Tamari lattice, right weak order and intervals

Viviane Pons

Universität Wien

Strobl, December 16, 2013

Definition

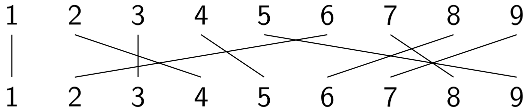
A permutation is a word of size n on the alphabet $\{1, \dots, n\}$ where each letter appears exactly once.

Example : 143592867

Definition

A permutation is a word of size n on the alphabet $\{1, \ldots, n\}$ where each letter appears exactly once.

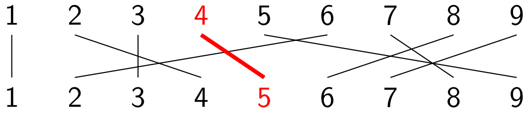
Example : 143592867



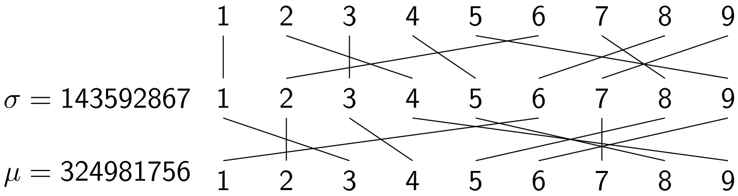
Definition

A permutation is a word of size n on the alphabet $\{1, \ldots, n\}$ where each letter appears exactly once.

Example : 143592867



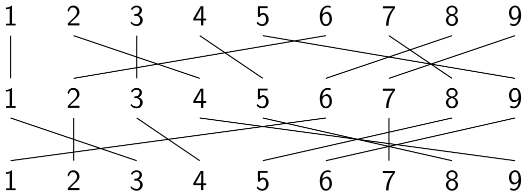
Group structure



Group structure

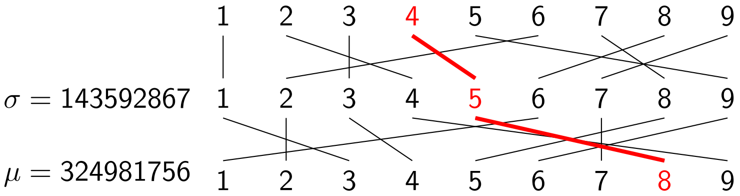
$\sigma = 143592867$

$\mu = 324981756$



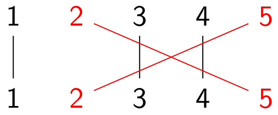
$$\mu.\sigma = 394862517$$

Group structure



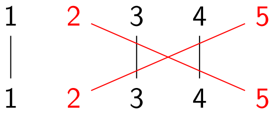
$\mu.\sigma = 394862517$

Transpositions



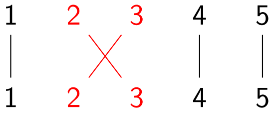
$(2,5)$

Transpositions



$(2,5)$

Simple transpositions



$(2,3) = s_2$

Right weak order

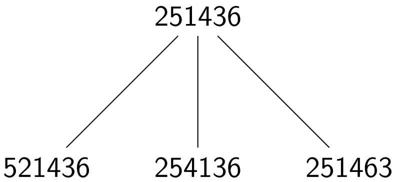
$$\sigma$$

$$|$$

$$\sigma s_i$$

$$\ell(\sigma s_i) = \ell(\sigma) + 1$$

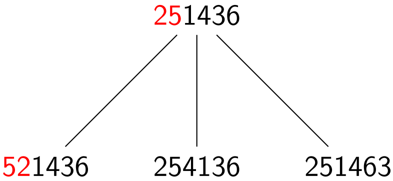
Right weak order



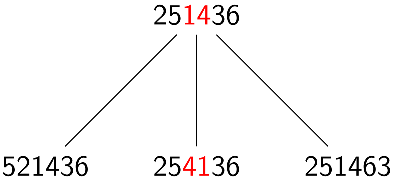
$$\ell(\sigma s_i) = \ell(\sigma) + 1$$

Right weak order

$$\ell(\sigma s_i) = \ell(\sigma) + 1$$



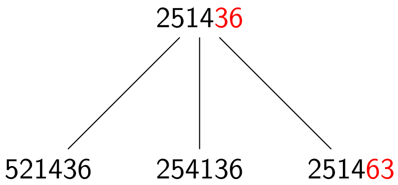
Right weak order



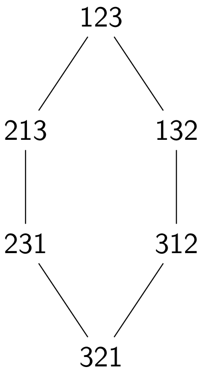
$$\ell(\sigma s_i) = \ell(\sigma) + 1$$

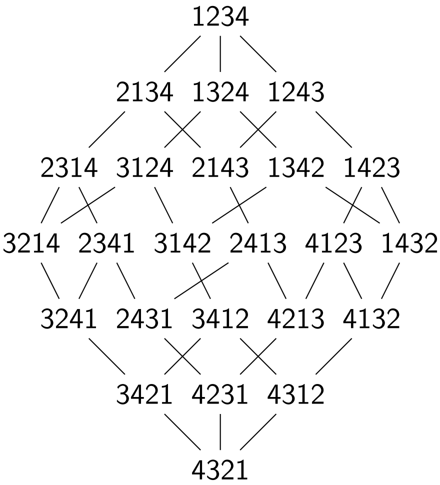
Right weak order

$$\ell(\sigma s_i) = \ell(\sigma) + 1$$

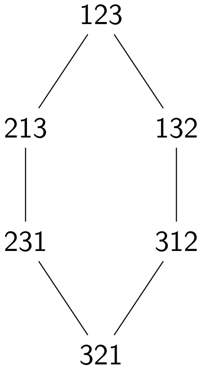


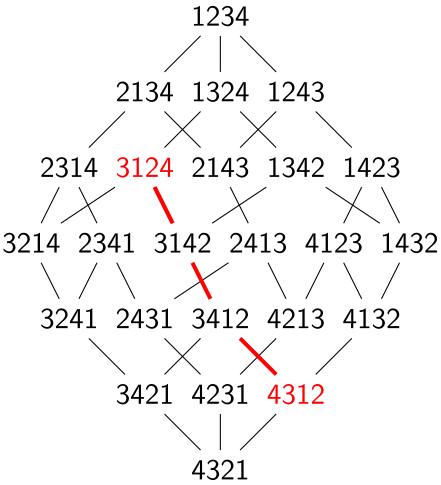
Right weak order



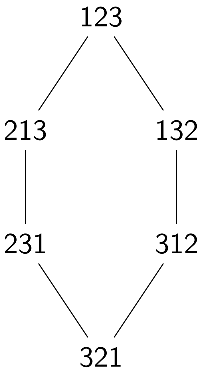


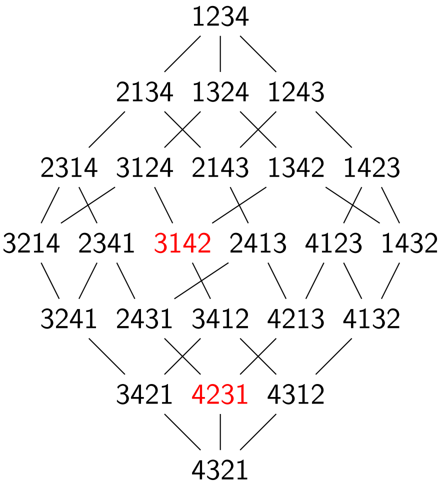
Right weak order



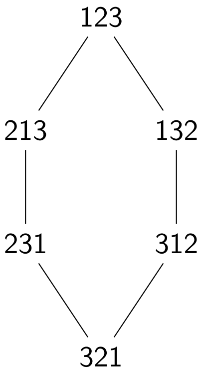


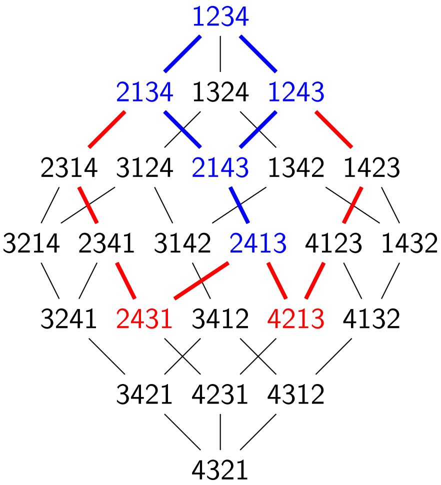
Right weak order





Right weak order





Tamari lattice

Tamari lattice

- 1962, Tamari : poset of formal bracketing

Tamari lattice

- 1962, Tamari : poset of formal bracketing
- 1972, Huang, Tamari : lattice structure

