Interval-posets of Tamari

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Tamari lattice

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- 1962, Tamari : poset of formal bracketing

Tamari lattice

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- 1972, Huang, Tamari : lattice structure

Tamari lattice

- 1962, Tamari : poset of formal bracketing
- 1972, Huang, Tamari : lattice structure
- 2007, Chapoton : number of intervals

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$$

m-Tamari lattices

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- Bergeron, Préville-Ratelle : m-Tamari posets

m-Tamari lattices

- Bergeron, Préville-Ratelle : m-Tamari posets
- Bousquet-Mélou, Fusy, Préville-Ratelle : lattice structure and number of intervals

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n + m}{n-1}$$

Binary trees

Recursive definition :

- the empty tree or
- a left subtree and a right subtree grafted to a root node

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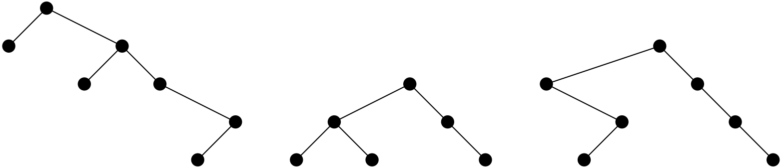
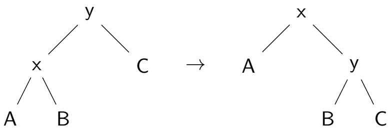
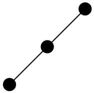
Examples



Right rotation





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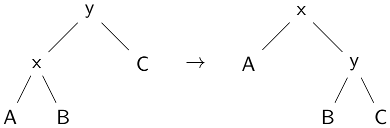

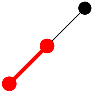

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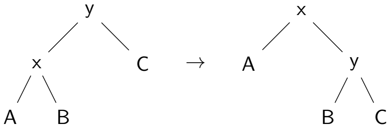

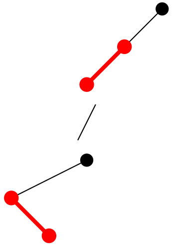

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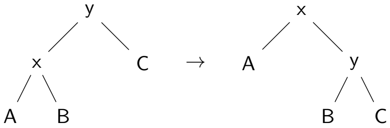

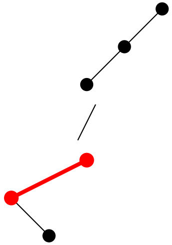

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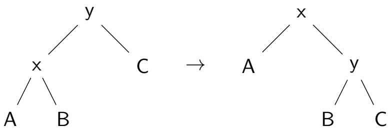

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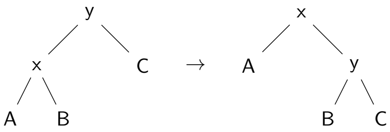

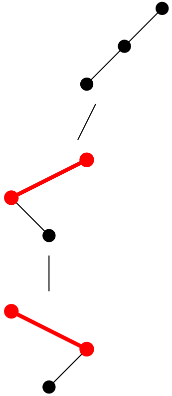

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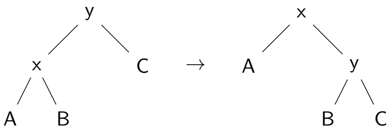

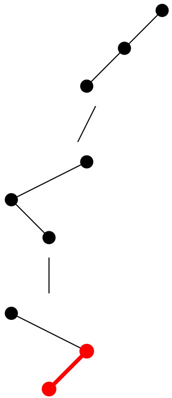

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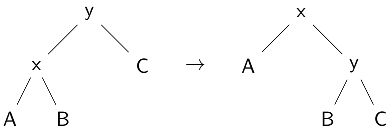

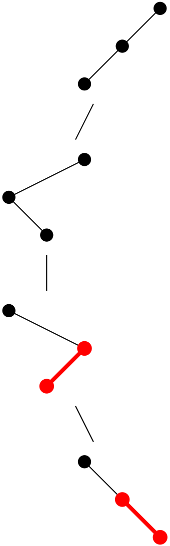

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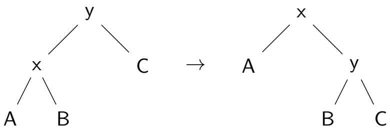

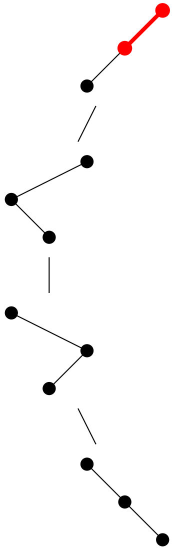

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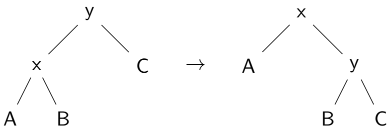

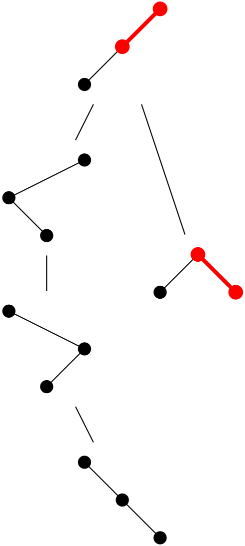

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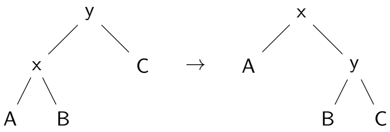

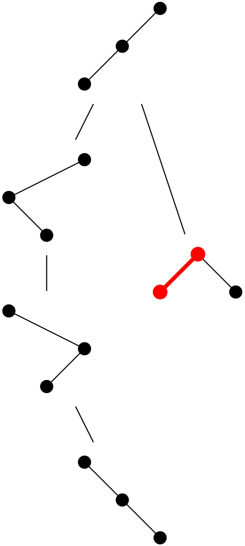

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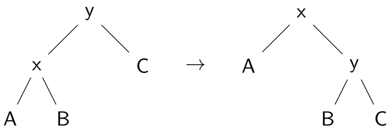

