Tamari lattice and weak order on permutations

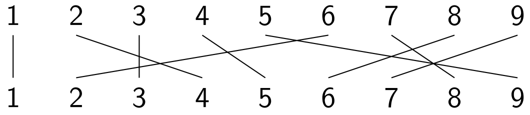
Viviane Pons

Universität Wien

Lisbon, May 20, 2014

Permutations group

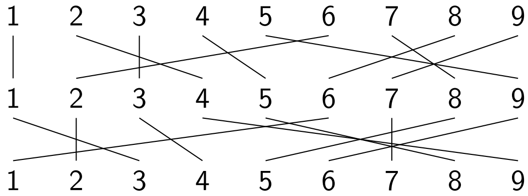
$\sigma = 143592867$



Permutations group

$\sigma = 143592867$

$\mu = 324981756$



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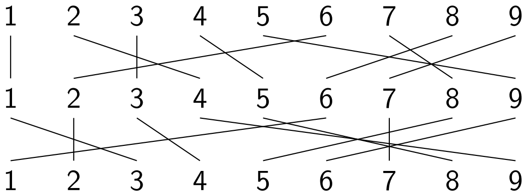

$$\mu.\sigma = 394862517$$

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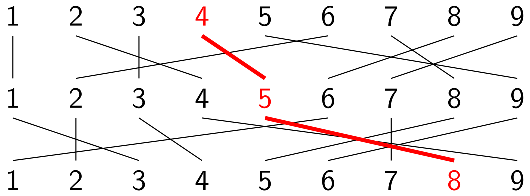

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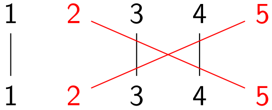
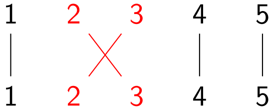
Transpositions



$$(2, 5)$$

Simple transpositions



$$(2, 3) = s_2$$

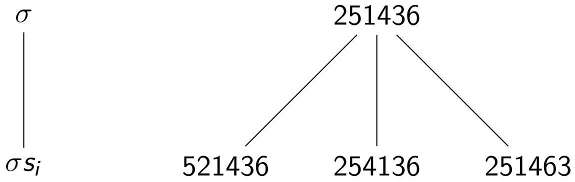
Right weak order

$$\sigma$$

$$\sigma s_i$$

$$\ell(\sigma s_i) = \ell(\sigma) + 1$$

Right weak order



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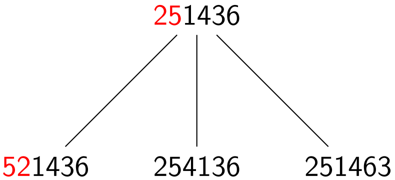

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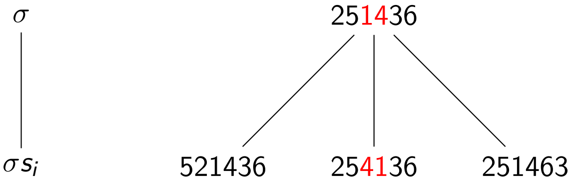

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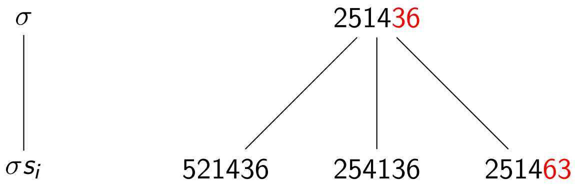