Tamari lattice

- 1962, Tamari : poset of formal bracketing
- 1972, Huang, Tamari : lattice structure
- 2007, Chapoton : number of intervals

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$$

m-Tamari lattices

m-Tamari lattices

- Bergeron, Préville-Ratelle : m-Tamari posets

m-Tamari lattices

- Bergeron, Préville-Ratelle : m-Tamari posets
- Bousquet-Mélou, Fusy, Préville-Ratelle : lattice structure and number of intervals

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n + m}{n-1}$$

Binary trees

Recursive definition :

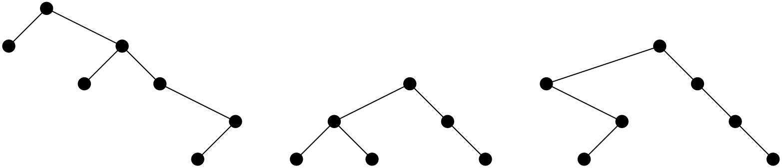
- the empty tree or
- a left subtree and a right subtree grafted to a root node

Binary trees

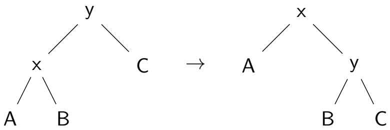
Recursive definition :

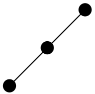
- the empty tree or
- a left subtree and a right subtree grafted to a root node