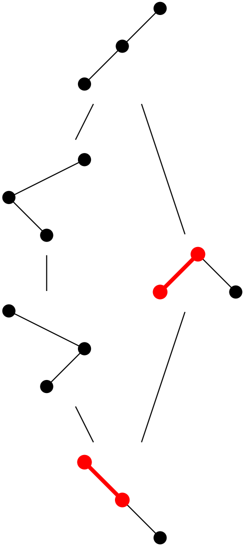

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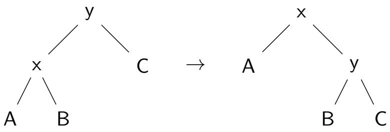
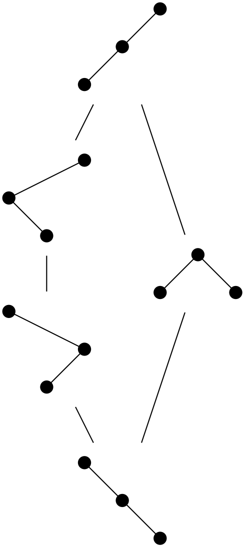
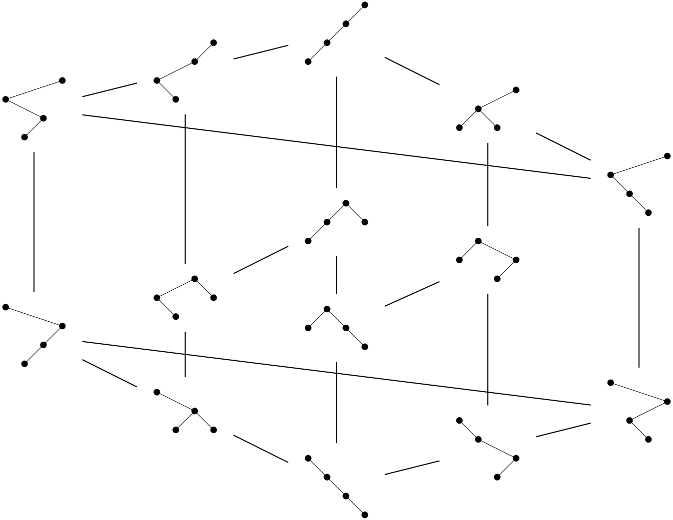
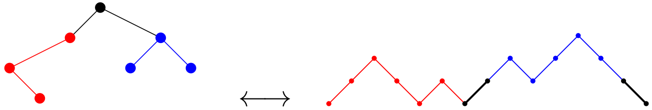


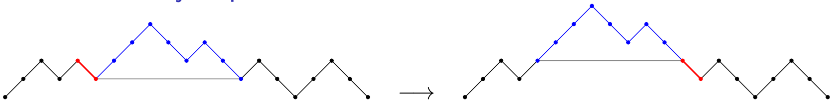


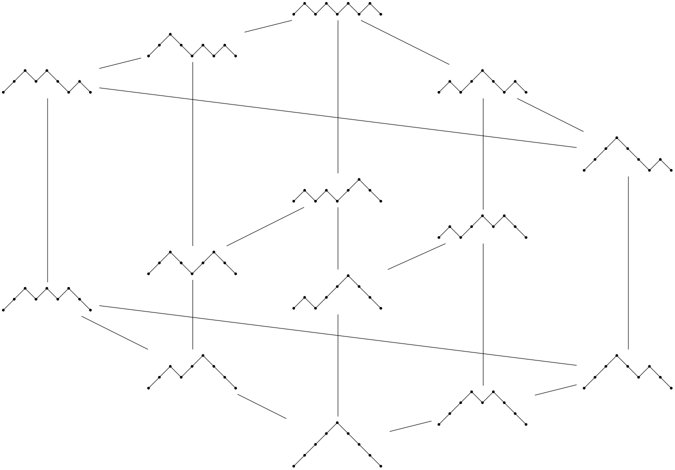


Bijection binary trees - Dyck paths

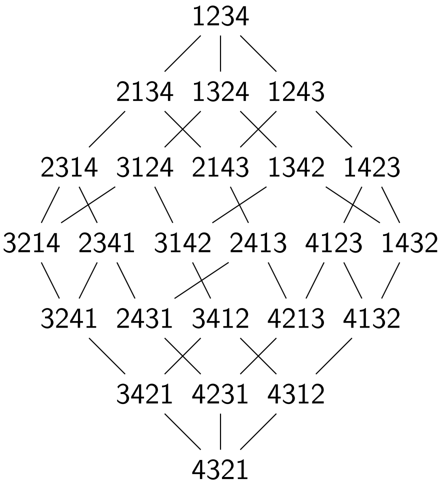


Rotation on Dyck paths

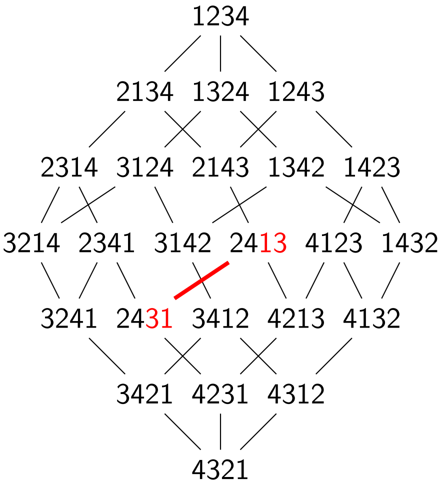




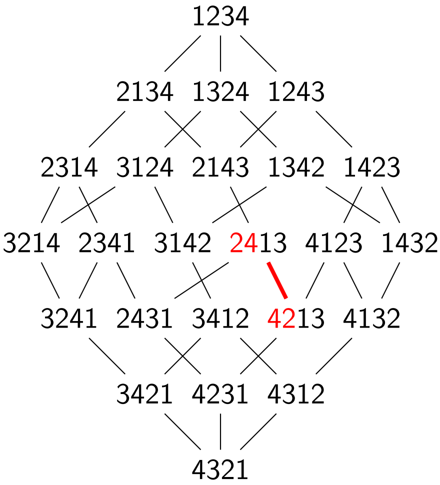
Right weak order



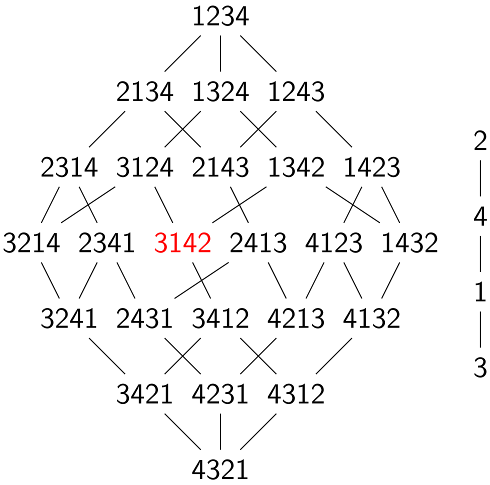
Right weak order



Right weak order



Right weak order



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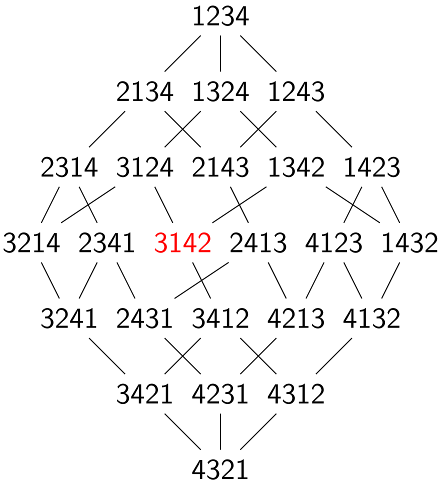

