Énumération des intervalles du treillis de Tamari

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Tamari lattice

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- 1962, Tamari : poset of formal bracketing

Tamari lattice

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- 1972, Huang, Tamari : lattice structure

Tamari lattice

- 1962, Tamari : poset of formal bracketing
- 1972, Huang, Tamari : lattice structure
- 2007, Chapoton : number of intervals

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$$

m-Tamari lattices

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- Bergeron, Préville-Ratelle : *m*-Tamari posets

m-Tamari lattices

- Bergeron, Préville-Ratelle : m-Tamari posets
- Bousquet-Mélou, Fusy, Préville-Ratelle : lattice structure and number of intervals

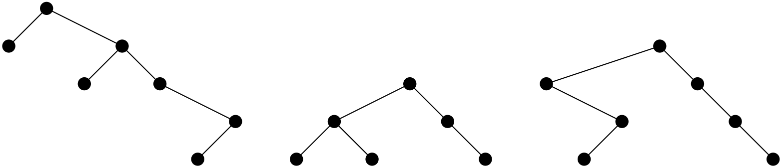
$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1}$$

Binary trees

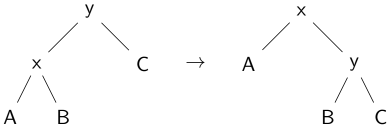
Recursive definition :

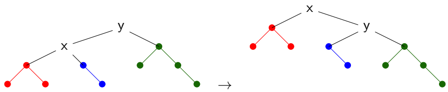
- the empty tree or
- a left subtree and a right subtree grafted to a root node

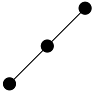
Examples



Right rotation







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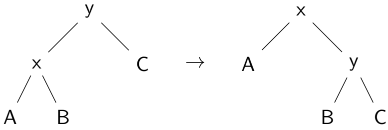

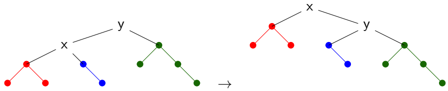

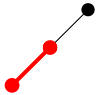

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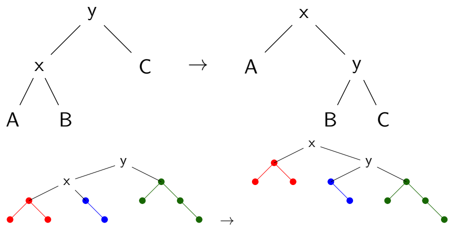

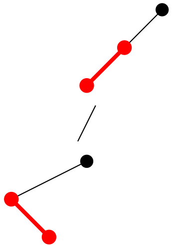

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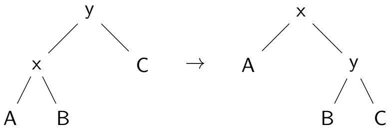

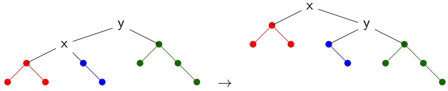

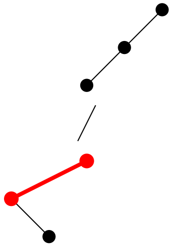

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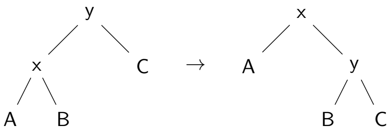

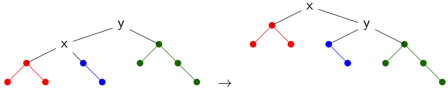

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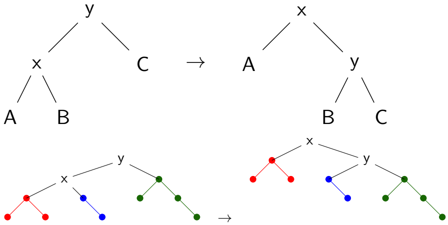

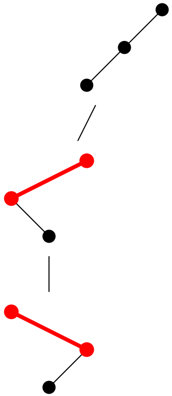

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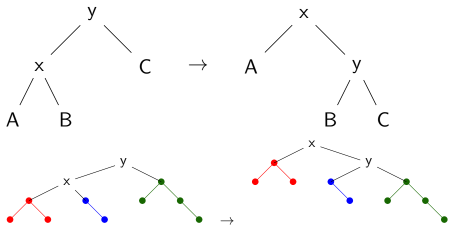

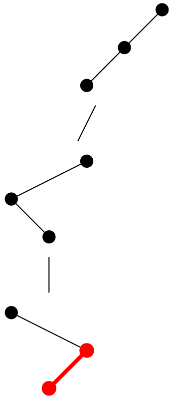

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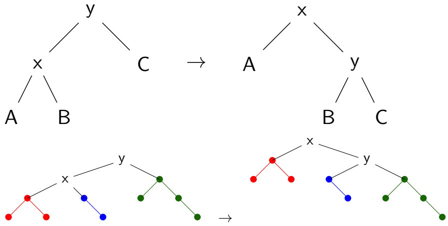

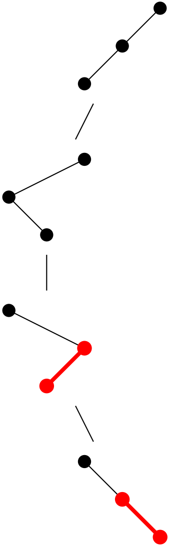

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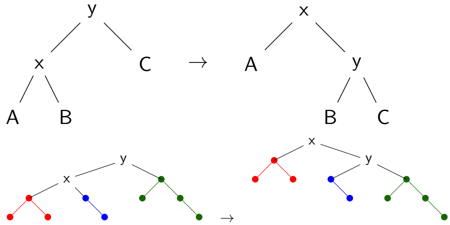

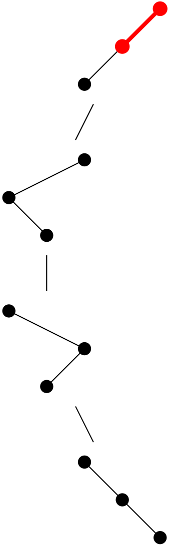

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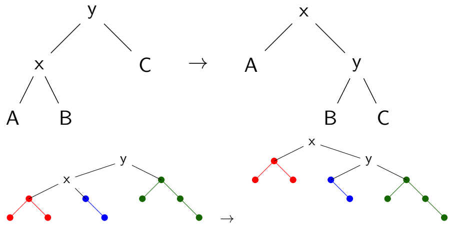

