari lattices [1] opens a natural question: what are the geometrical, combinatorial, and algebraic relations between a m-version of the weak order and the m-Tamari lattice? In [2], the authors study m-permutations and describe a new congruence relation. They use it to define a new Hopf Algebra. **We prove here that this congruence also leads to the definition of a new lattice.**



Three realizations of the metasylvester lattice; from top to bottom: on m-permutations, on decreasing trees, and on metasylvester chains.

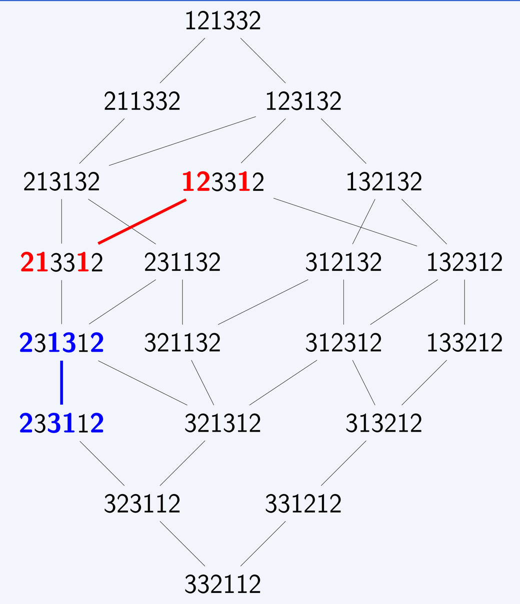
The m-permutations lattice



m-permutations of size n:
permutations of the word $1^m 2^m \cdots n^m$.
Example:
- **122313** is a **2**-permutation of size 3,
- **211212** is a **3**-permutation of size **2**.

The set of m-permutations admits a lattice structure (induced from the weak order on classical permutations).
On the left: the lattice of 2-permutations of size 2.

The metasylvester congruence



The metasylvester congruence is defined on m-permutations as the reflexive transitive closure of the relations

$$ac \cdots a \equiv ca \cdots a$$
$$b \cdots ac \cdots b \equiv b \cdots ca \cdots b$$

with $a < b < c$.
See an example of a class on the left.

Number of classes

The number of metasylvester classes is given by the product

$$(1+m)(1+2m)\cdots(1+(n-1)m).$$

For $m = 2$, it is the product of odd numbers, which gives

$$1, 3, 15, 105, 945, \ldots$$

Some properties [2]

- Each class possesses a unique maximal element, which is the only element avoiding a subword of the form $\cdots a \cdots b \cdots a \cdots$ with $a < b$.
- The set of maximal elements is in bijection with $(m+1)$-ary decreasing trees.

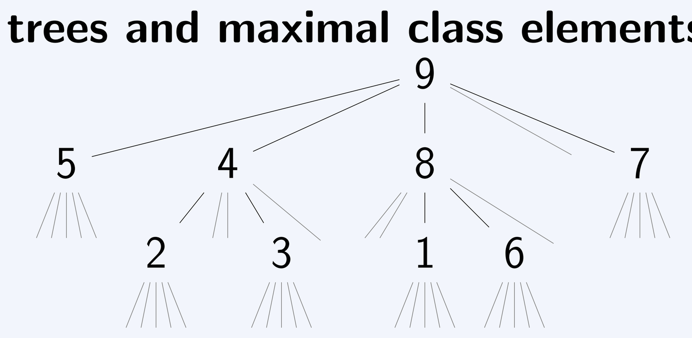
New results

- The metasylvester classes form intervals of the m-permutations lattice.
- The set of maximal elements form a sublattice of the m-permutations lattice.

Another realization: metasylvester m-chains

Metasylvester m-chain of size n:
a list $c = (\sigma^{(m)}, \sigma^{(m-1)}, \ldots, \sigma^{(1)})$ of permutations with
- $\sigma^{(m)} \leq \sigma^{(m-1)} \leq \cdots \leq \sigma^{(1)}$ (in right weak order),
- for all $i < j$, we have $(\sigma^{(j)})^{-1}\sigma^{(i)}$ avoids the pattern 231.