Examples

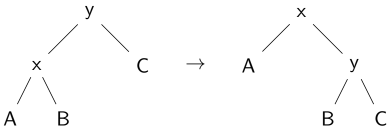


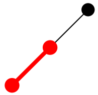
Right rotation



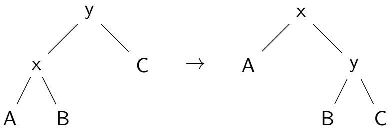


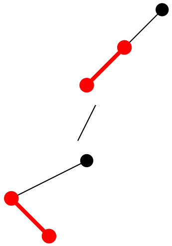
Right rotation



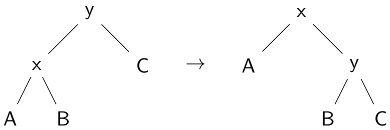


Right rotation

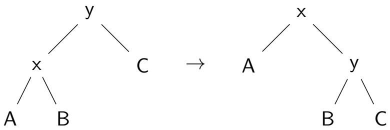


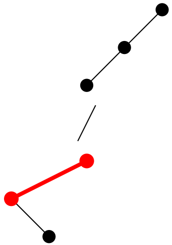


Right rotation

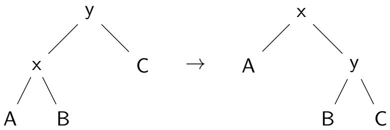


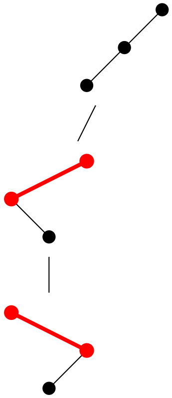
Right rotation



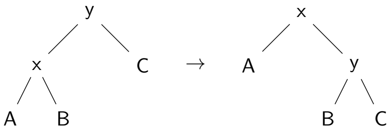


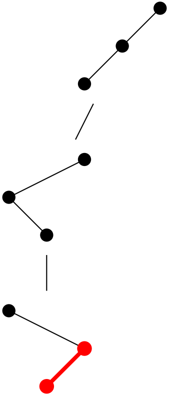
Right rotation



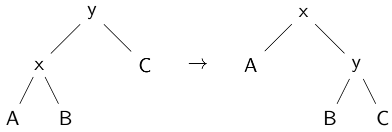


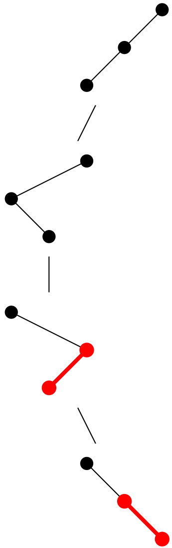
Right rotation



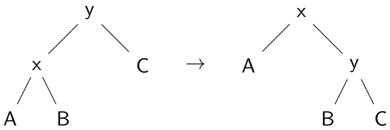


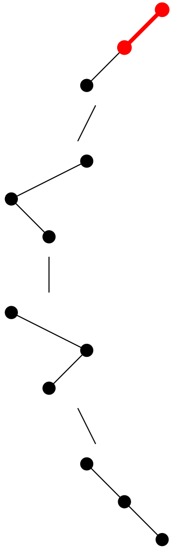
Right rotation



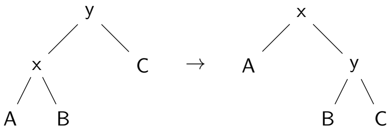


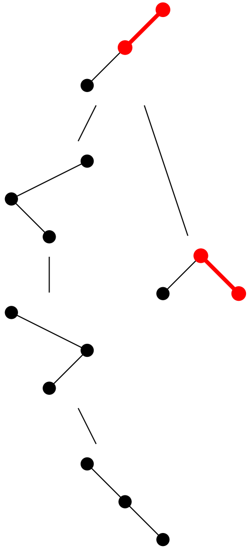
Right rotation



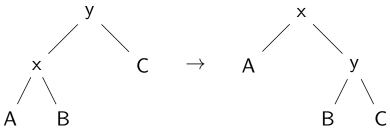


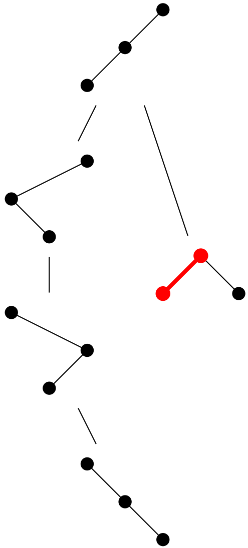
Right rotation



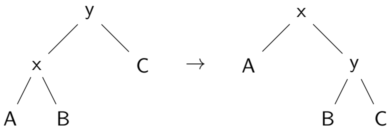


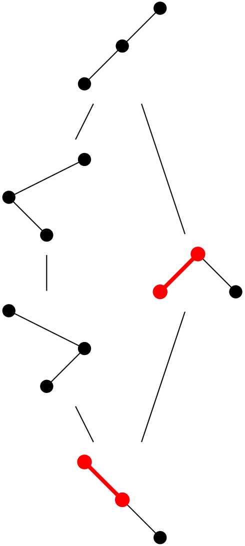
Right rotation



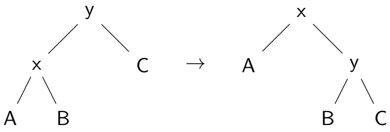


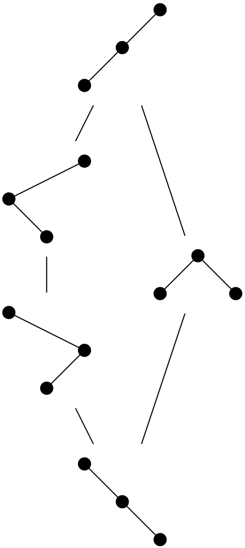
Right rotation

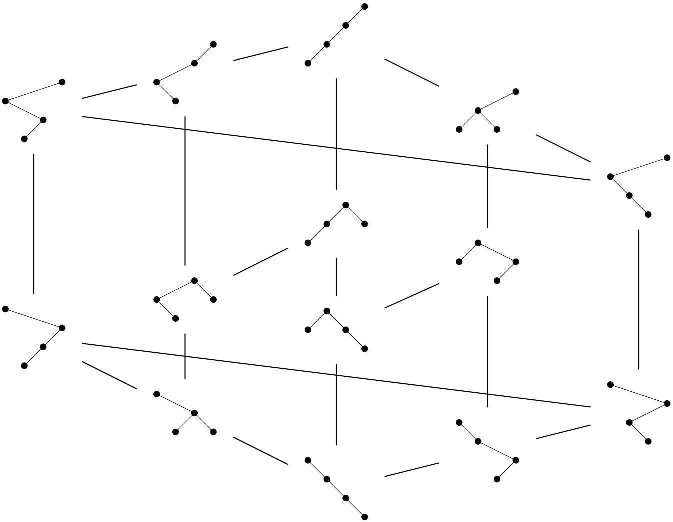




Right rotation

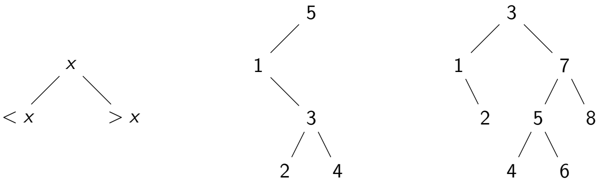






Link between the right weak order and the Tamari order

canonical binary search tree labelling

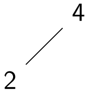


Binary search tree insertion

$$15324 \quad \rightarrow \quad 4$$

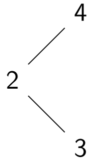
Binary search tree insertion

15324 $\rightarrow$



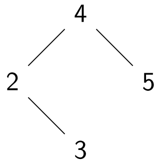
Binary search tree insertion

15324 $\rightarrow$



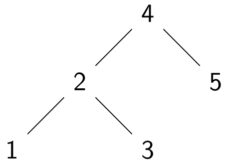
Binary search tree insertion

15324 $\rightarrow$



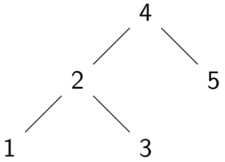
Binary search tree insertion

15324 $\rightarrow$



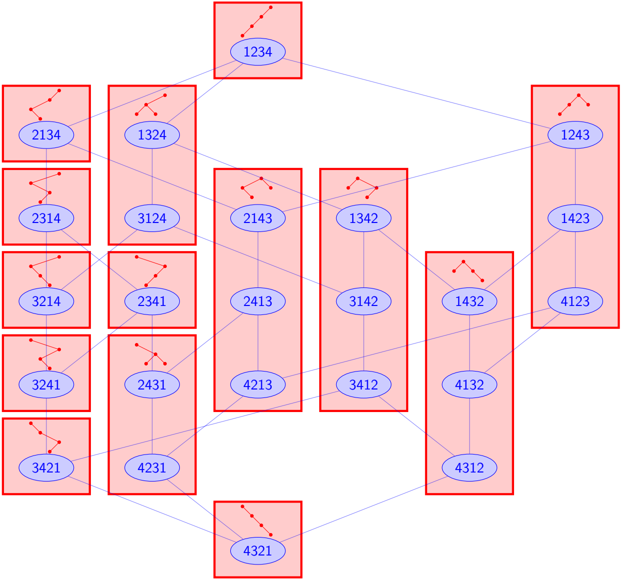
Binary search tree insertion

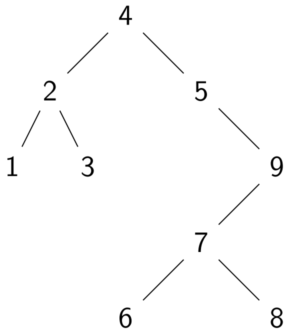
15324 $\rightarrow$

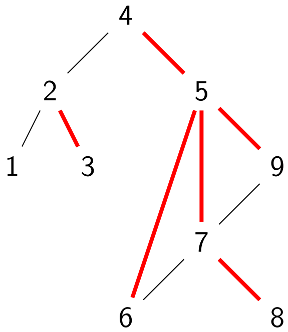


Characterization : the permutations sent to a given tree are its linear extensions