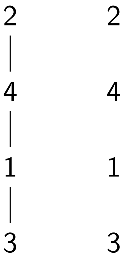

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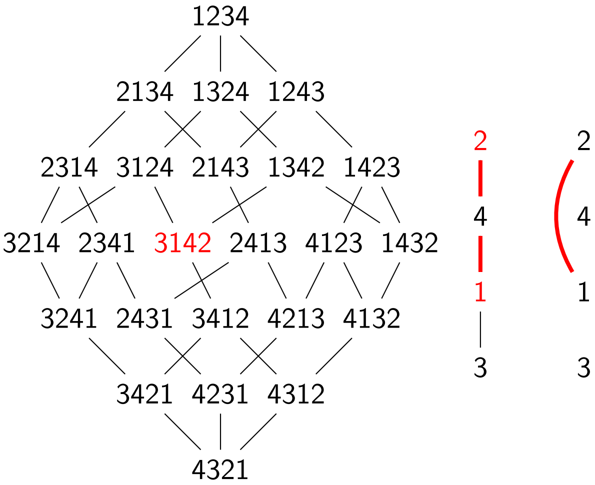

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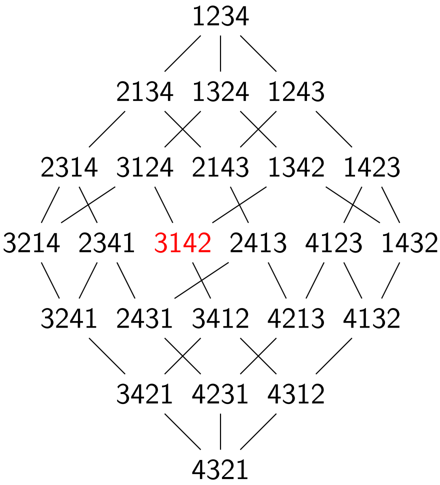

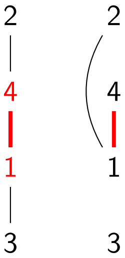

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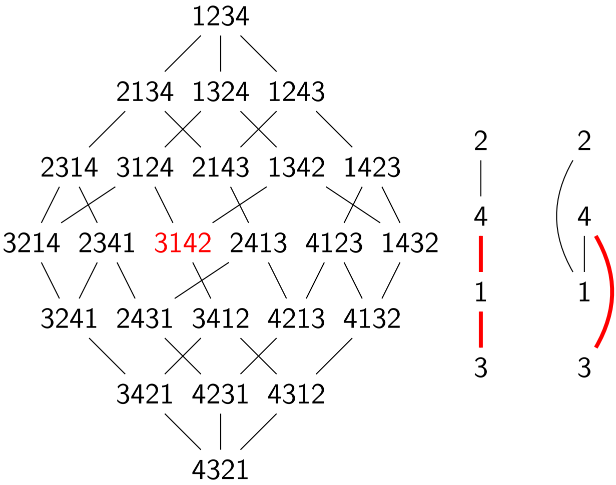

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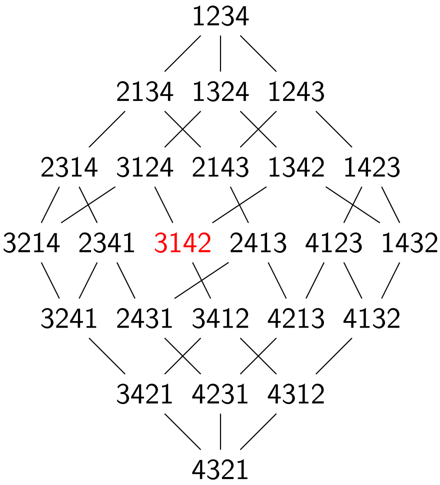

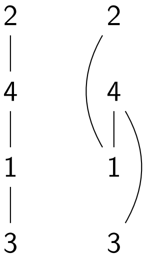

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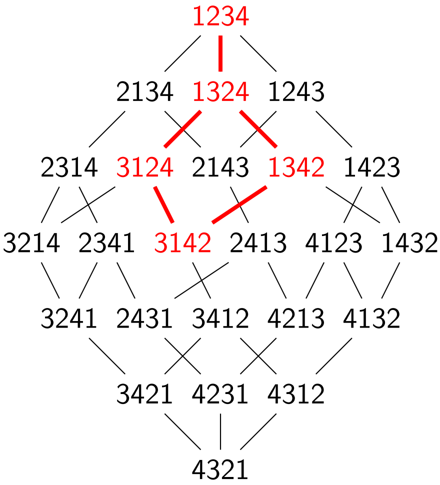
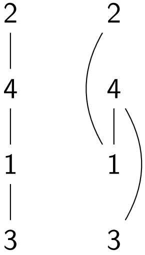
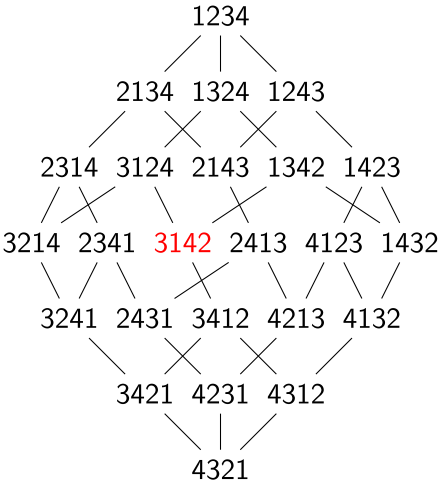
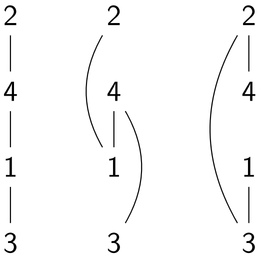




Right weak order





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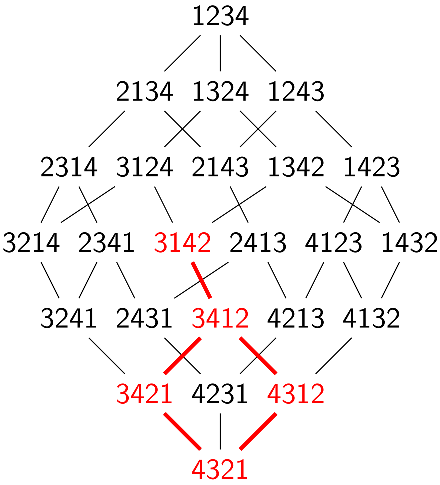
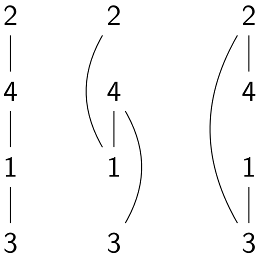
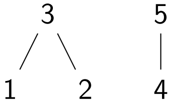