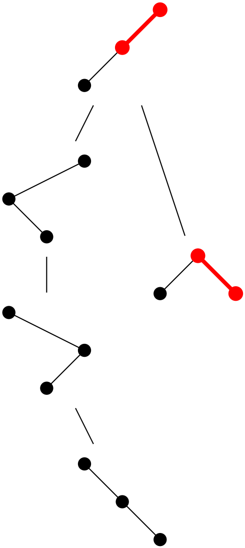

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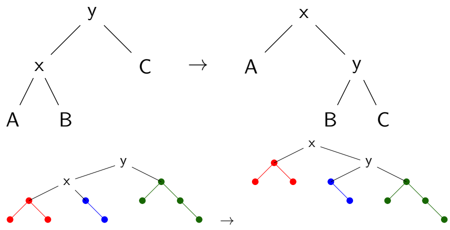

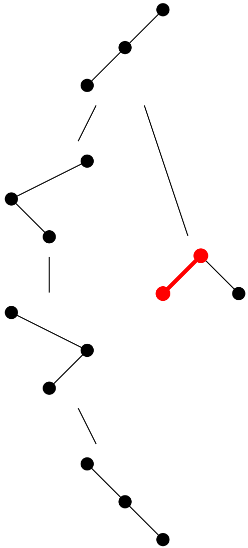

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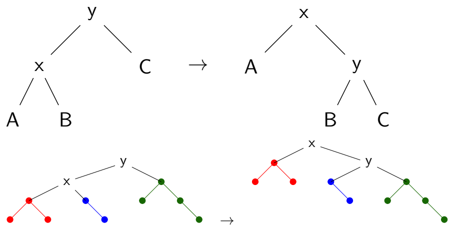

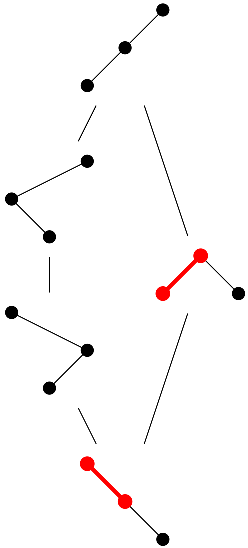

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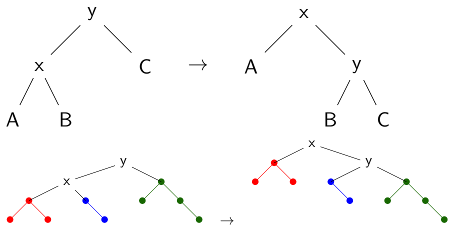
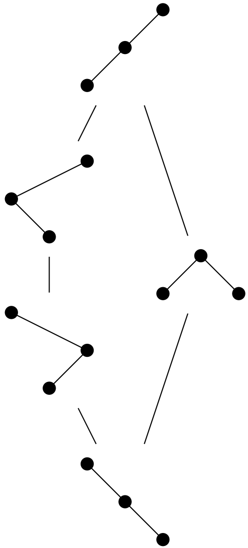
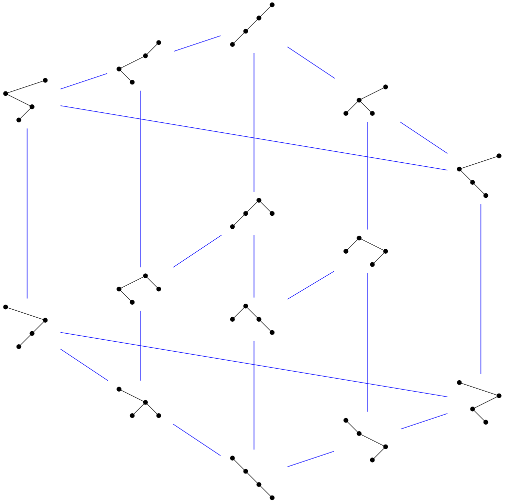
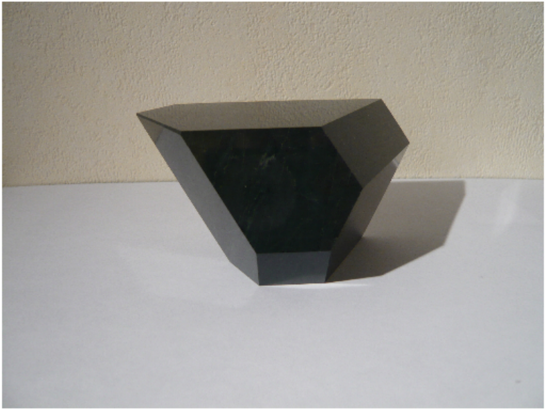
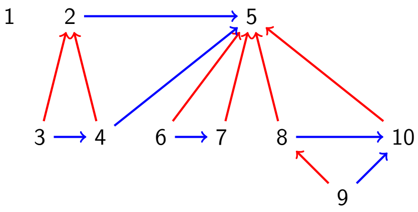






Associahedron



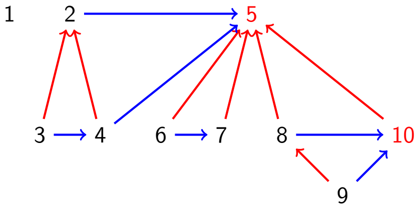


Definition

An interval-poset is a poset of size n, labelled with $1, \dots, n$ such that

- if $a < c$ and $c \lhd a$ then $b \lhd a$ for all $a < b < c$,
- if $a < c$ and $a \lhd c$ then $b \lhd c$ for all $a < b < c$.

We write $a \lhd b$ for a lower than b in the poset.



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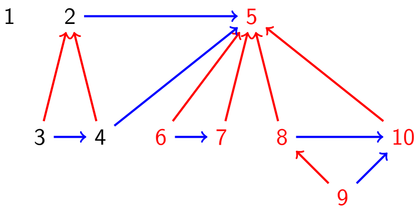

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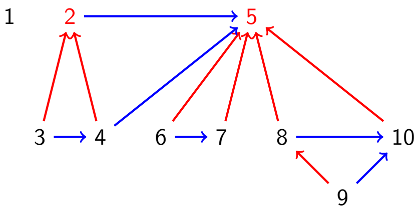

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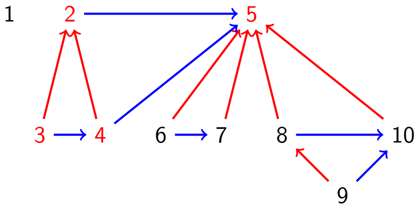

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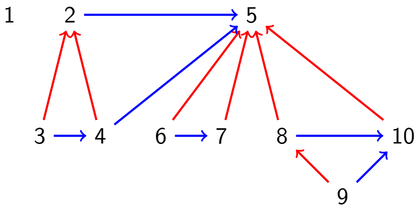
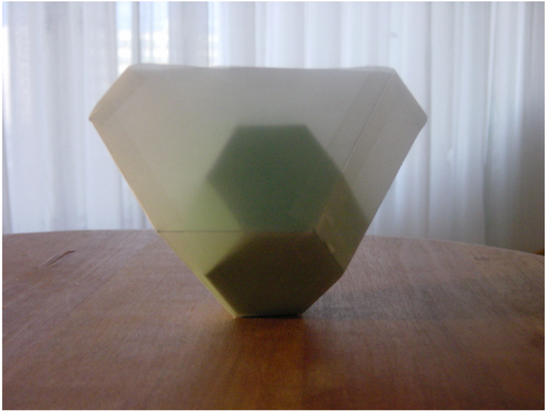


Theorem (Châtel, P.)