Bijections between metasylvester classes, $(m+1)$-ary decreasing trees and maximal class elements:

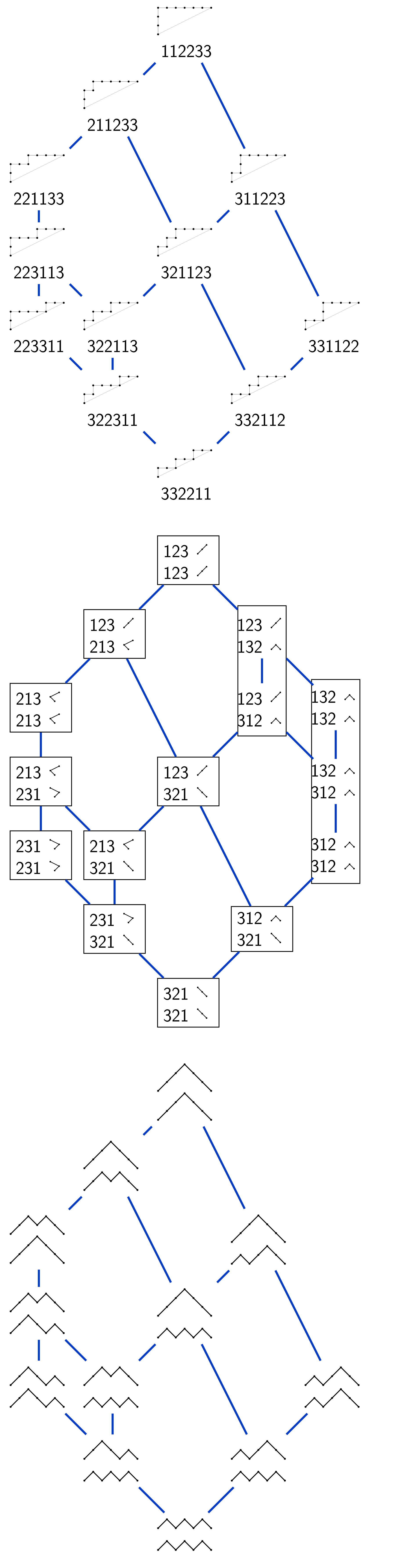


$\sigma^{(4)} = 5\,2\,3\,4\,1\,6\,8\,9\,7$
$\sigma^{(3)} = 5\,2\,4\,3\,1\,8\,6\,9\,7$
$\sigma^{(2)} = 5\,2\,4\,3\,9\,8\,1\,6\,7$
$\sigma^{(1)} = 5\,9\,2\,4\,3\,8\,1\,6\,7$

5 5 5 5 9 2 2 2 2 4 4 4 3 3 3 3 4 9 8 8 1 1 1 1 8 6 6 6 6 8 9 9 7 7 7 7

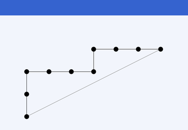
Connection with the m-Tamari lattice

The m-Tamari lattice is both a quotient lattice and a sublattice of the metasylvester lattice. See figures on the right.
It leads to a new realization of the m-Tamari lattice, as shown on the bottom figure on the right. The explicit bijection is explained in the bellow box.

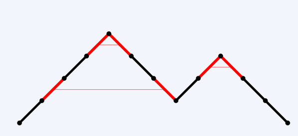


The m-Tamari lattice as a sublattice and quotient lattice of the metasylvester lattice. And a new realization of the m-Tamari lattice.

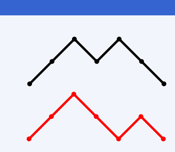
Bijection between m-paths and certain m-chains of the Tamari lattice



Replace each up-step by two up-steps to obtain a classical Dyck path.



Color in red every other up-step and connect it to its corresponding down-step.



The red steps form one Dyck path of the chain and the black ones form the other one.

References

- F. Bergeron and L.-F. Préville-Ratelle, *Higher trivariate diagonal harmonics via generalized Tamari posets*, J. Comb. **3** (2012), no. 3, 317–341. MR 3029440
- J.-C. Novelli and J.-Y. Thibon, *Hopf Algebras of m-permutations, $(m+1)$-ary trees, and m-parking functions*, arXiv:1403.5962 preprint (2014).