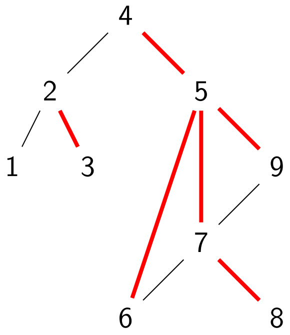
$15324, 31254, 35124, 51324, \dots$


1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

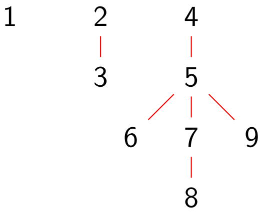

4
2
5
1
3
9
7
6
8

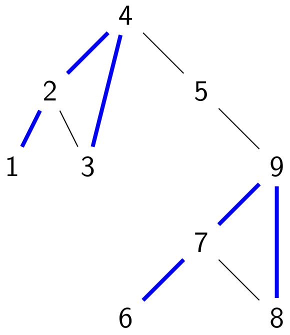


final forest $F_{\geq}(T)$

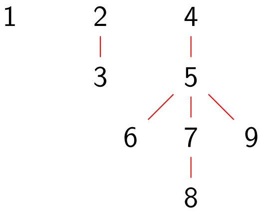


final forest $F_{\geq}(T)$

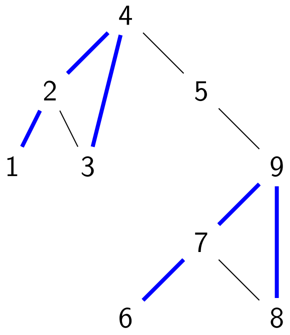




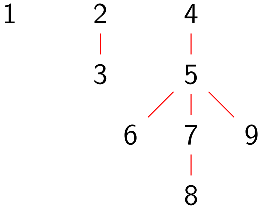
final forest $F_{\geq}(T)$



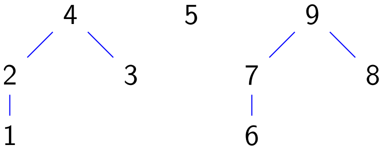
Initial forest $F_{\leq}(T)$

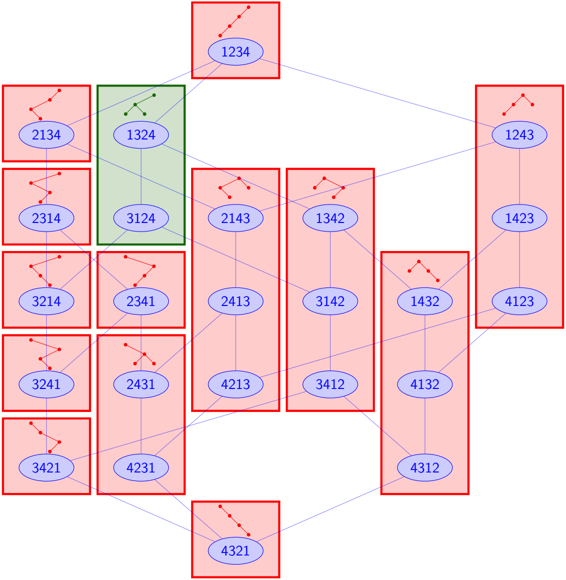


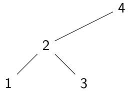
final forest $F_{\geq}(T)$

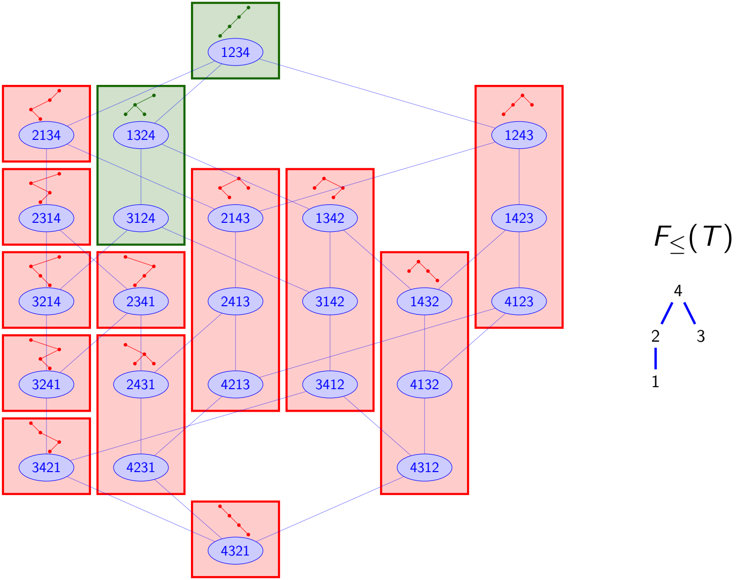


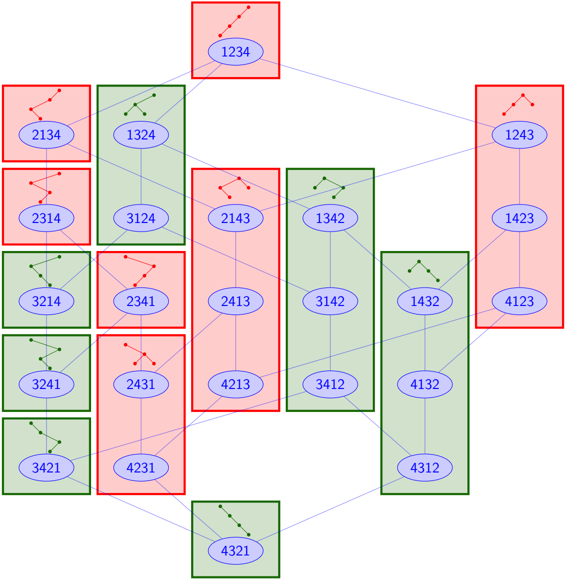
Initial forest $F_{\leq}(T)$




1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

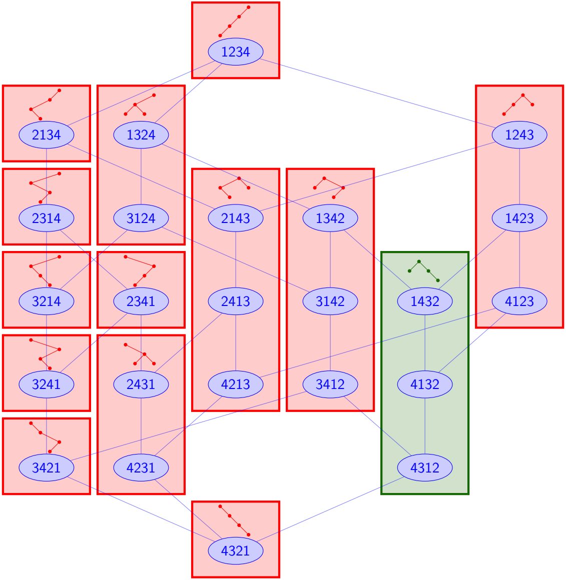

4
2
1
3

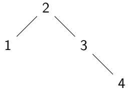

1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321
$F_{\leq}(T)$
4
2
3
1

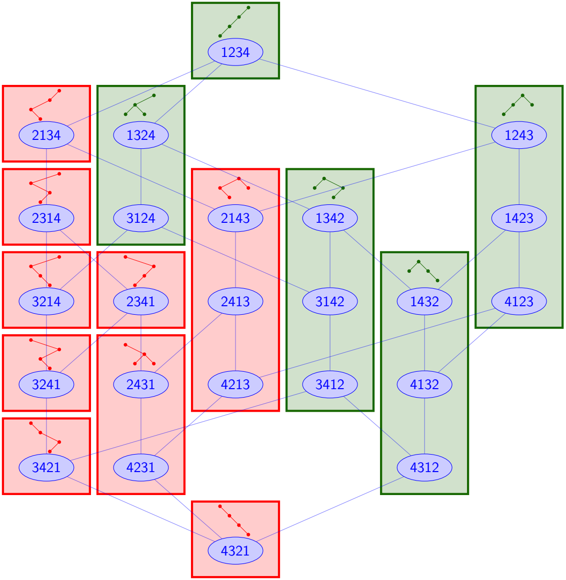

1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

$F_{\geq}(T)$

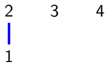

1
2
4
3

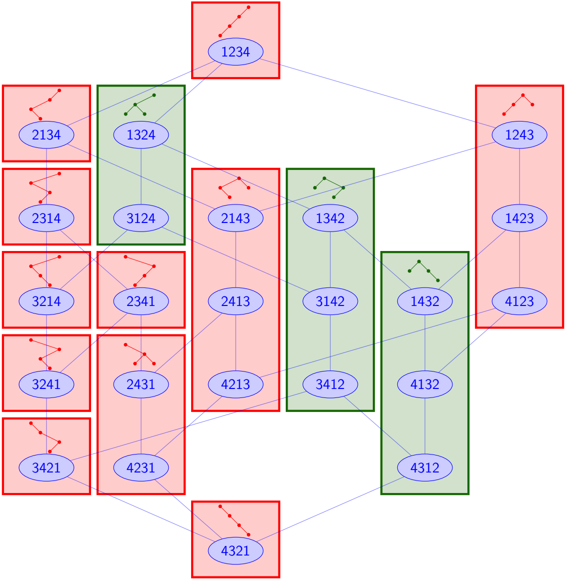

1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321


2
1
3
4


1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
4312
3421
4231
4321

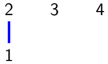
$F_{\leq}(T')$


2
3
4
1



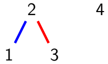
$F_{\geq}(T)$

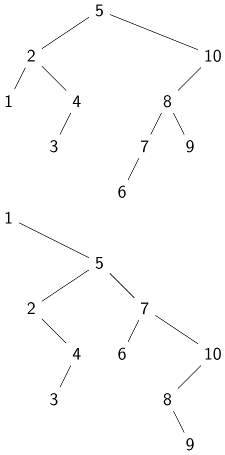
$F_{\leq}(T')$

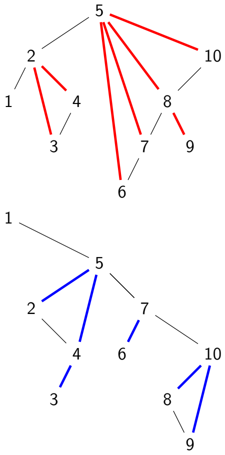


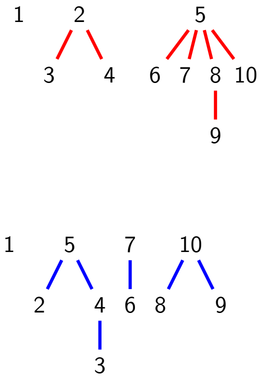
Intervalle-poset

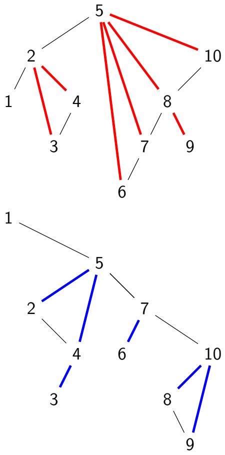
$[T, T']$

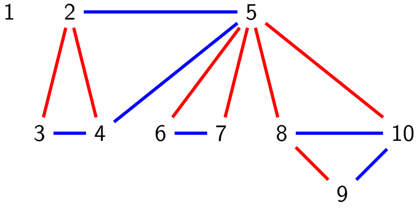


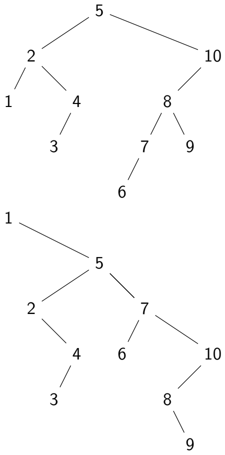

5
2
10
1
4
8
3
7
9
6
1
5
2
7
4
6