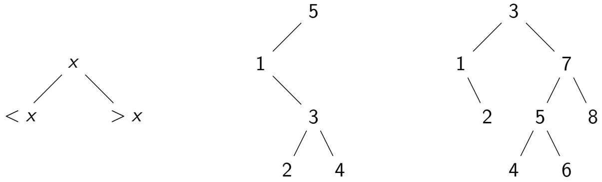
Interval-posets are in bijections with intervals of the Tamari lattice.

Link with the weak order



(image from Jean-Louis Loday)

Binary search tree

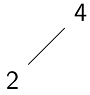


Binary search tree insertion

$$15324 \quad \rightarrow \quad 4$$

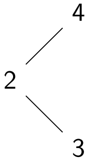
Binary search tree insertion

15324 →



Binary search tree insertion

15324 $\rightarrow$



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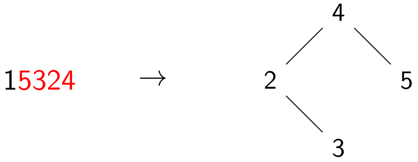

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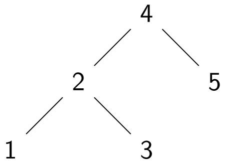
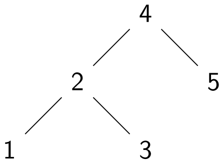
15324 →



Binary search tree insertion

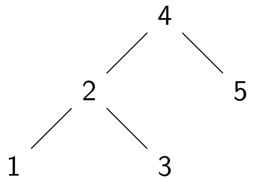
15324 $\rightarrow$

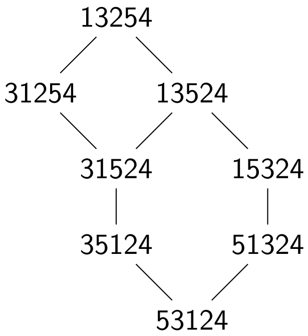


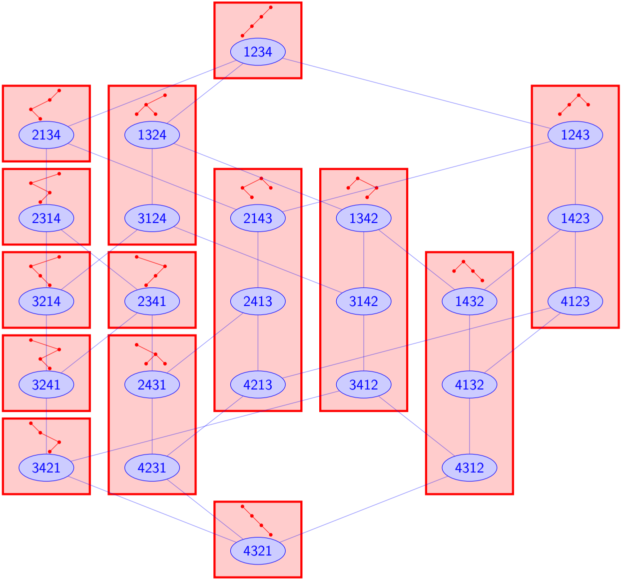
Characterization : the permutations sent to a given tree are its linear extensions

$15324, 31254, 35124, 51324, \ldots$

Binary search tree insertion



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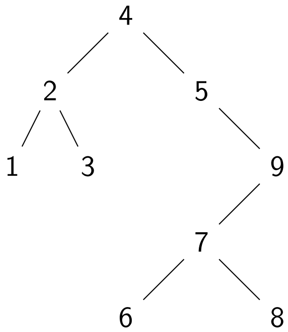
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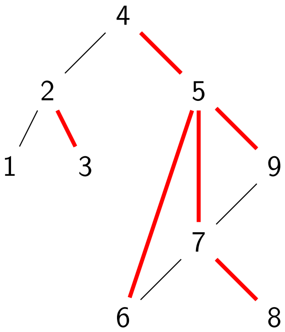
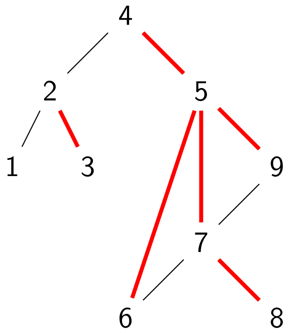


final forest $F_{\geq}(T)$



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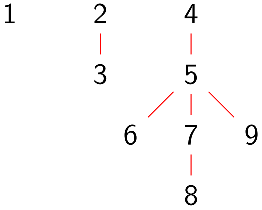

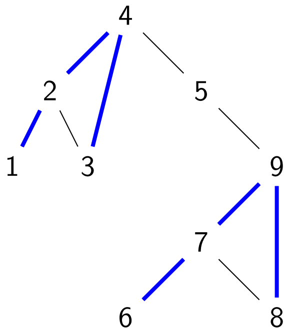

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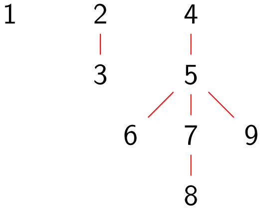
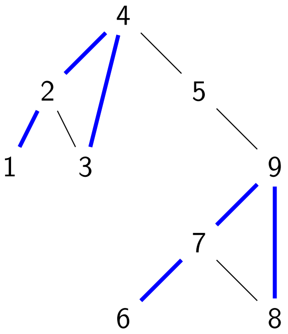
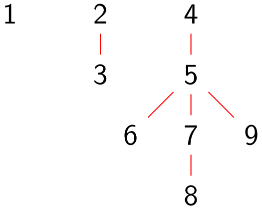


Initial forest $F_{\leq}(T)$



final forest $F_{\geq}(T)$



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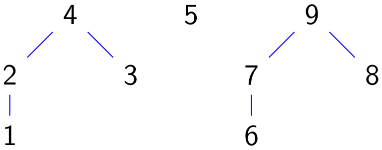