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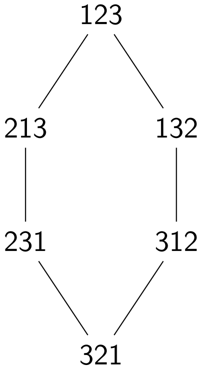
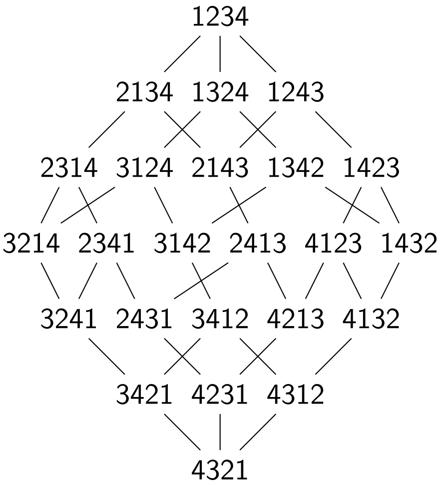
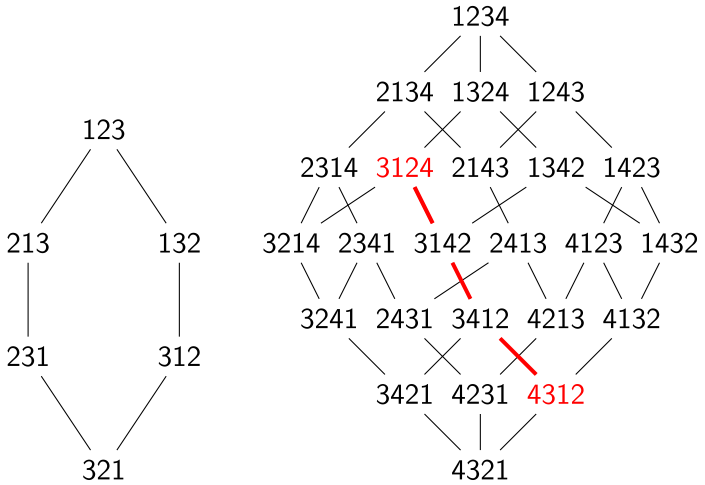
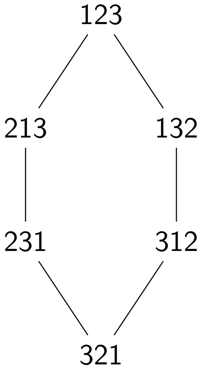


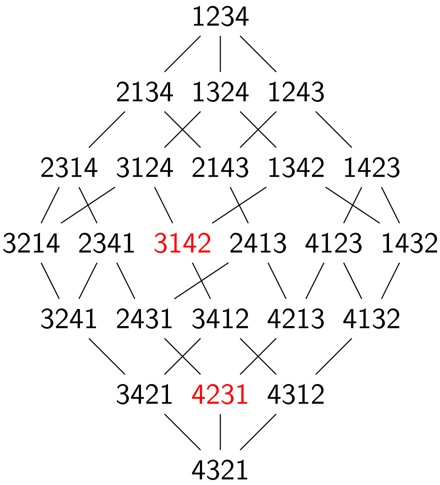


Right weak order

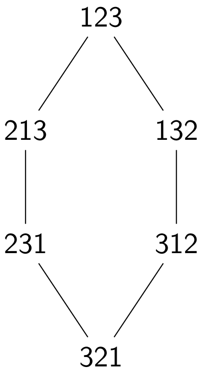


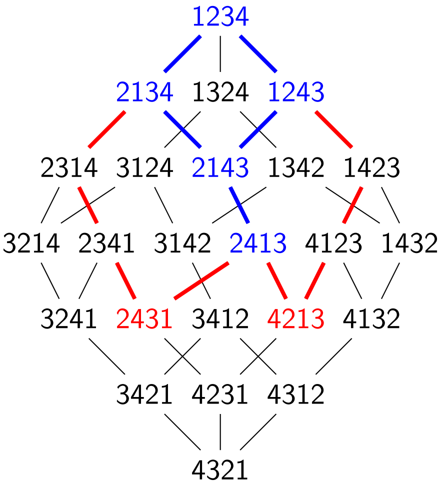
Right weak order



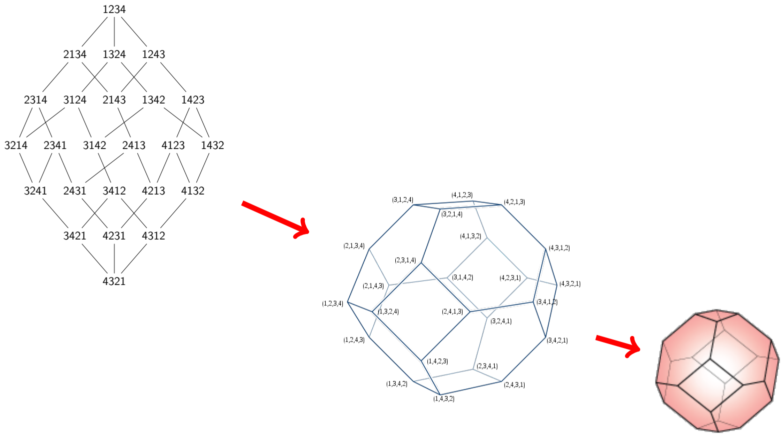


Right weak order





Permutohedron



Malvenuto-Reutenauer Algebra

The basis elements are indexed by permutations: $(\mathbf{F}_\sigma)$.
The product is defined by a *shuffle* operation :

$$\mathbf{F}_{21} . \mathbf{F}_{12} = \mathbf{F}_{21 \overline{\sqcup\!\sqcup} 12}$$
$$=$$