10
3
8
9

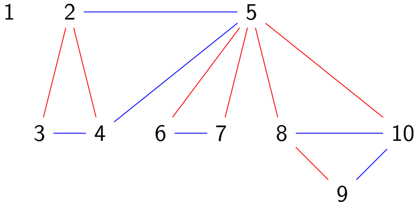

5
2
10
1
4
8
3
7
9
6
1
5
2
7
4
6
10
3
8
9

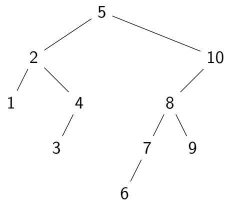

1
2
5
3
4
6
7
8
10
9
1
5
7
10
2
4
6
8
9
3

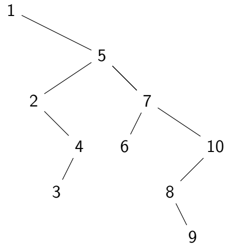

5
2
10
1
4
8
3
7
9
6
1
5
2
7
4
6
10
3
8
9

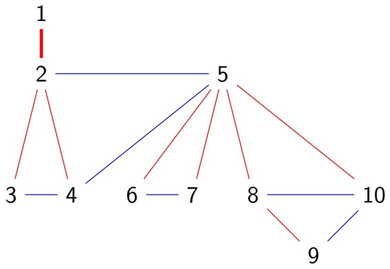

1
2
5
3
4
6
7
8
10
9

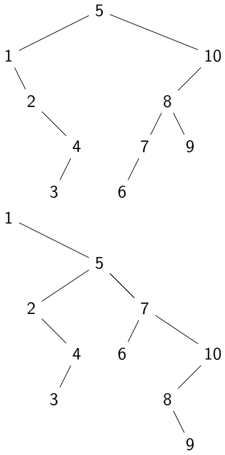

5
2
10
1
4
8
3
7
9
6
1
5
2
7
4
6
10
3
8
9

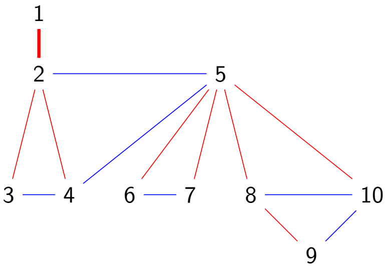

1
2
5
3
4
6
7
8
10
9

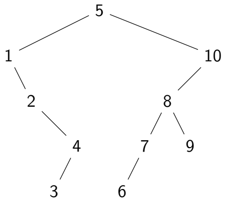

5
2
10
1
4
8
3
7
9
6

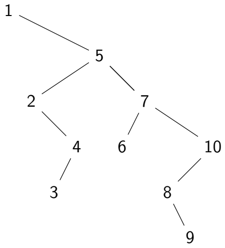

1
5
2
7
4
6
10
3
8
9

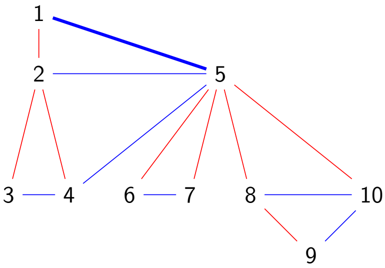

1
2
5
3
4
6
7
8
10
9

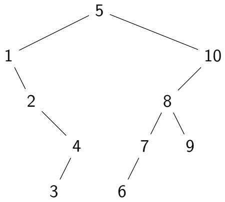

5
1
10
2
8
4
7
9
3
6
1
5
2
7
4
6
10
3
8
9

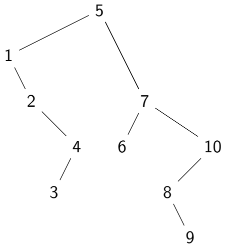

1
2
5
3
4
6
7
8
10
9

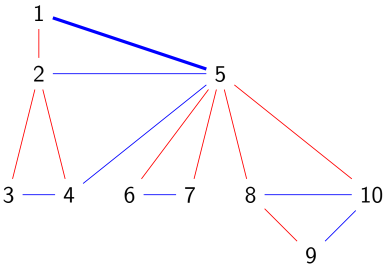

5
1
10
2
8
4
7
9
3
6


1
5
2
7
4
6
10
3
8
9


1
2
5
3
4
6
7
8
10
9


5
1
10
2
8
4
7
9
3
6


5
1
2
7
4
6
10
3
8
9


1
2
5
3
4
6
7
8
10
9

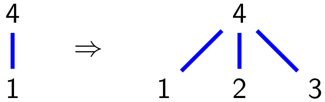
Theorem (Châtel, P.)

Intervals of Tamari are in bijection with labelled posets of size n and labels $1, \dots, n$ such that

Theorem (Châtel, P.)

Intervals of Tamari are in bijection with labelled posets of size n and labels $1, \dots, n$ *such that*

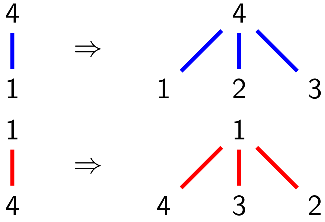
- *If* $a < c$ *and* $a \lhd c$ *then* $b \lhd c$ *for all* $a < b < c$.



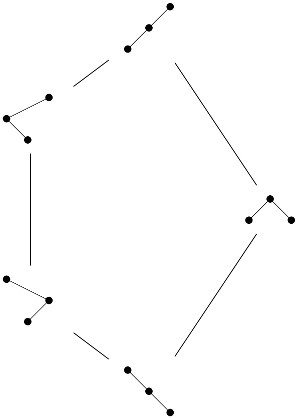
Theorem (Châtel, P.)

Intervals of Tamari are in bijection with labelled posets of size n and labels $1, \ldots, n$ such that

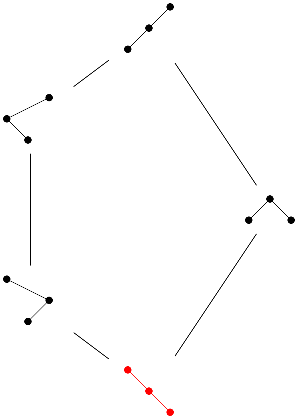
- *If $a < c$ and $a \triangleleft c$ then $b \triangleleft c$ for all $a < b < c$.*
- *If $a < c$ and $c \triangleleft a$ then $b \triangleleft a$ for all $a < b < c$.*