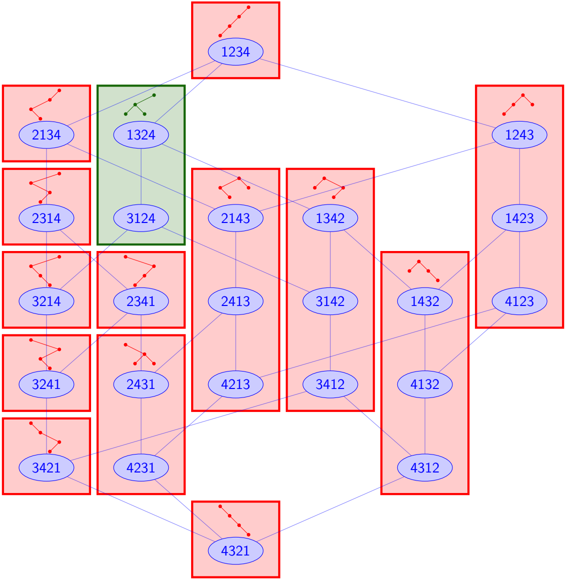
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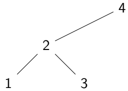
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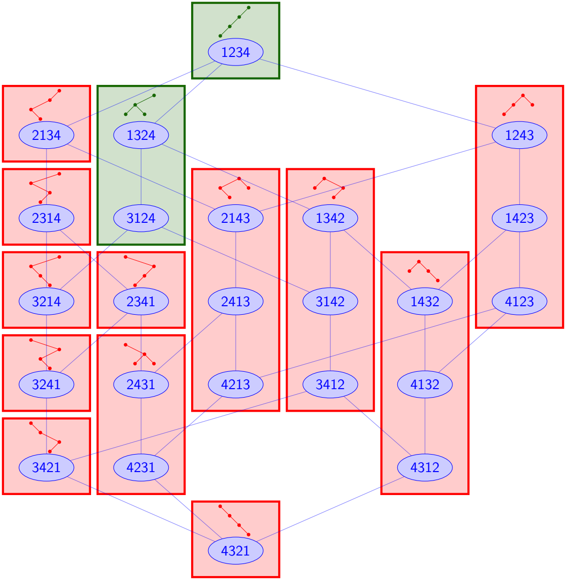
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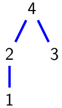
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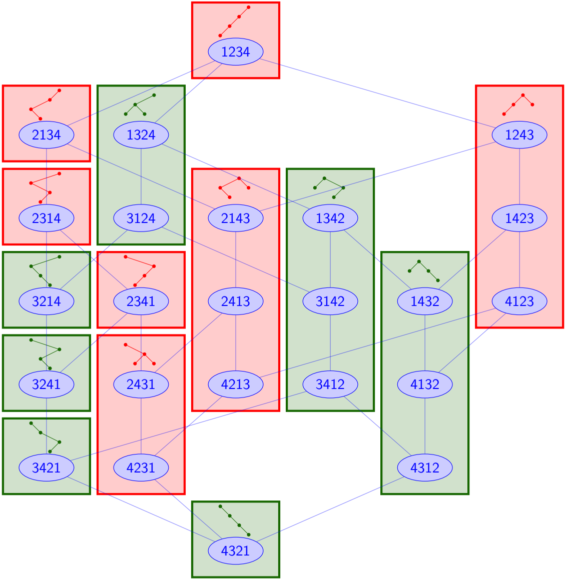
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$F_{\geq}(T)$


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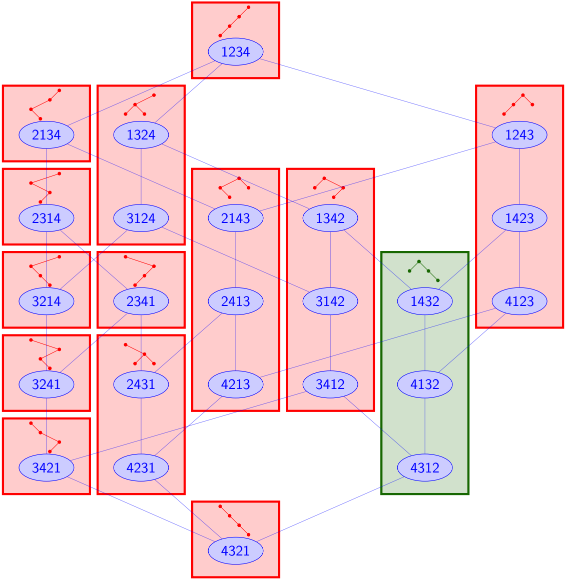
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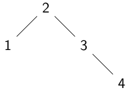
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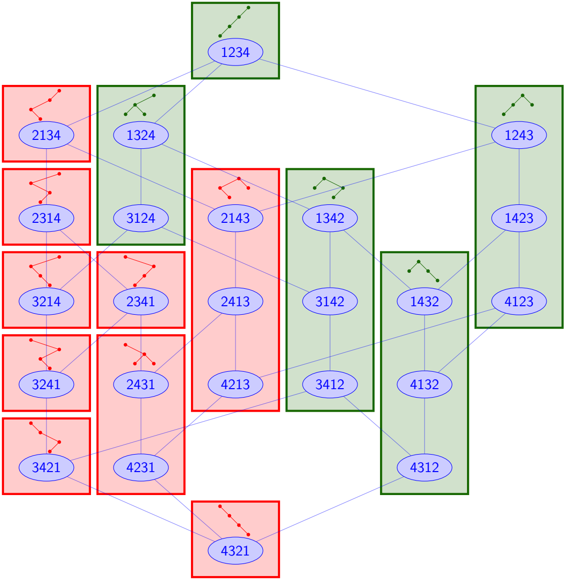
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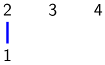
$F_{\leq}(T')$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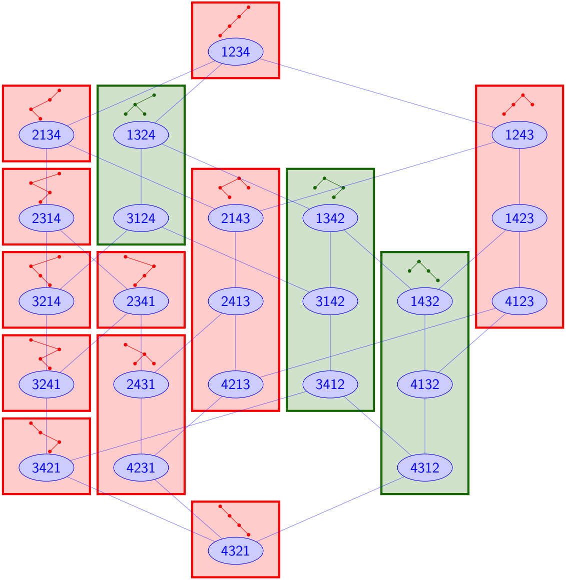
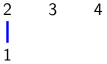