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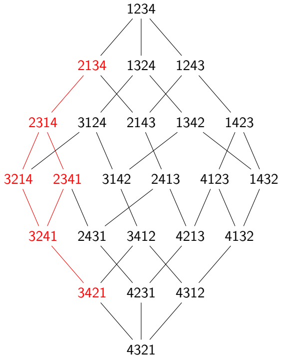

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Tamari Lattice

- 1962, Tamari : partial order on formal bracketing
- 1972, Huang, Tamari : lattice structure

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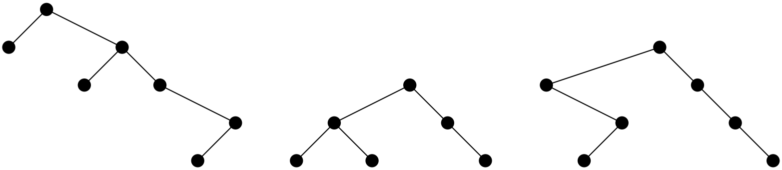
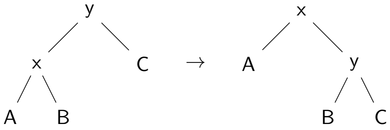
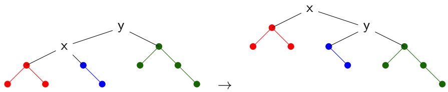
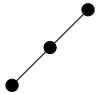
Binary trees



Right rotation







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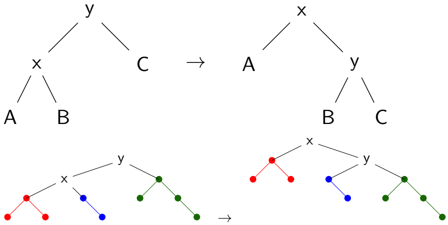

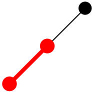

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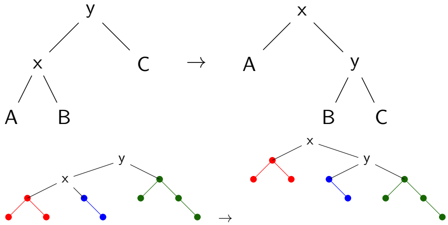

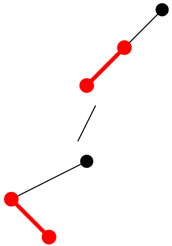

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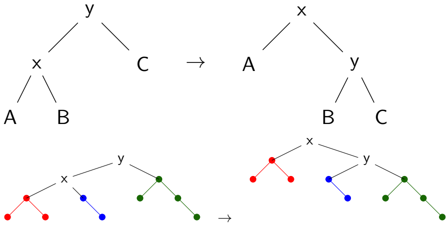

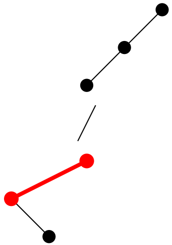

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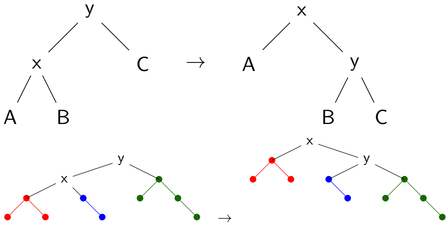

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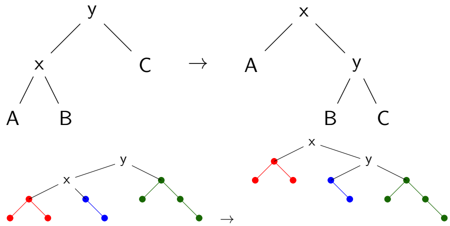