Number of intervals

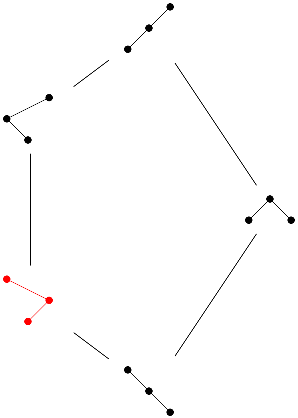


Number of intervals



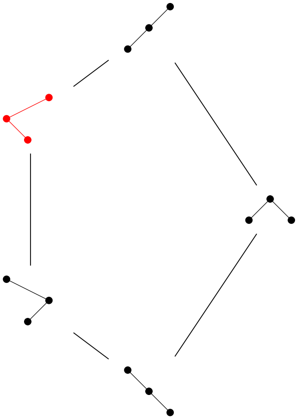
5

Number of intervals



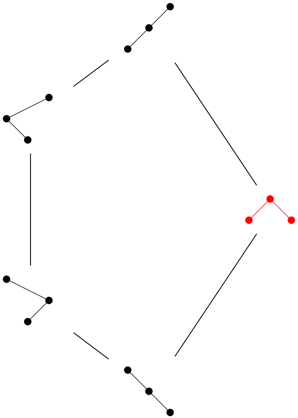
$5 + 3$

Number of intervals



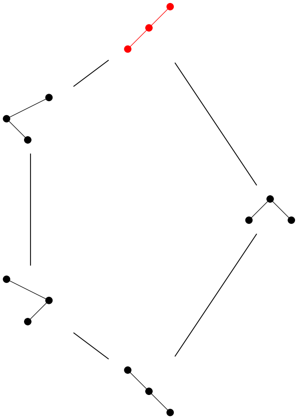
$5 + 3 + \color{red}{2}$

Number of intervals



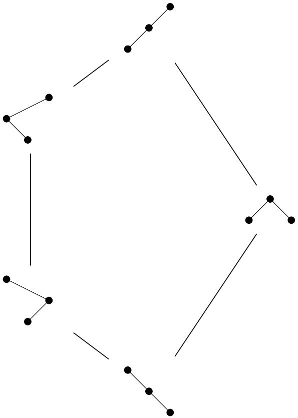
$5 + 3 + 2 + 2$

Number of intervals



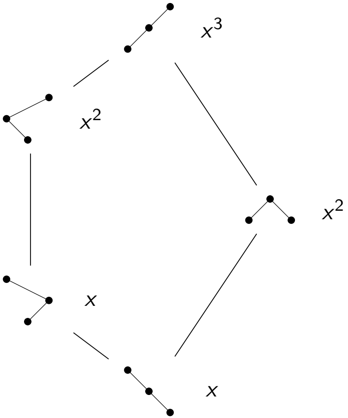
$$5 + 3 + 2 + 2 + 1$$

Number of intervals



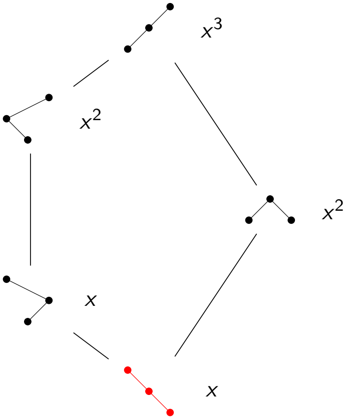
$$5 + 3 + 2 + 2 + 1 = 13$$

Number of intervals



$$5 + 3 + 2 + 2 + 1 = 13$$

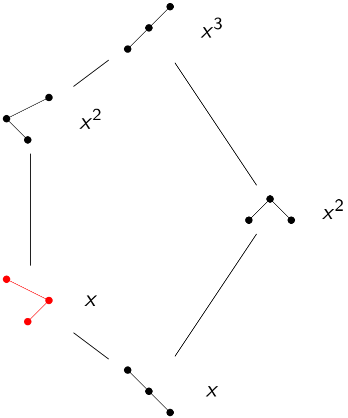
Number of intervals



$$5 + 3 + 2 + 2 + 1 = 13$$

$$(2x + 2x^2 + x^3)$$

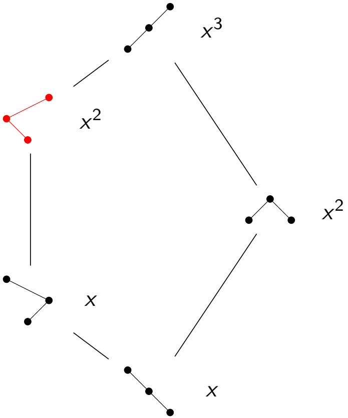
Number of intervals



$$5 + 3 + 2 + 2 + 1 = 13$$

$$(2x + 2x^2 + x^3)$$
$$+ (x + x^2 + x^3)$$

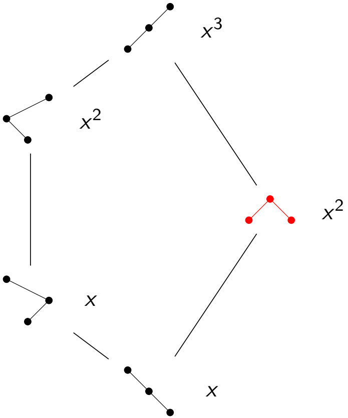
Number of intervals



$$5+3+2+2+1=13$$

$$(2x+2x^2+x^3)$$
$$+(x+x^2+x^3)$$
$$+(x^2+x^3)$$

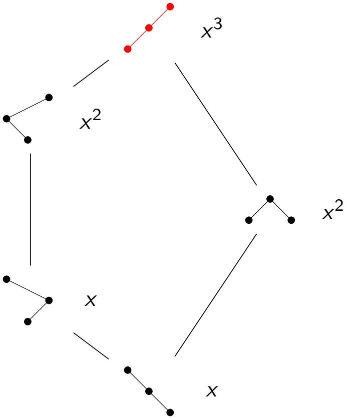
Number of intervals



$$5 + 3 + 2 + 2 + 1 = 13$$

$$(2x + 2x^2 + x^3)$$
$$+ (x + x^2 + x^3)$$
$$+ (x^2 + x^3)$$
$$+ (x^2 + x^3)$$

Number of intervals



$$5 + 3 + 2 + 2 + 1 = 13$$

$$(2x + 2x^2 + x^3)$$
$$+ (x + x^2 + x^3)$$
$$+ (x^2 + x^3)$$
$$+ (x^2 + x^3)$$
$$+ \color{red}{x^3}$$

Theorem (Chapoton)

The generating functions of Tamari intervals satisfy the functional equation

$$\Phi(x,y) = \mathrm{B}(\Phi,\Phi) + 1$$

where

$$\mathrm{B}(f,g) = xyf(x,y)\frac{xg(x,y) - g(1,y)}{x-1}$$

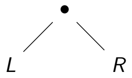
Tamari Polynomials

$\mathcal{B}_T$ is recursively defined by

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

with T =



Theorem (Châtel, P.)

$\mathcal{B}_T$ counts the number of trees smaller than or equal to T in the Tamari lattice according to the number of nodes on their left border.

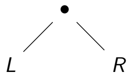
Tamari Polynomials

$\mathcal{B}_T$ is recursively defined by

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

with T =



Theorem (Châtel, P.)

$\mathcal{B}_T$ counts the number of trees smaller than or equal to T in the Tamari lattice according to the number of nodes on their left border.

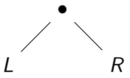
Tamari Polynomials

$\mathcal{B}_T$ is recursively defined by

$$\mathcal{B}_\emptyset := 1$$

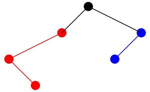
$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

with T =



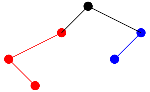
Theorem (Châtel, P.)

$\mathcal{B}_T$ counts the number of trees smaller than or equal to T in the Tamari lattice according to the number of nodes on their left border.



$$\mathcal{B}_\emptyset := 1$$

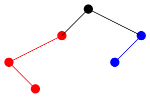
$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$



$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

$$\mathcal{B}_L(x) = x^3 + x^2$$

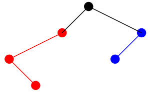


$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

$$\mathcal{B}_L(x) = x^3 + x^2$$

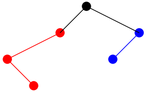
$$\mathcal{B}_R(x) = x^2$$



$$\mathcal{B}_{\emptyset} := 1$$

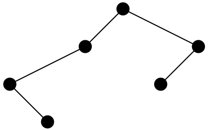
$$\mathcal{B}_T(x) := x(x^3 + x^2)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

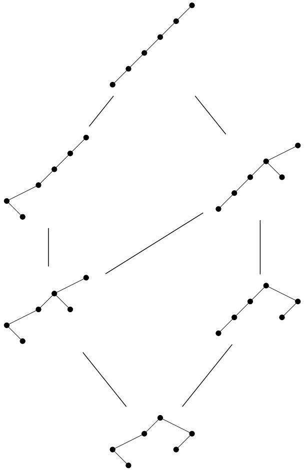
$$\mathcal{B}_R(x) = x^2$$



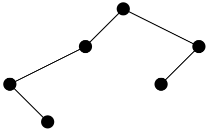
$$\mathcal{B}_\emptyset := 1$$

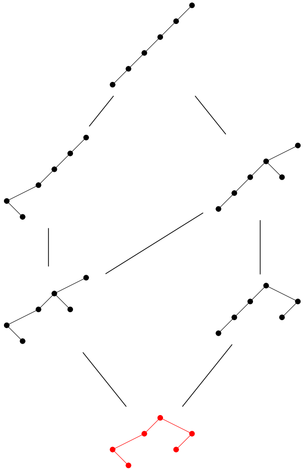
$$\mathcal{B}_T(x) := x(x^3 + x^2)(1 + x + x^2)$$



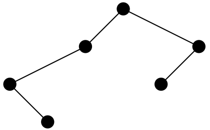


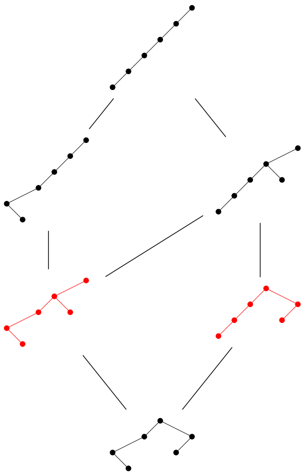
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



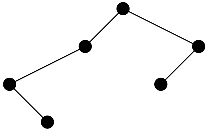


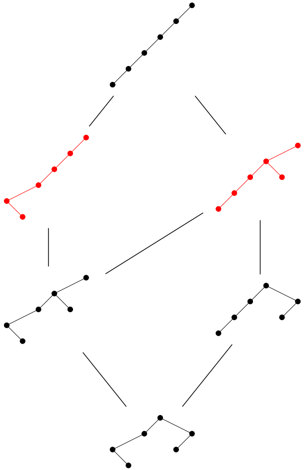
$$\mathcal{B}_T(x) = {\color{red}x^3} + 2x^4 + 2x^5 + x^6$$