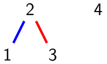
$F_{\geq}(T)$

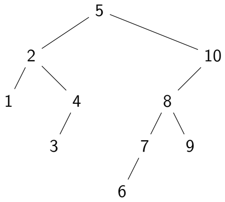
$F_{\leq}(T')$



Intervalle-poset

$[T, T']$

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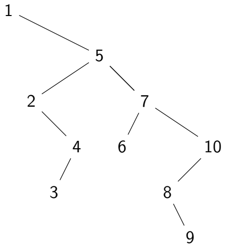
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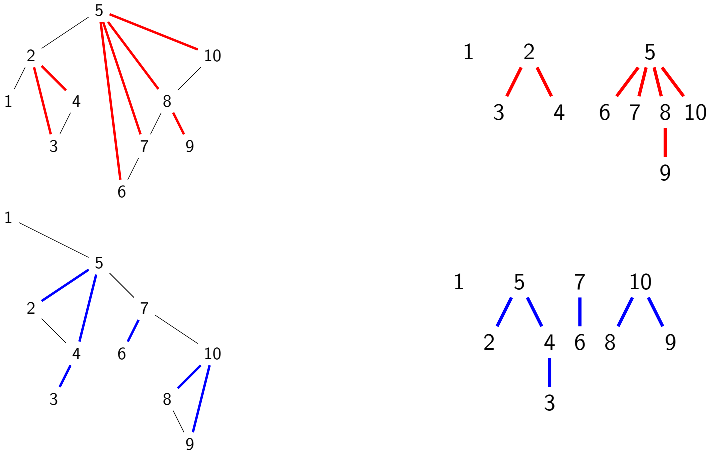
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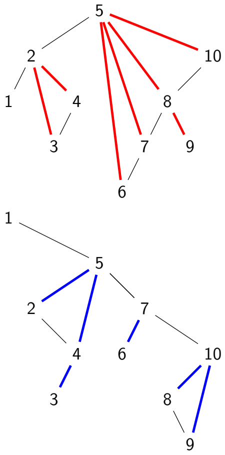
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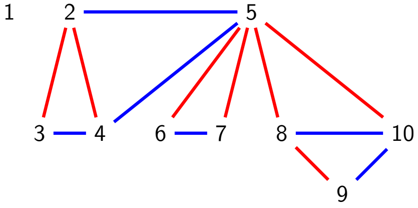
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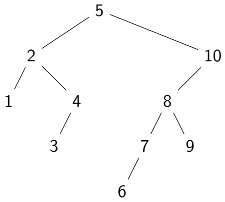
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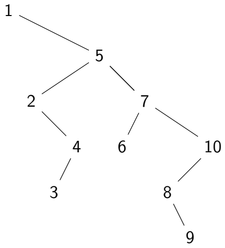
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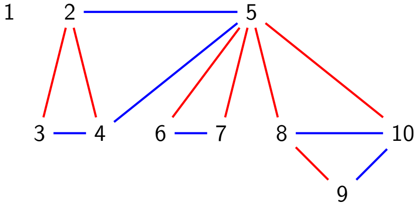
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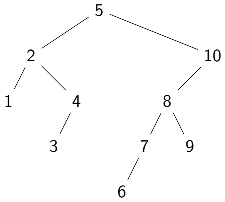
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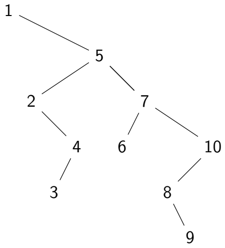
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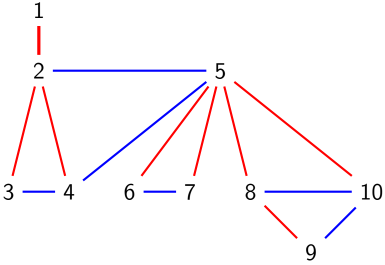
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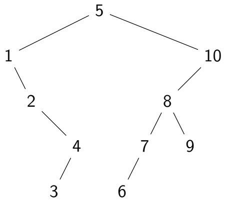
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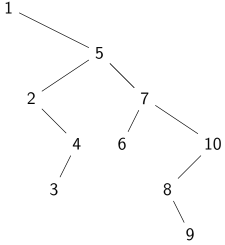
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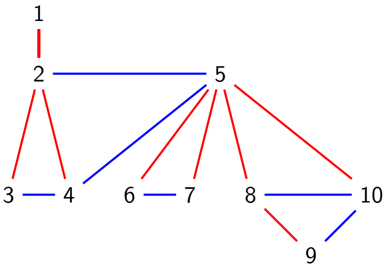
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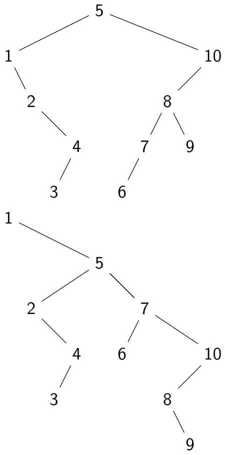

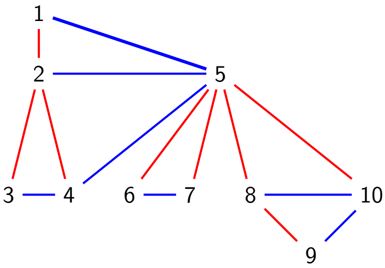

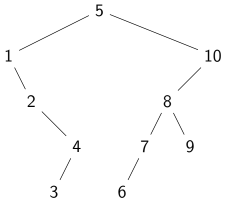
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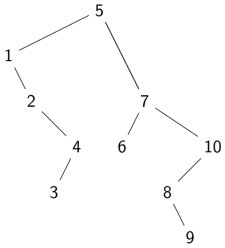
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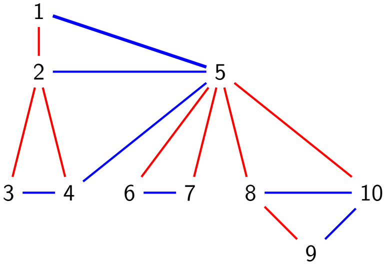
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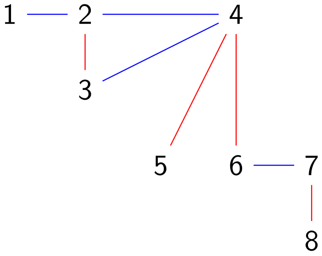
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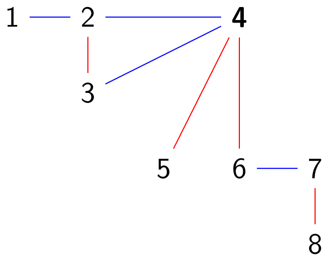
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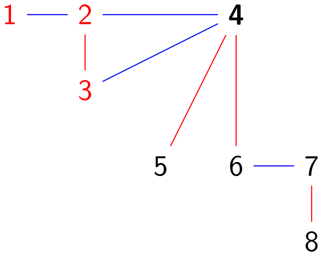
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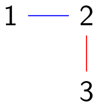
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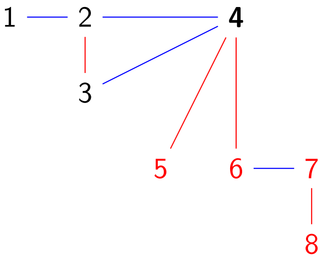
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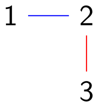
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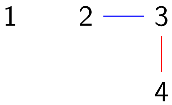
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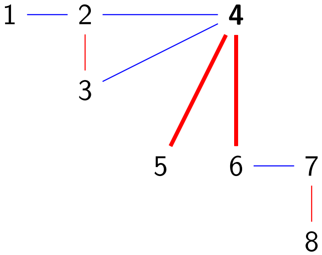