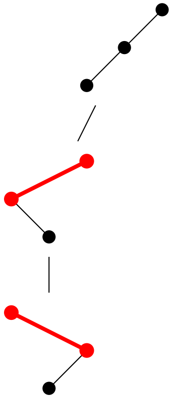

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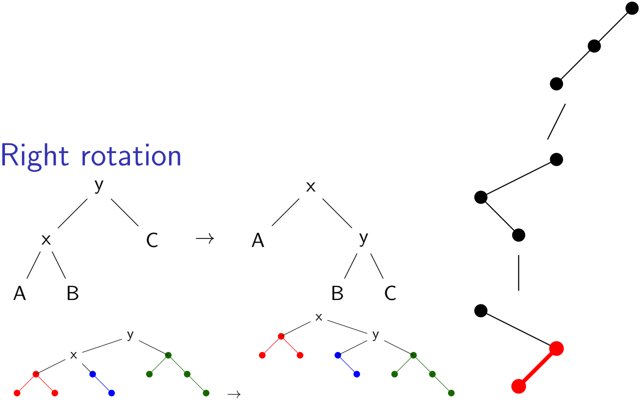

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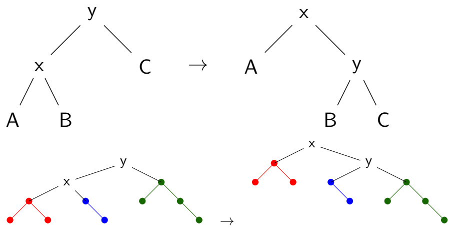

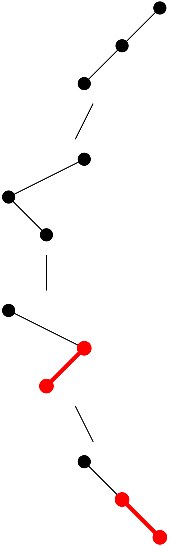

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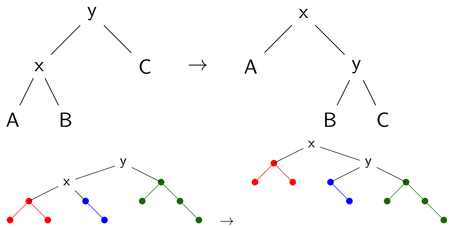

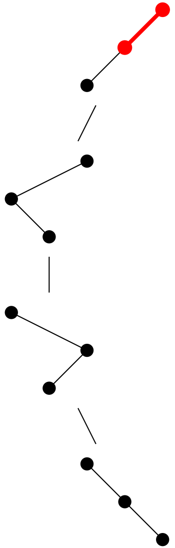

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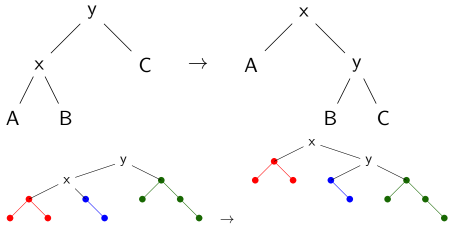

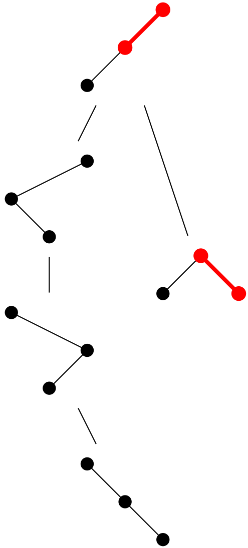

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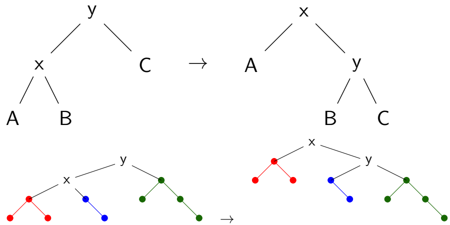