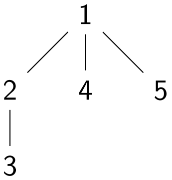


Avoids 132 ⇔ Increasing poset is a forest



45213

Avoids 312 ⇔ Decreasing poset is a forest



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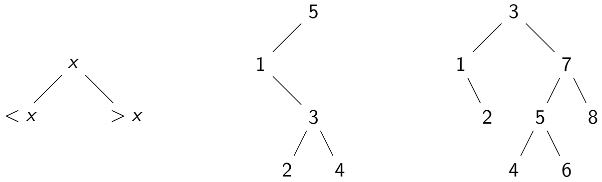
Link between the right weak order and the Tamari order

canonical binary search tree labelling

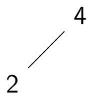


Binary search tree insertion

$$15324 \quad \rightarrow \quad 4$$

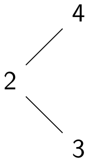
Binary search tree insertion

15324 $\rightarrow$



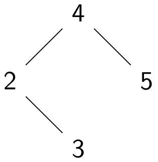
Binary search tree insertion

15324 $\rightarrow$



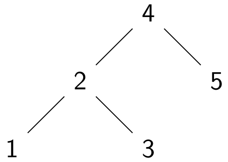
Binary search tree insertion

15324 $\rightarrow$

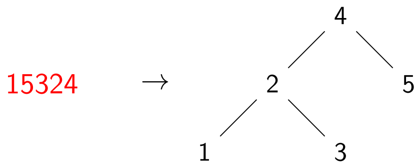


Binary search tree insertion

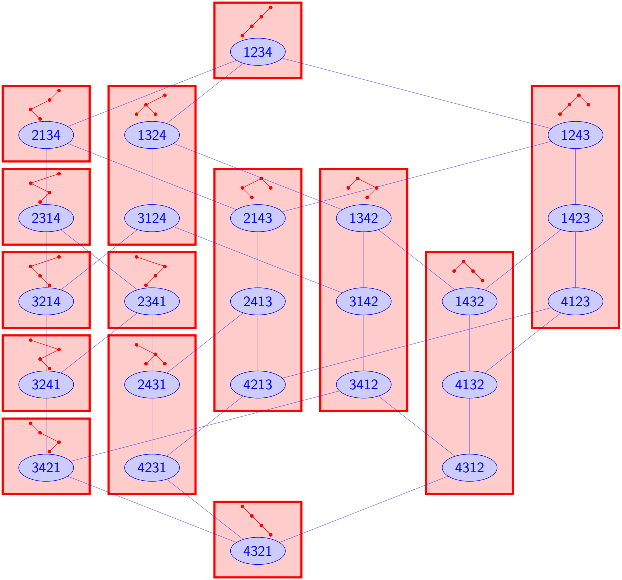
15324 $\rightarrow$



Binary search tree insertion



Characterization : the permutations sent to a given tree are its linear extensions
$15324, 31254, 35124, 51324, \ldots$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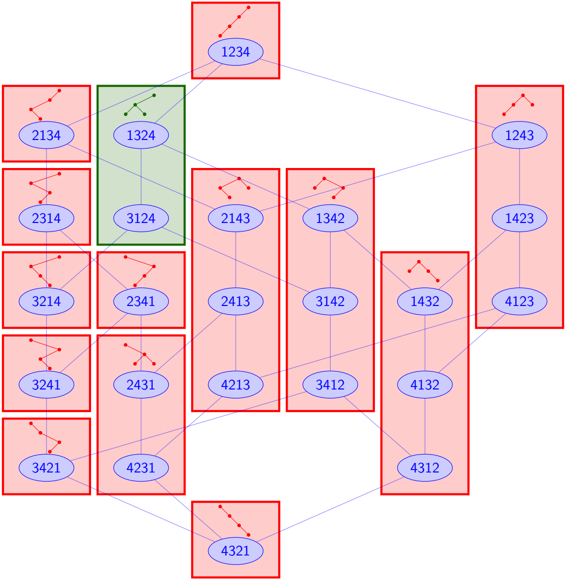
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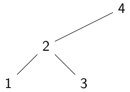
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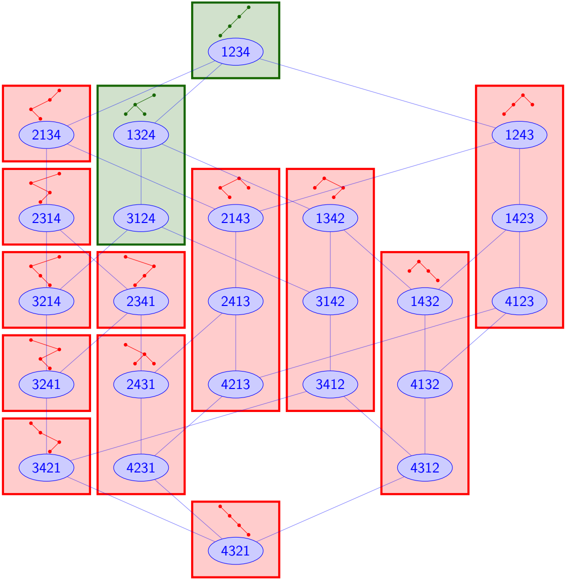
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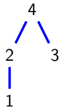
$F_{\leq}(T)$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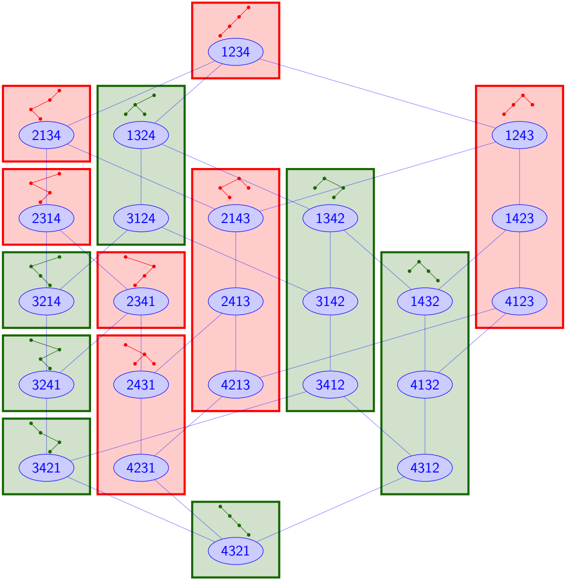
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$F_{\geq}(T)$