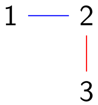

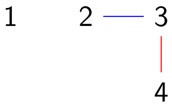

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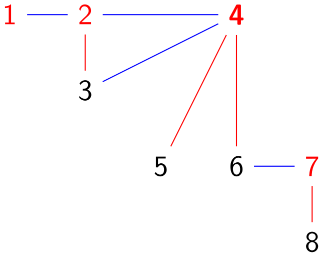
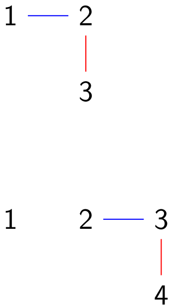


x^4



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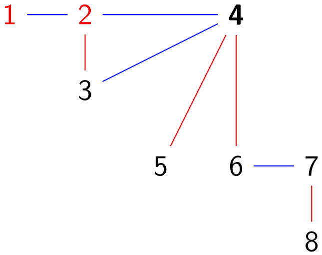
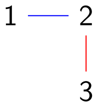
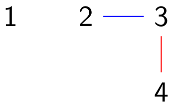




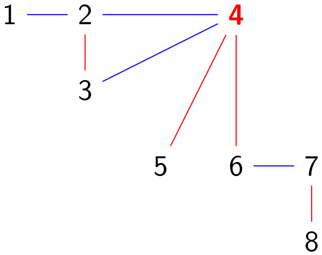
x^2



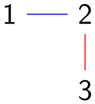
x^3

$$x^4 = x^2 . x . \frac{x^3}{x^2}$$

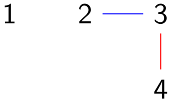
2



$$x^4 = x^2 . x . \frac{x^3}{x^2}$$

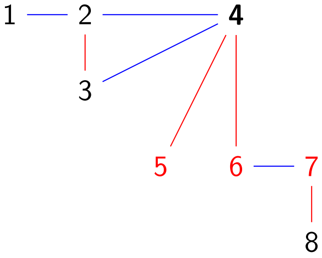


x^2

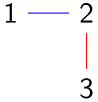


x^3

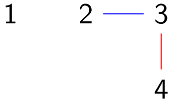
2



$$x^4 = x^2.x.\frac{x^3}{x^2}$$



x^2



x^3

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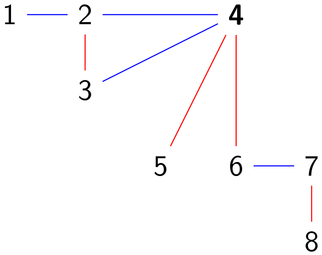
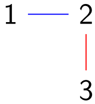
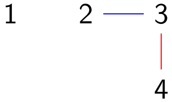




$x^2 \; z$

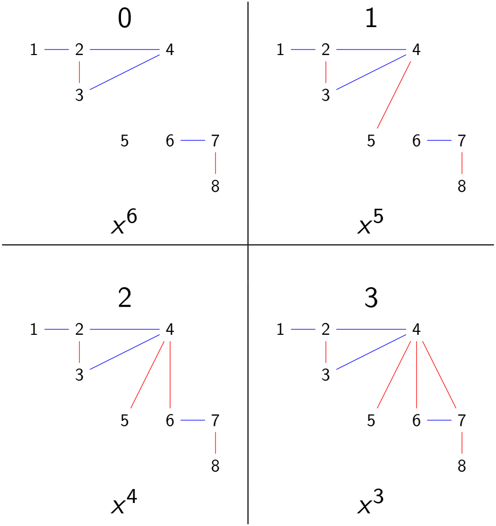


$x^3 \; z^2$

$$x^4 = x^2.x.\frac{x^3}{x^2}$$

z

2


0
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x^6
1
1 2 3 4 5 6 7 8
x^5
2
1 2 3 4 5 6 7 8
x^4
3
1 2 3 4 5 6 7 8
x^3

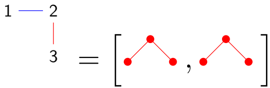
Theorem (Chapoton)

The generating functions of Tamari intervals satisfy the functional equation

$$\Phi(x, y) = \mathrm{B}(\Phi, \Phi) + 1$$

where

$$\mathrm{B}(f, g) = xyf(x, y)\frac{xg(x, y) - g(1, y)}{x - 1}$$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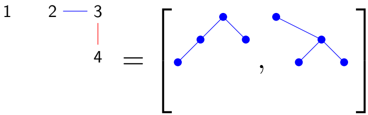
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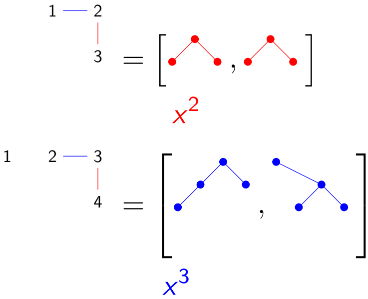
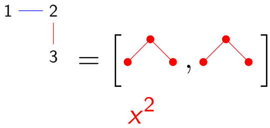

1 2
3
=
x^2
1 2 3
4
=
x^3


1 2
3
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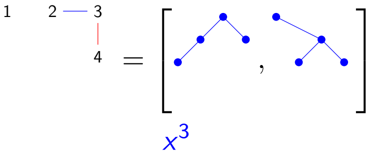
x2


1 2 3
4
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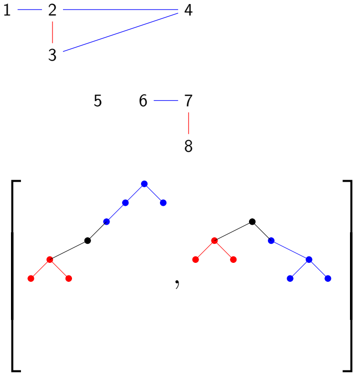
x3


1 2 4
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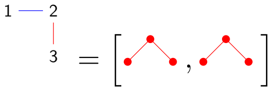
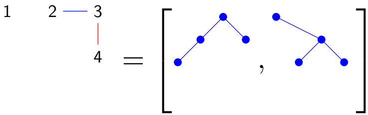
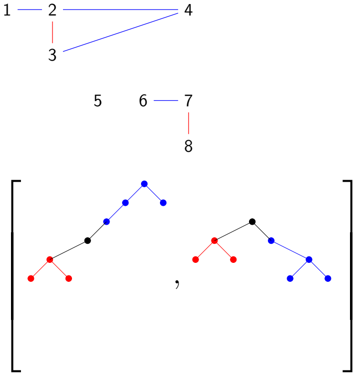