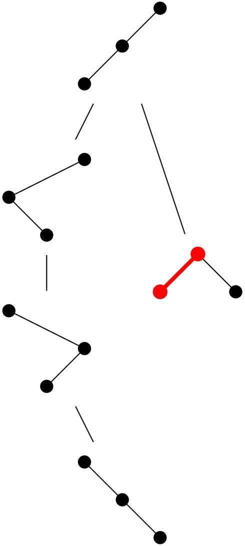

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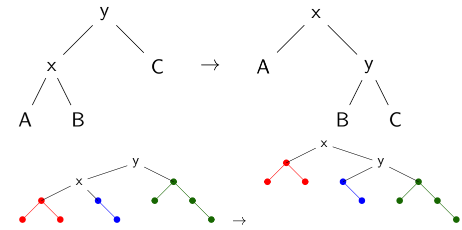

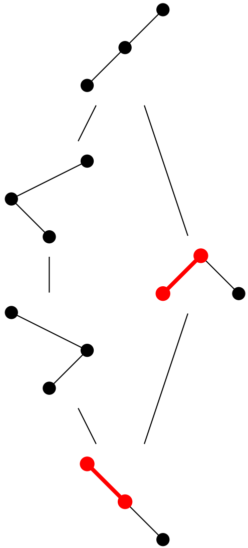

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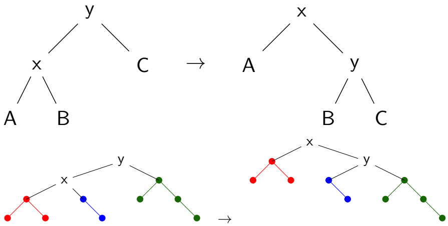
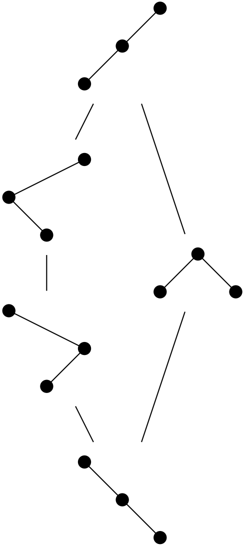
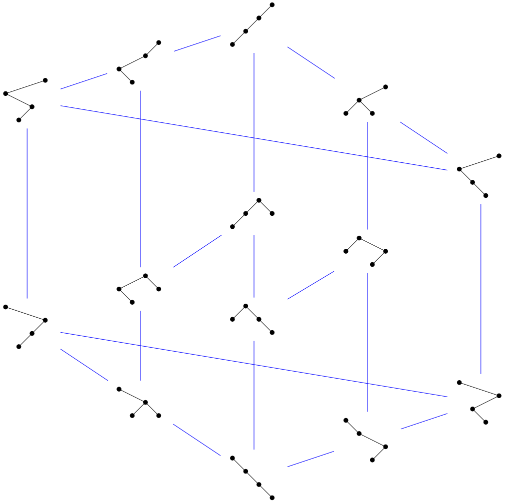
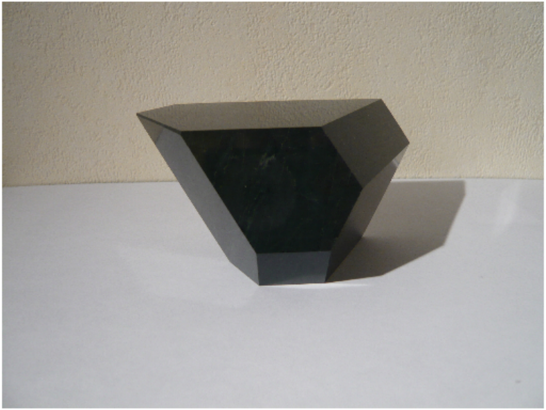