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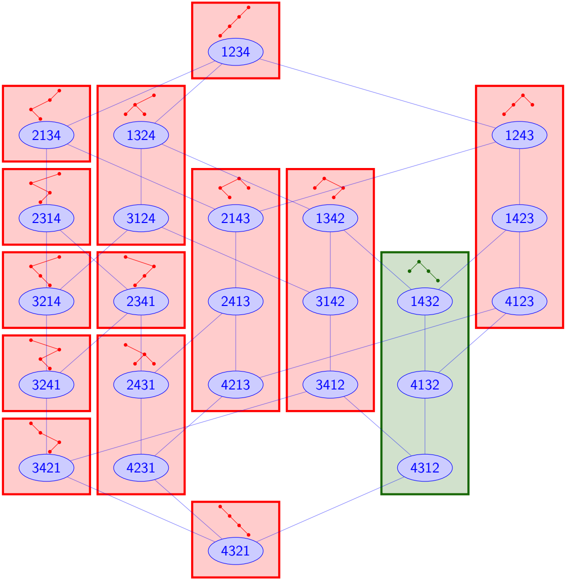
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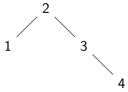
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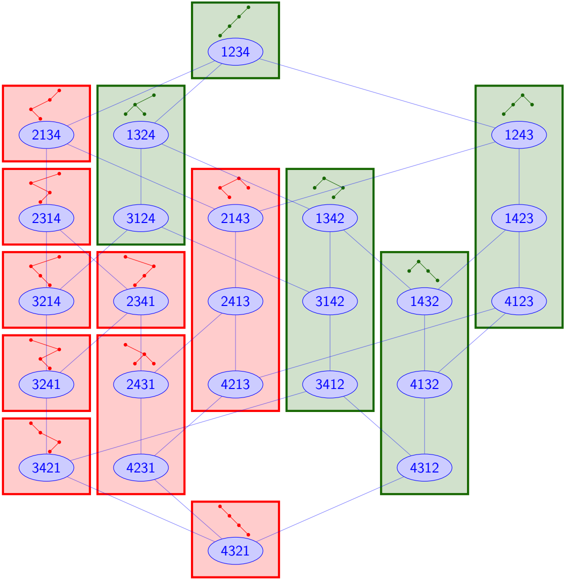
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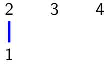
$F_{\leq}(T')$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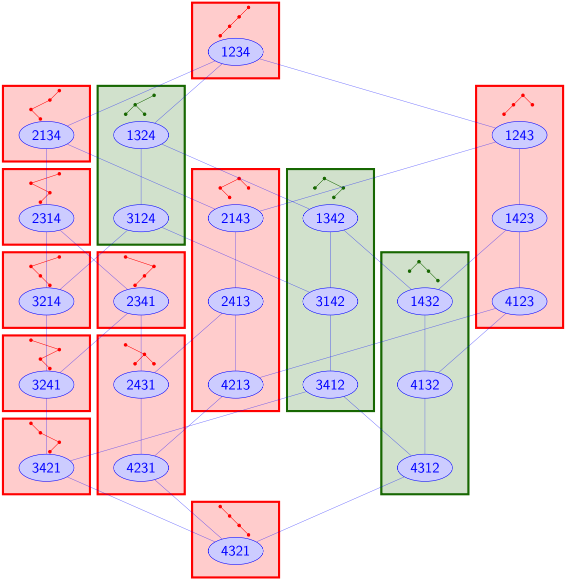
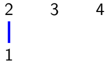


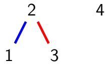
$F_{\geq}(T)$

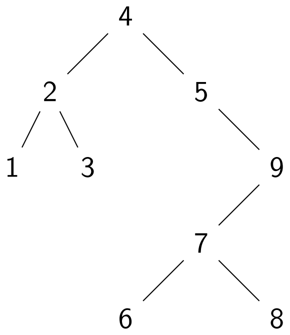
$F_{\leq}(T')$



Intervalle-poset

$[T, T']$

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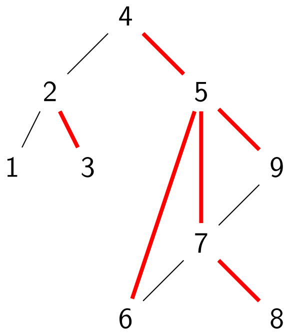
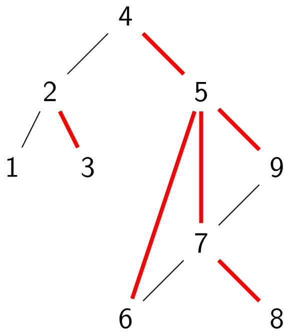
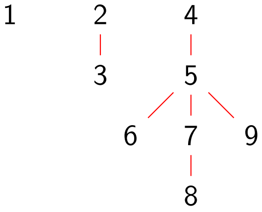
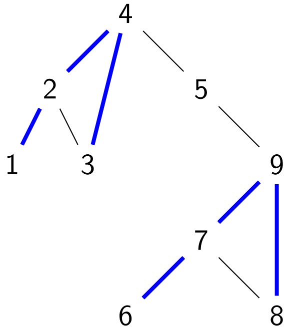
final forest $F_{\geq}(T)$