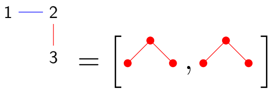
x^2



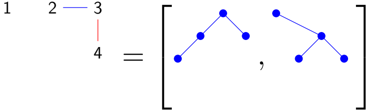
x^3



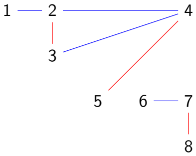
$x^2.x.x^3$

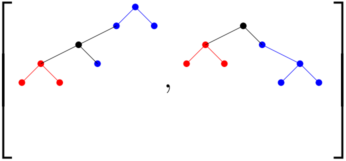


x^2

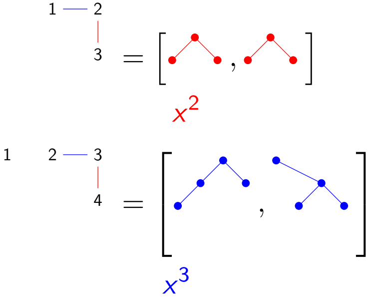


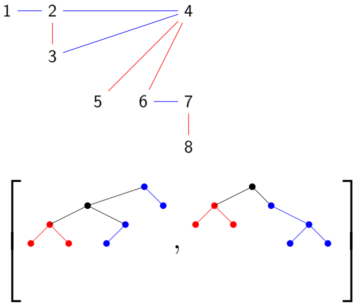
x^3



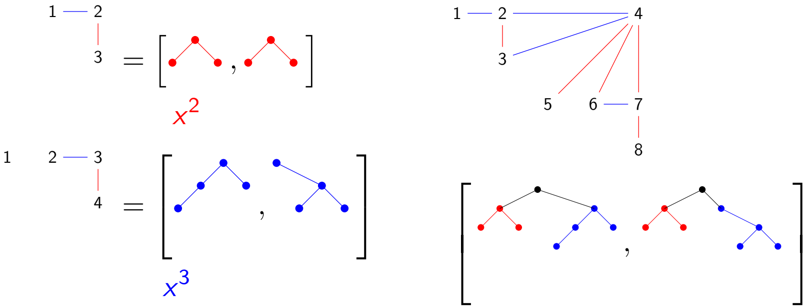


$$x^2.x.x^3 + x^2.x.x^2$$

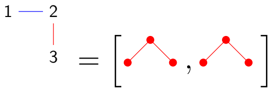




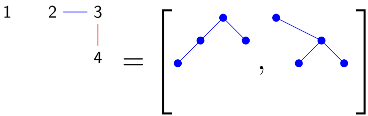
$$x^2.x.x^3 + x^2.x.x^2 + x^2.x.x$$



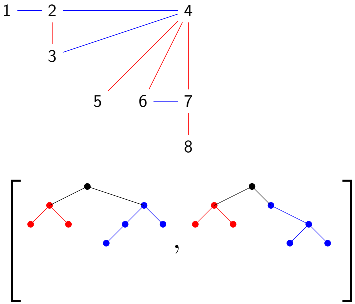
$$x^2.x.x^3 + x^2.x.x^2 + x^2.x.x + x^2.x$$



x^2



x^3



$$x^2.x.(x^3 + x^2 + x + 1)$$

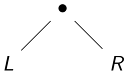
Tamari Polynomials

$\mathcal{B}_T$ is recursively defined by

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

avec T =



Theorem (Châtel, P.)

$\mathcal{B}_T$ counts the number of trees smaller than or equal to T in the Tamari lattice according to the number of nodes on their left border.

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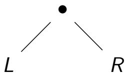

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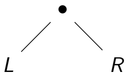

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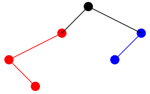
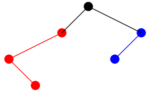


$$\mathcal{B}_{\emptyset} := 1$$

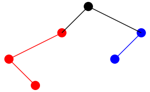
$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$



$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

$$\mathcal{B}_L(x) = x^3 + x^2$$

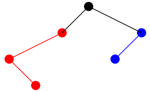


$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

$$\mathcal{B}_L(x) = x^3 + x^2$$

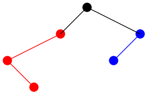
$$\mathcal{B}_R(x) = x^2$$



$$\mathcal{B}_\emptyset := 1$$

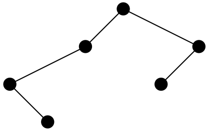
$$\mathcal{B}_T(x) := x(x^3 + x^2)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

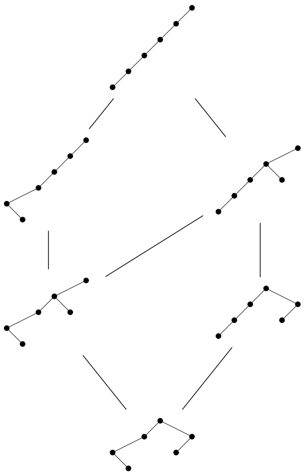
$$\mathcal{B}_R(x) = x^2$$



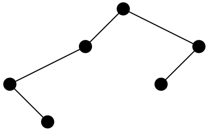
$$\mathcal{B}_\emptyset := 1$$

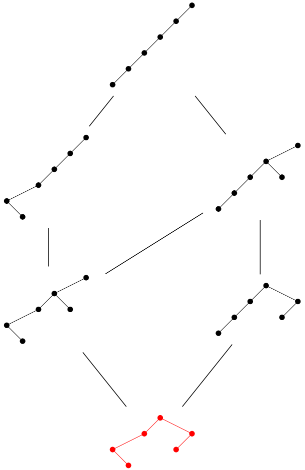
$$\mathcal{B}_T(x) := x(x^3 + x^2)(1 + x + x^2)$$



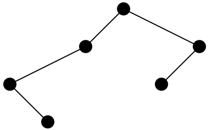


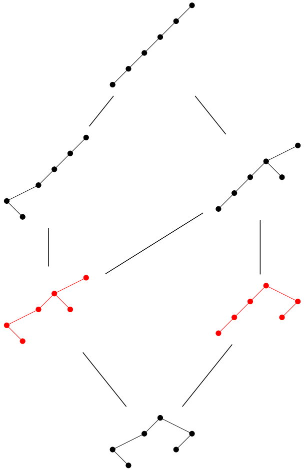
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



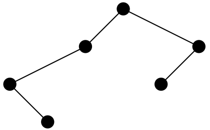


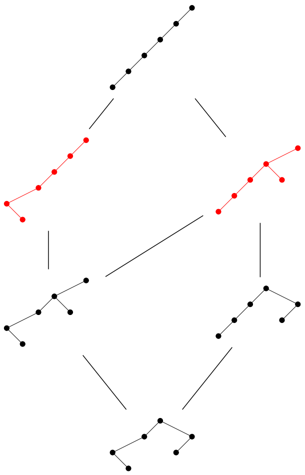
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