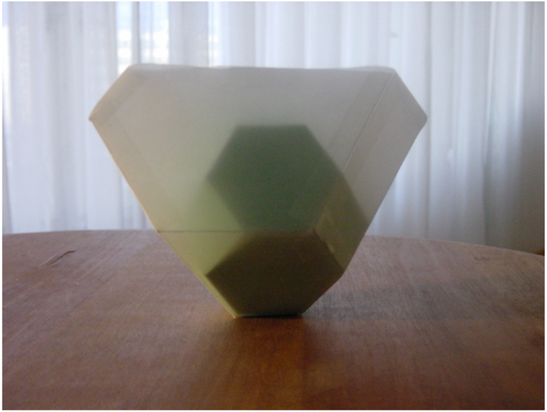




Associahedron

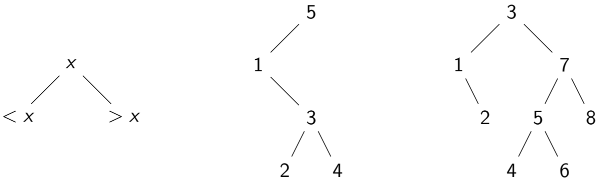


Associahedron and permutohedron



Link with the weak order

Canonical labelling

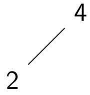


Binary search tree insertion

$$1532\textcolor{red}{4} \quad \rightarrow \quad 4$$

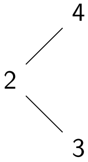
Binary search tree insertion

15324 $\rightarrow$



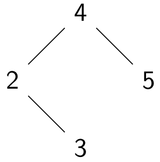
Binary search tree insertion

15324 $\rightarrow$



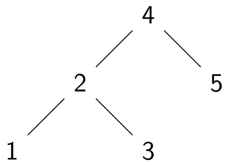
Binary search tree insertion

15324 →

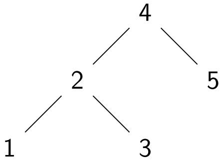


Binary search tree insertion

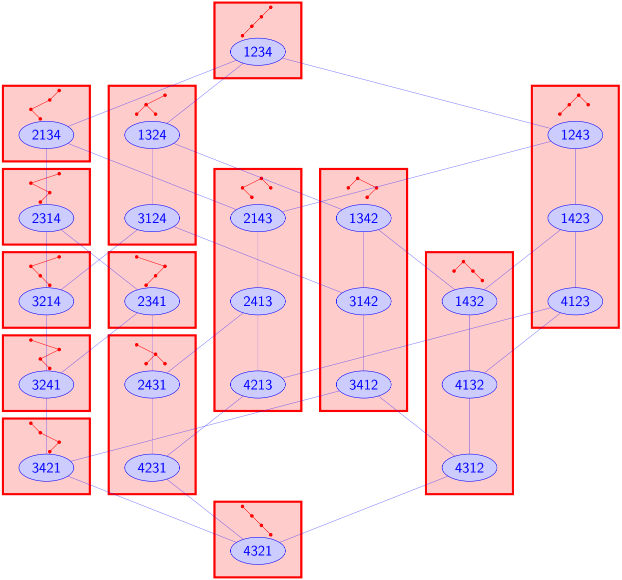
15324 $\rightarrow$



Binary search tree insertion

15324 $\rightarrow$ 

Characterisation : permutations corresponding to a given tree are its linear extension
$15324, 31254, 35124, 51324, \ldots$

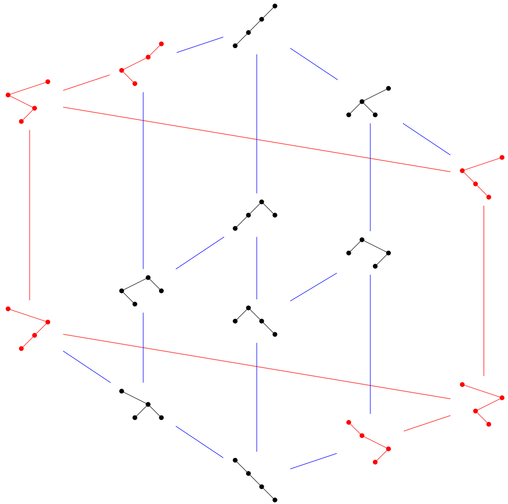

1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

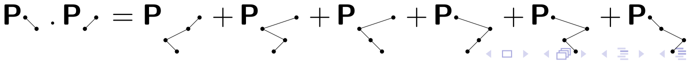
Binary trees algebra

$$\mathbf{P}_T = \sum_{\mathrm{BST}(\sigma)=T} \mathbf{F}_\sigma$$

$\mathbf{P}$  $= \mathbf{F}_{2143} + \mathbf{F}_{2413} + \mathbf{F}_{4213}$

- Loday, Ronco, 1998.
- Hivert, Novelli, Thibon, 2005.