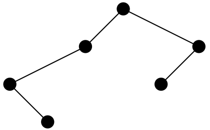


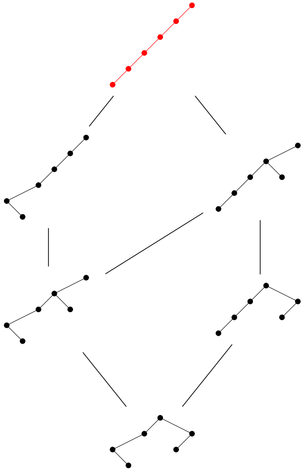
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



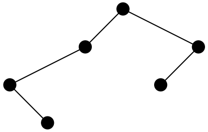


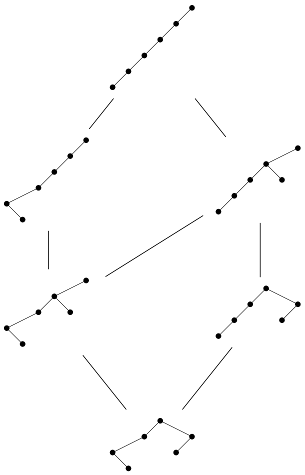
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



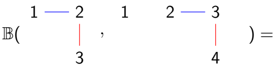


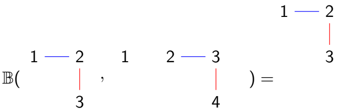
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$

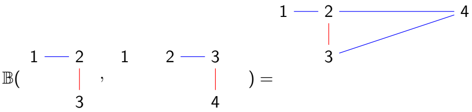


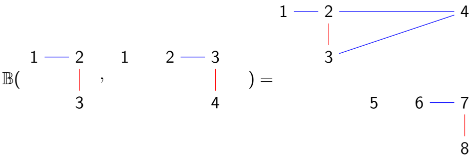


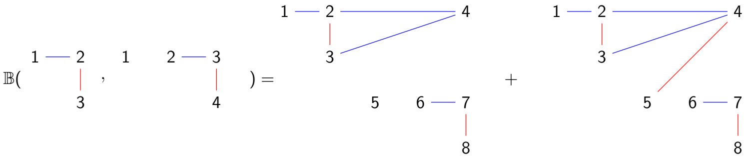
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$
$$\mathcal{B}_T(1) = 6$$