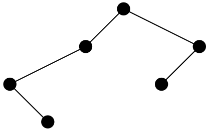


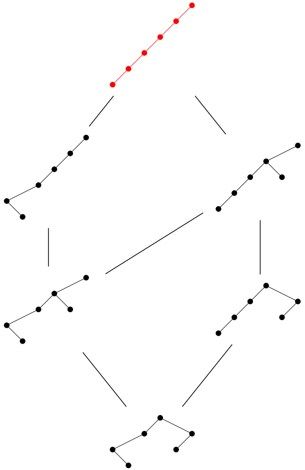
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



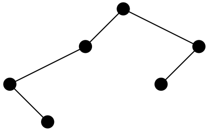


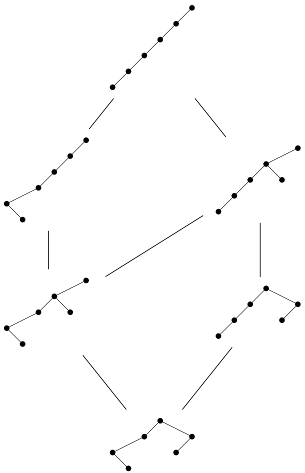
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



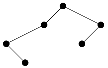


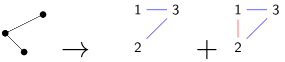
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$





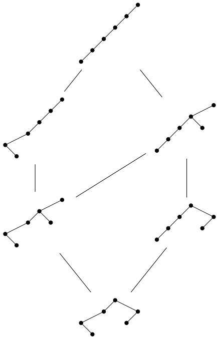
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$
$$\mathcal{B}_T(1) = 6$$

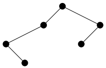


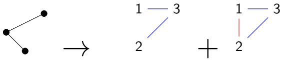


$$x^3 + x^2$$

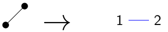
$$x^2$$



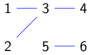




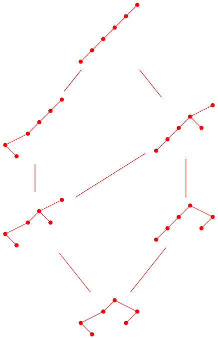
$x^3 + x^2$

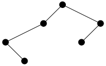


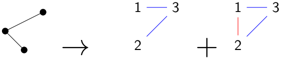
x^2



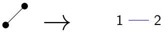
$x^3.x.x^2$





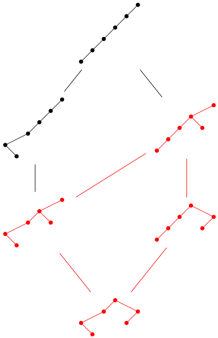


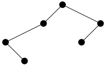
$x^3 + x^2$

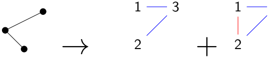


x^2

$x^3.x.x^2 \quad + x^3.x.x$

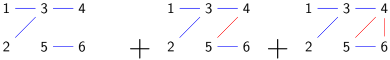




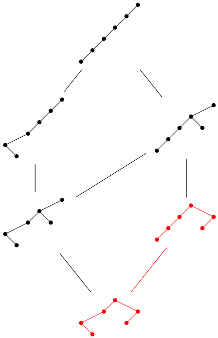


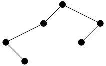
$$x^3 + x^2$$

$$x^2$$



$$x^3.x.x^2 \quad + x^3.x.x + x^3.x$$





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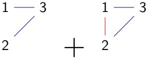
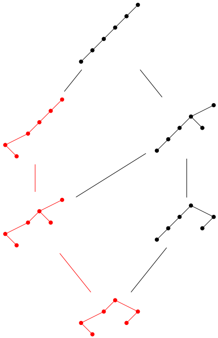
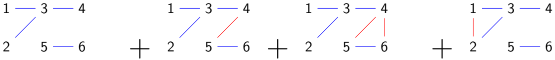


$$x^3 + x^2$$

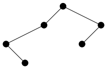
$\rightarrow$

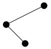
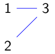
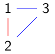
$$x^2$$





$$x^3.x.x^2 + x^3.x.x + x^3.x + x^2.x.x^2$$

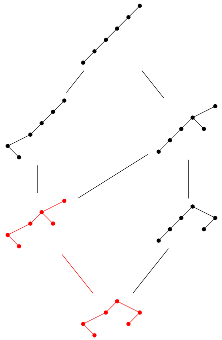


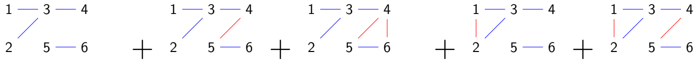
 $\rightarrow$  $+$ 

$$x^3 + x^2$$

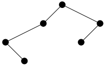
 $\rightarrow$ 

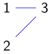
$$x^2$$

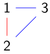




$$x^3.x.x^2 + x^3.x.x + x^3.x + x^2.x.x^2 + x^2.x.x$$



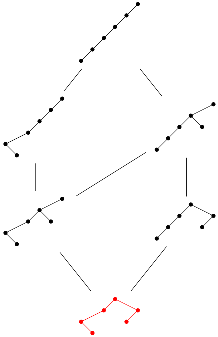
 $\rightarrow$ 

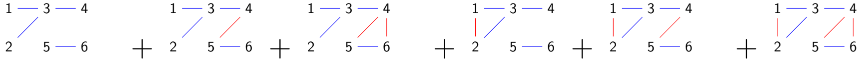
$+$ 

$$x^3 + x^2$$

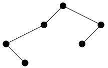
 $\rightarrow$ 

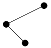
$$x^2$$





$$x^3.x.x^2 \quad + x^3.x.x + x^3.x \quad + x^2.x.x^2 + x^2.x.x \quad + x^2.x$$





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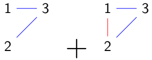
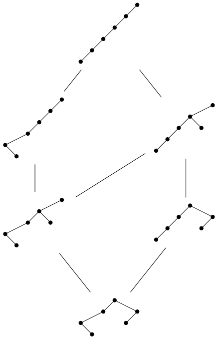


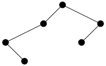
$$x^3 + x^2$$

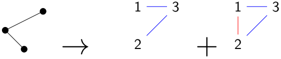
$\rightarrow$

$$x^2$$



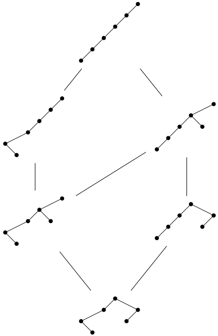
$$(x^3 + x^2).x.(x^2 + x + 1) =$$





$$x^3 + x^2$$

$$x^2$$



$$(x^3 + x^2).x.(x^2 + x + 1) = x\,\mathcal{B}_L(x)\frac{\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$