Intervals of the Tamari lattice

- Enumeration: Chapoton 2007

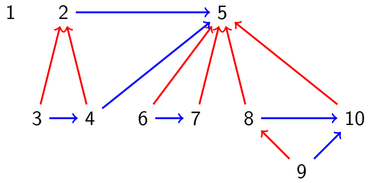
$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$$

- Bijection with triangulations: Bernardi, Bonichon 2009
- Bijection with flows on forests: Chapoton, Châtel, P., 2013

Interval-posets

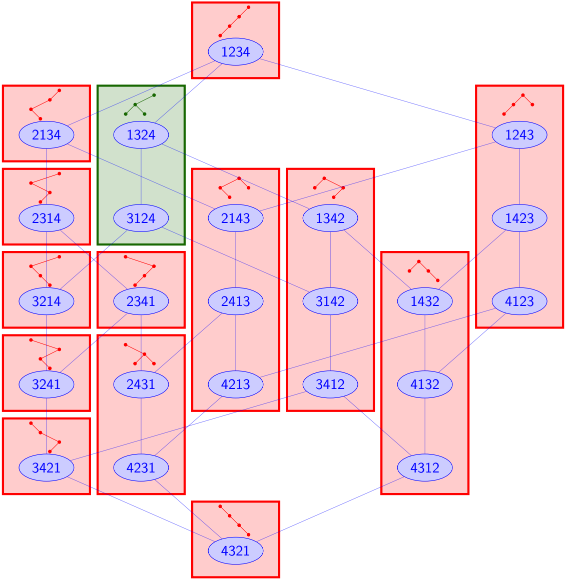
An interval-poset is a size n poset labelled by $1, \ldots, n$ such that:

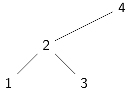
- if $a < c$ and a precedes c then b precedes c for all $a < b < c$;
- if $a < c$ and c precedes a then b precedes a for all $a < b < c$.

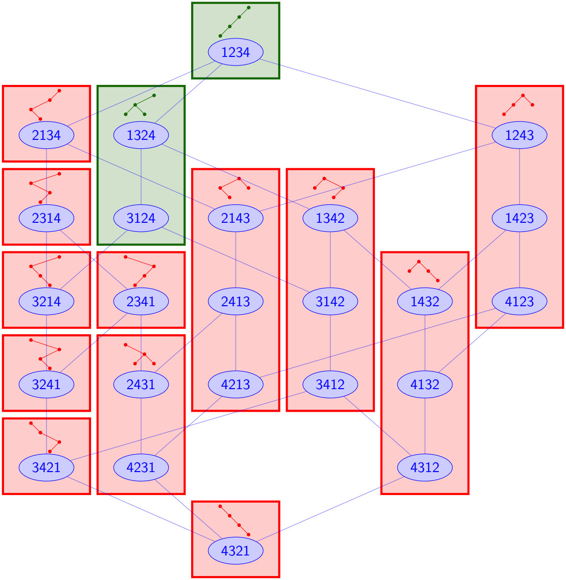


Theorem (2013 – Châtel, P.)

Interval-posets are in bijection with intervals of the Tamari lattice.


1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321


4
2
1
3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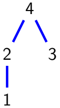
$F_{\leq}(T)$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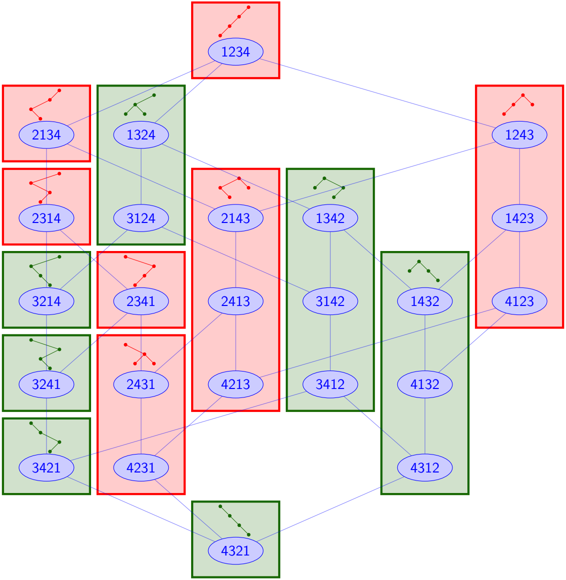
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$F_{\geq}(T)$