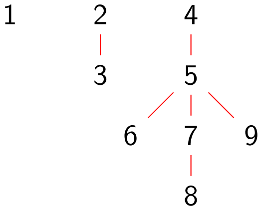


final forest $F_{\geq}(T)$

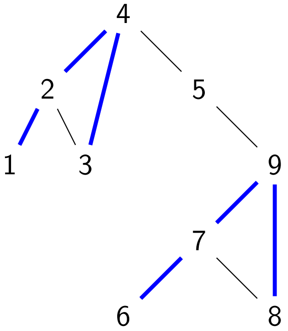




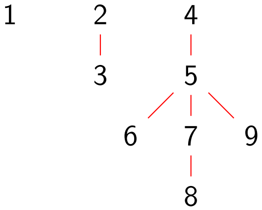
final forest $F_{\geq}(T)$



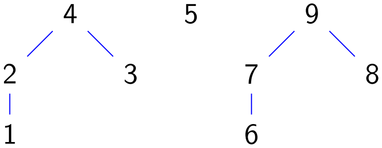
Initial forest $F_{\leq}(T)$

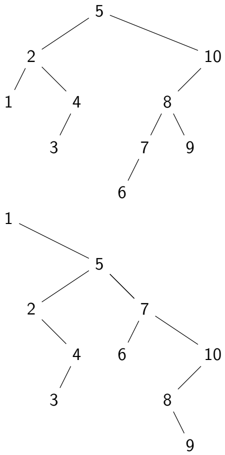


final forest $F_{\geq}(T)$



Initial forest $F_{\leq}(T)$

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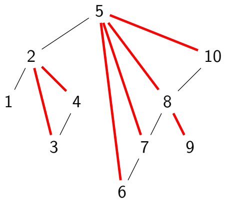
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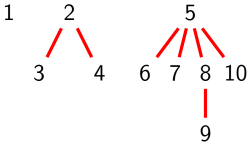
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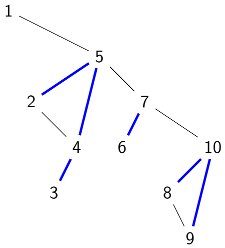
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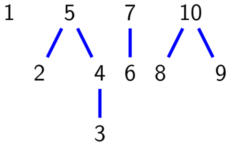
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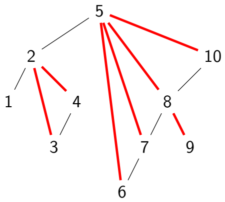
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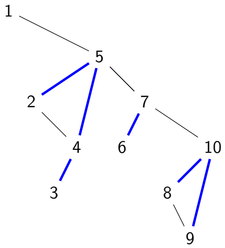
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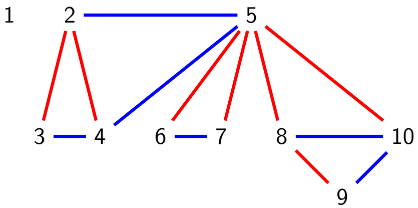
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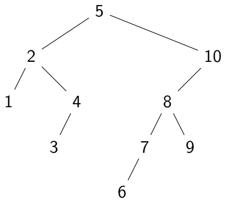
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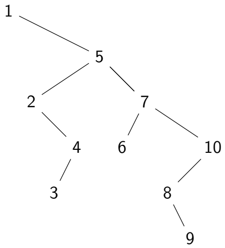
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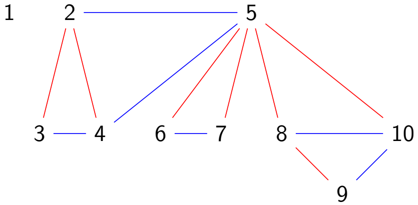
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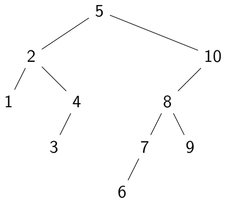
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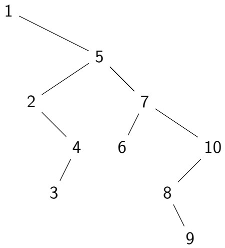
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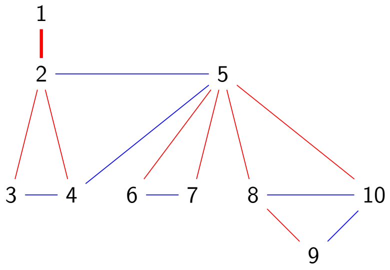
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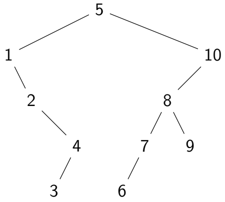
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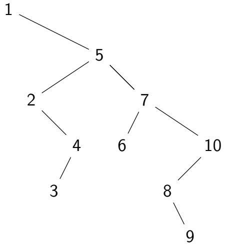
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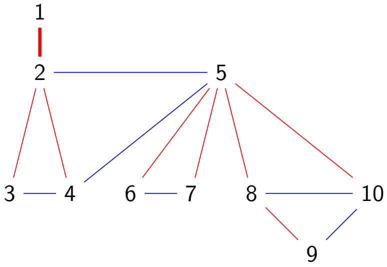
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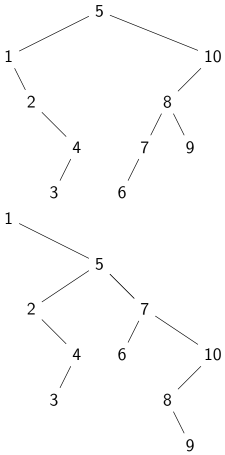
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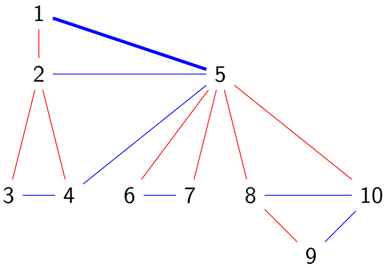
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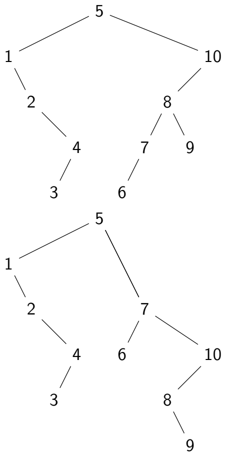
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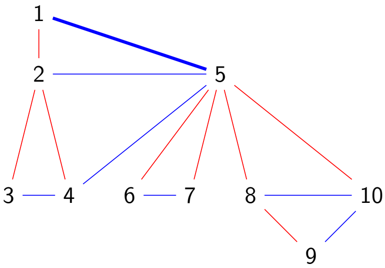
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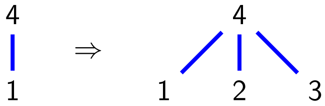
Theorem (Châtel, P.)

Intervals of Tamari are in bijection with labelled posets of size n and labels $1, \ldots, n$ such that

Theorem (Châtel, P.)

Intervals of Tamari are in bijection with labelled posets of size n and labels $1, \ldots, n$ such that

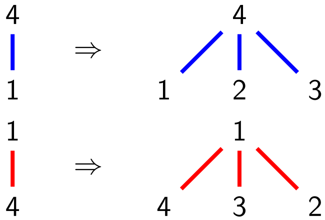
- *If $a < c$ and $a \lhd c$ then $b \lhd c$ for all $a < b < c$.*

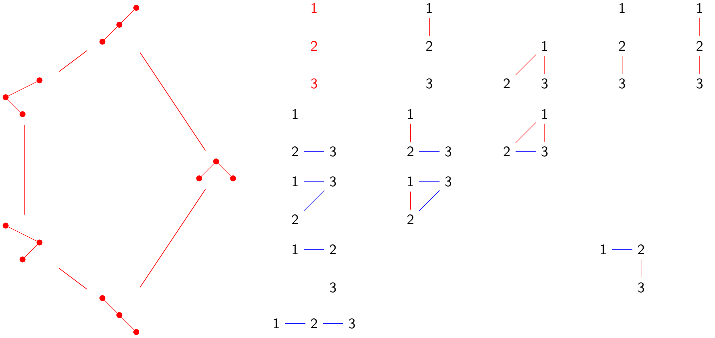


Theorem (Châtel, P.)