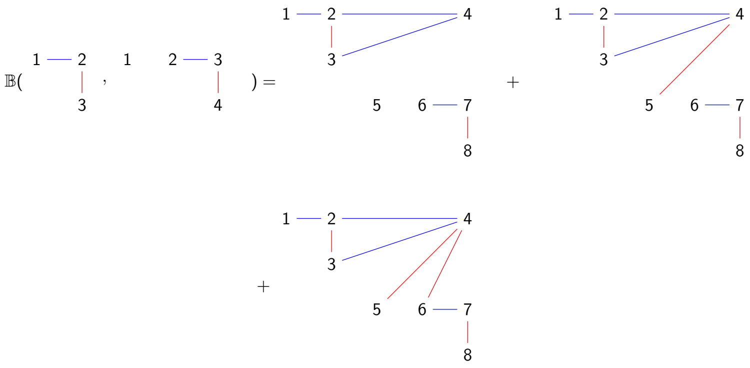

1 2 1 2 3
𝔹(,) =
3 4

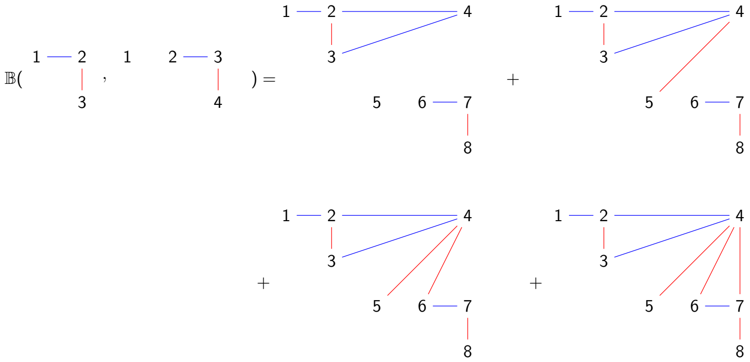

1 2
3
1
2 3
4
B(,) =
1 2
3

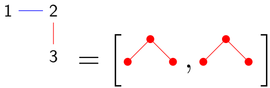

𝔹(
1 2
3
,
1
2 3
4
) =
1 2 4
3

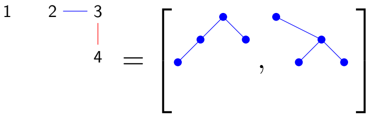

𝔹(
,
) =

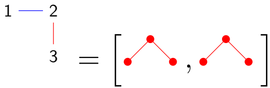

𝔹(
1 2 3
,
1 2 3 4
) =
1 2 3 4 5 6 7 8
+
1 2 3 4 5 6 7 8


𝔹(
1 2 3
,
1 2 3 4
) =
1 2 3 4 5 6 7 8
+
1 2 3 4 5 6 7 8
+
1 2 3 4 5 6 7 8

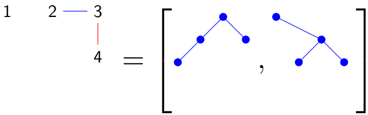

𝔹(
,
) =
+
+
+


1
2
3
=

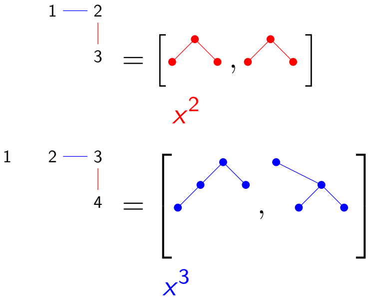

1
2
3
4
=

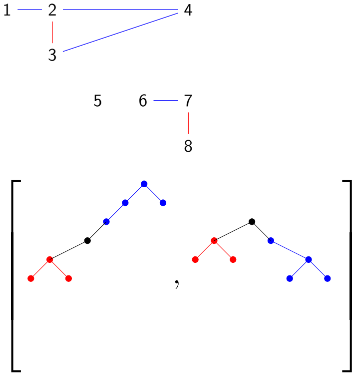


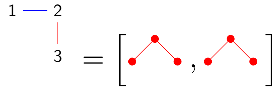
x^2



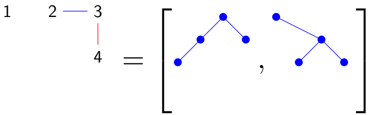
x^3


1
2
3
=
x^2
1
2
3
4
=
x^3

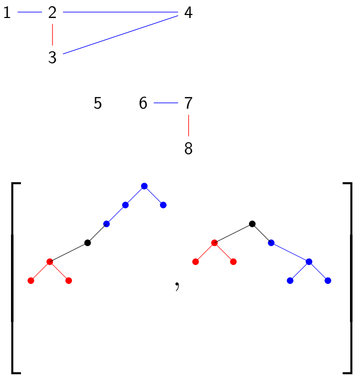

1
2
4
3
5
6
7
8



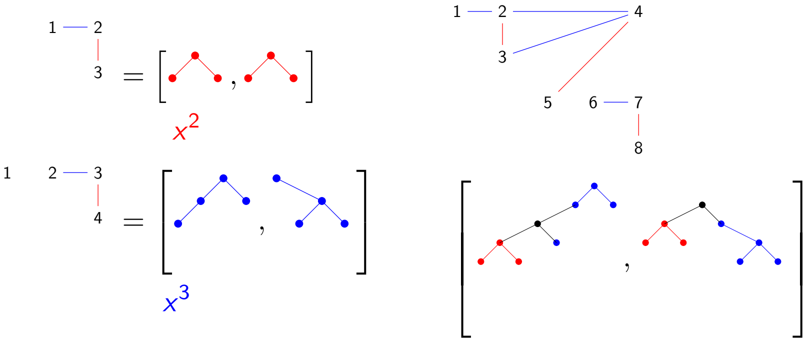
x^2



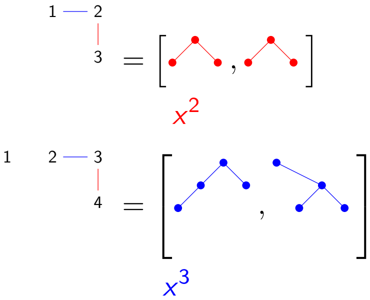
x^3

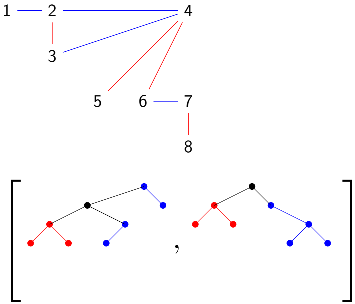


$x^2.x.x^3$

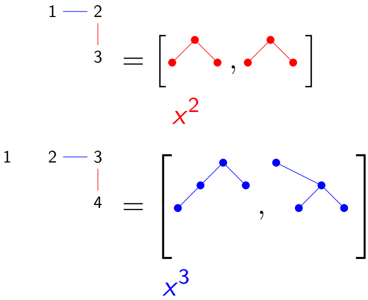


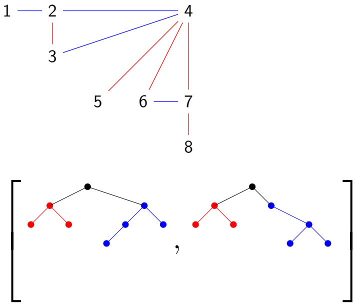
$$x^2.x.x^3 + x^2.x.x^2$$



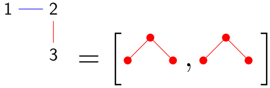


$$x^2 . x . x^3 + x^2 . x . x^2 + x^2 . x . x$$

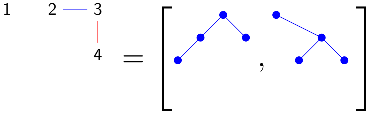




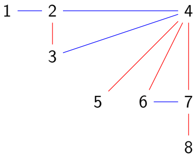
$$x^2.x.x^3 + x^2.x.x^2 + x^2.x.x + x^2.x$$

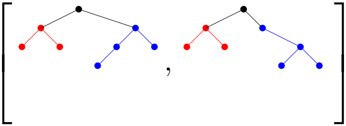


x^2



x^3



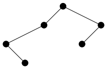


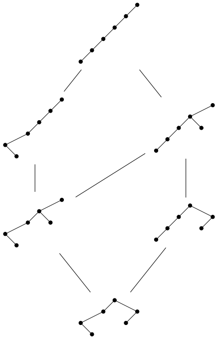
$$x^2.x.(x^3 + x^2 + x + 1)$$

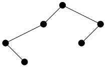
$$S_T := \sum_{T' \leq T} [T', T]$$

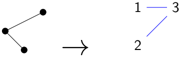
$$S_T = \mathbb{B}(S_L, S_R)$$

$$\rightarrow \mathcal{B}_T(x) = x\,\mathcal{B}_L(x) \frac{\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$



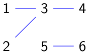




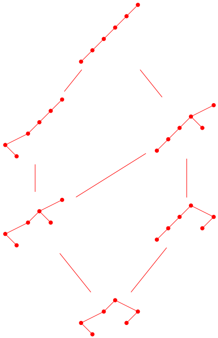


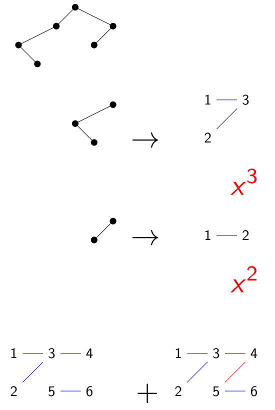
x^3

x^2

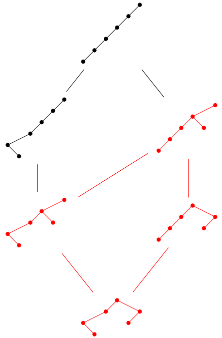


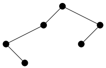
$x^3.x.x^2$

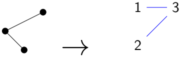




$x^3.x.x^2 + x^3.x.x$

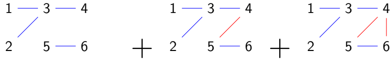




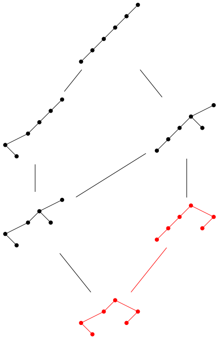


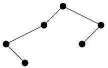
x^3

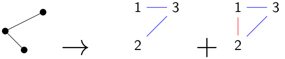
x^2



$x^3.x.x^2 \quad + x^3.x.x + x^3.x$

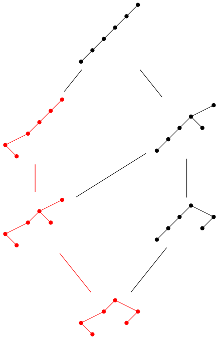


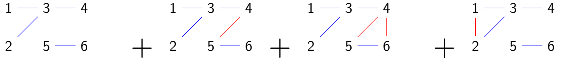




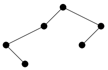
$$x^3 + x^2$$

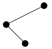
$$x^2$$

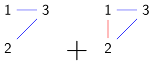




$$x^3.x.x^2 \quad + x^3.x.x + x^3.x \quad + x^2.x.x^2$$



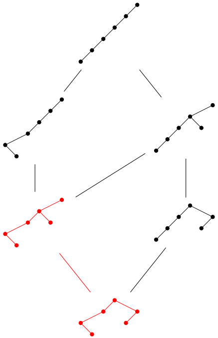
 →

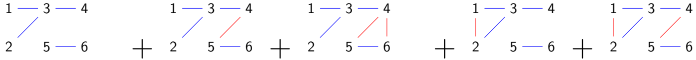


$x^3 + x^2$

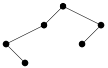
 →

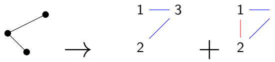
x^2





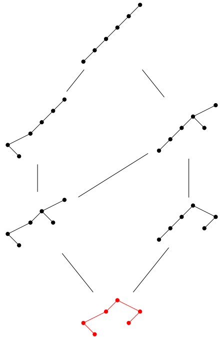
$x^3.x.x^2 + x^3.x.x + x^3.x + x^2.x.x^2 + x^2.x.x$

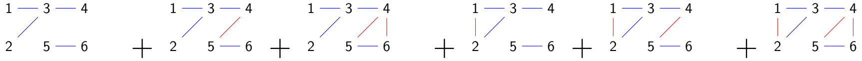




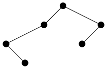
$$x^3 + {\color{red}x^2}$$

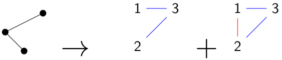
$${\color{red}x^2}$$



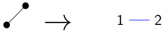


$$x^3.x.x^2 \quad + x^3.x.x + x^3.x \quad + x^2.x.x^2 + x^2.x.x \quad + {\color{red}x^2.x}$$

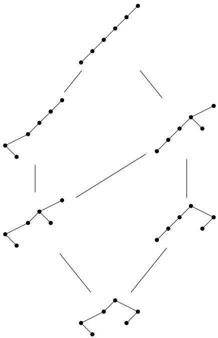




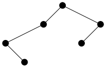
$$x^3 + x^2$$

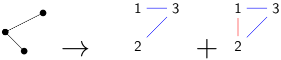


$$x^2$$



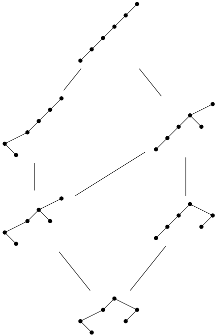
$$(x^3 + x^2).x.(x^2 + x + 1) =$$





$$x^3 + x^2$$

$$x^2$$



$$(x^3 + x^2).x.(x^2 + x + 1) = x\,\mathcal{B}_L(x)\frac{\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$

Other results

- Bijection with flows of forests
- Combinatorial proof of some equally distributed statistics
- Generalization to m-Tamari

Perspectives

- Bijection with rooted triangulations
- Generalization to other Cambrian lattices