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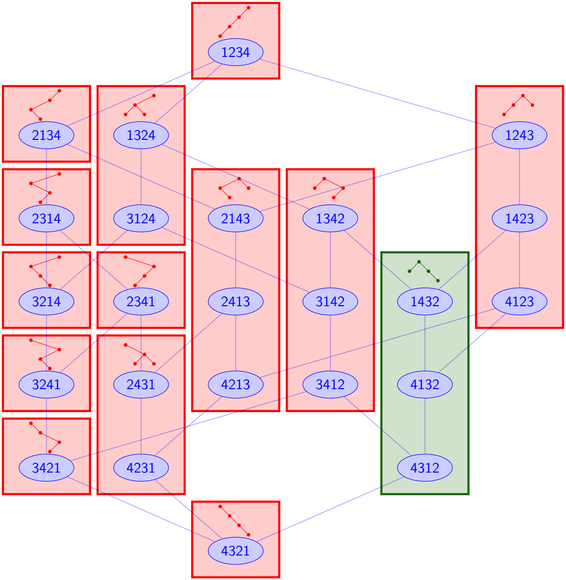
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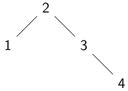
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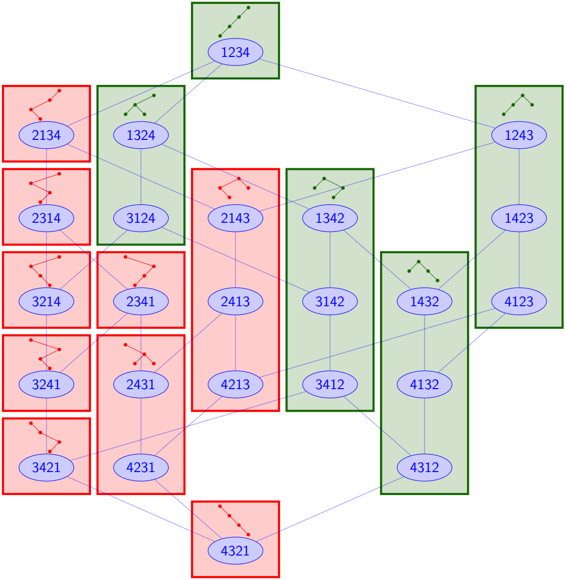
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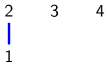
$F_{\leq}(T')$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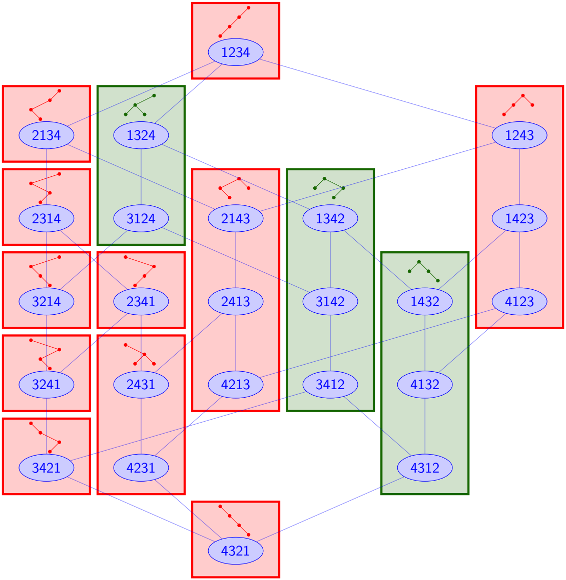
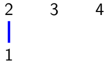
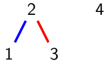


$F_{\geq}(T)$

$F_{\leq}(T')$



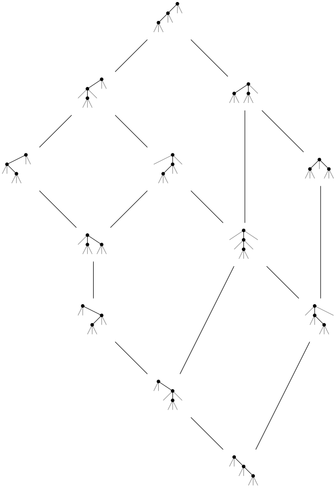
Interval-poset
$[T, T']$



m-Tamari lattices

- Bergeron, Préville-Ratelle: *m*-Tamari posets
- Bousquet-Mélou, Fusy, Préville-Ratelle: number of intervals

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n + m}{n-1}$$



Cambrian lattices

- introduced by Reading, 2006
- a lattice on "signed" binary trees
- signed permutations, Cambrian algebras