Intervals of Tamari are in bijection with labelled posets of size n and labels $1, \dots, n$ such that

- *If $a < c$ and $a \lhd c$ then $b \lhd c$ for all $a < b < c$.*
- *If $a < c$ and $c \lhd a$ then $b \lhd a$ for all $a < b < c$.*

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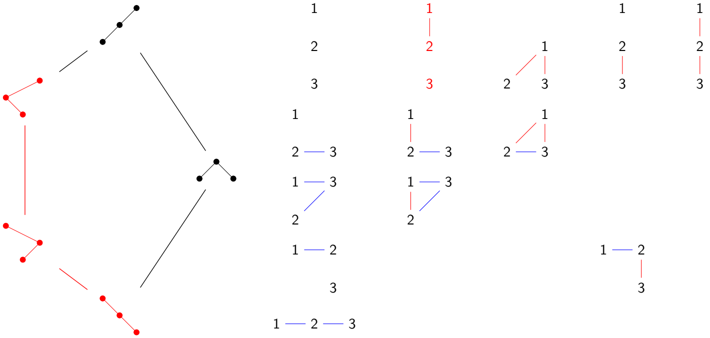

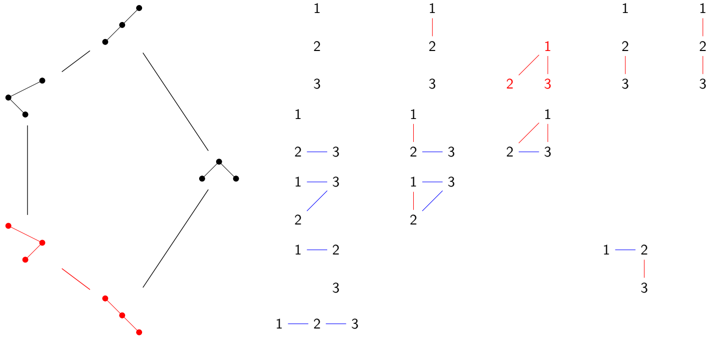

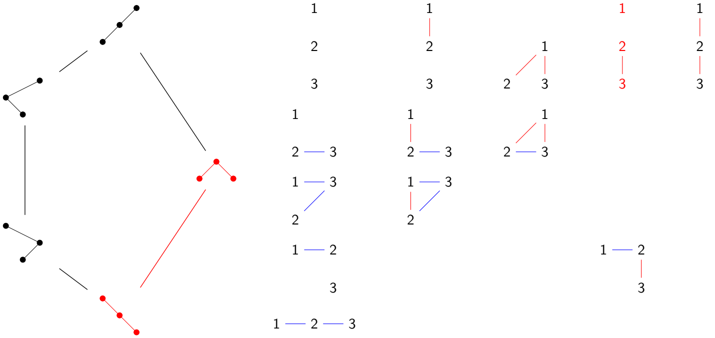

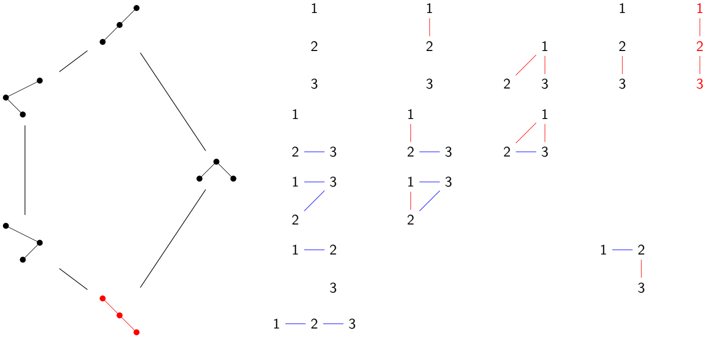

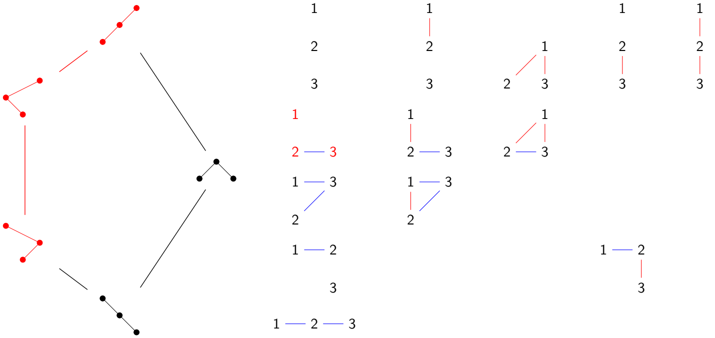

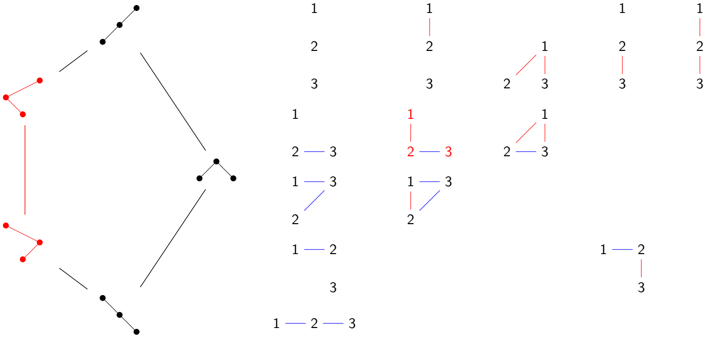

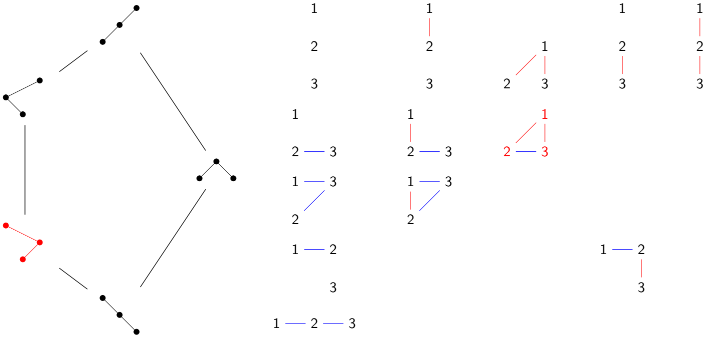
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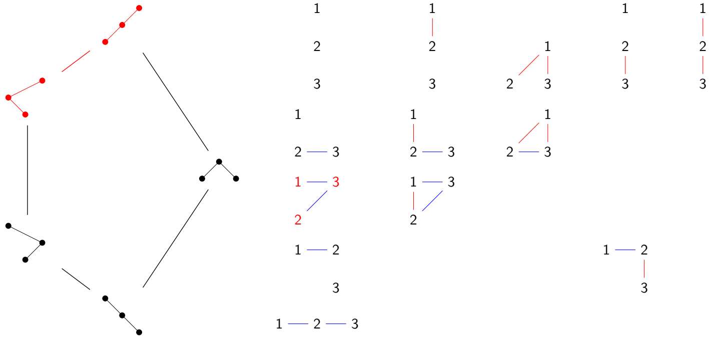

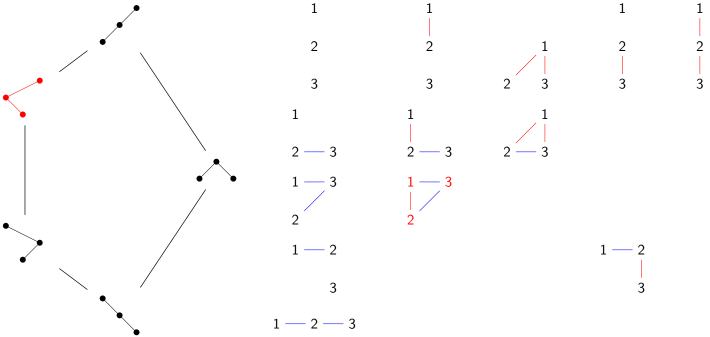

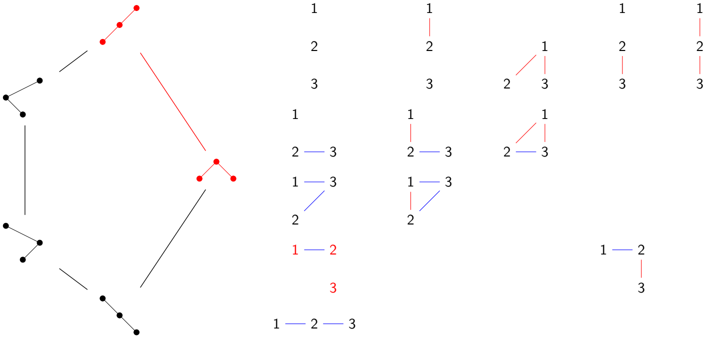
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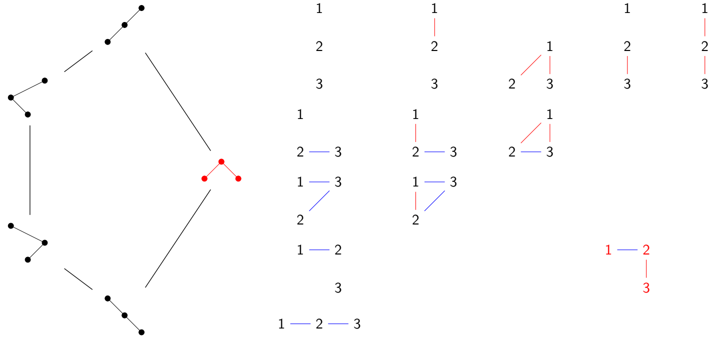
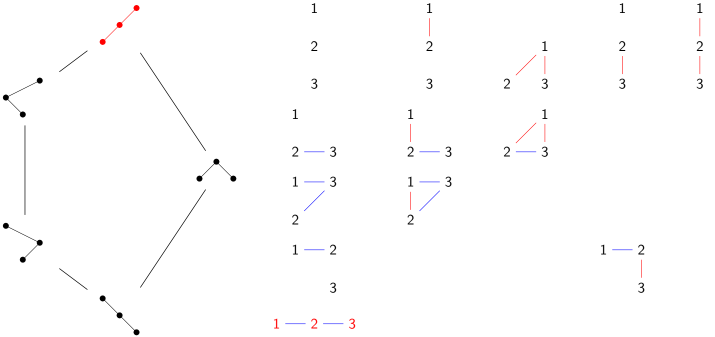
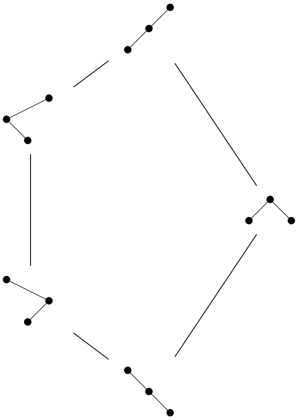




Number of intervals



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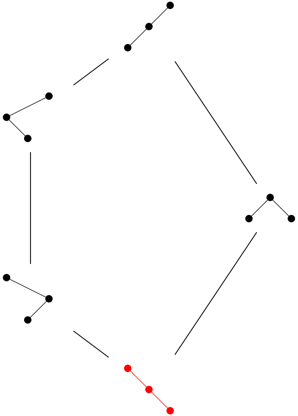

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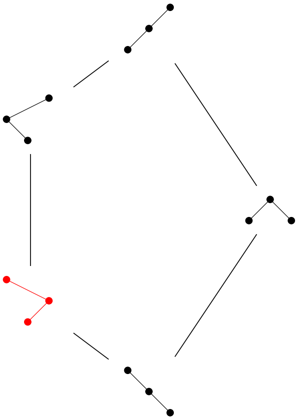
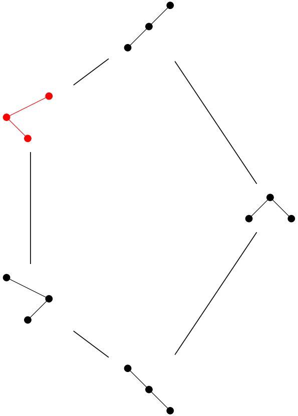
Number of intervals



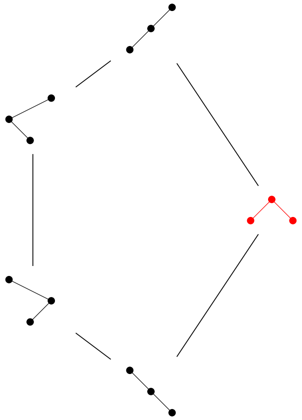
$5 + 3$

Number of intervals



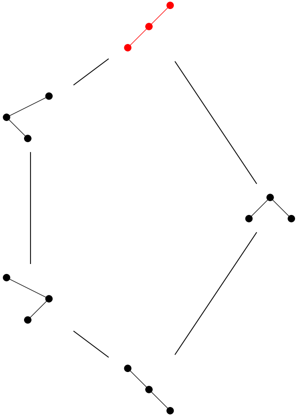
$5 + 3 + 2$

Number of intervals



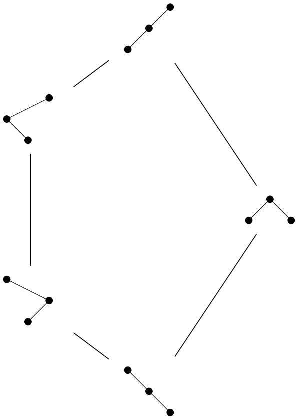
$$5 + 3 + 2 + 2$$

Number of intervals



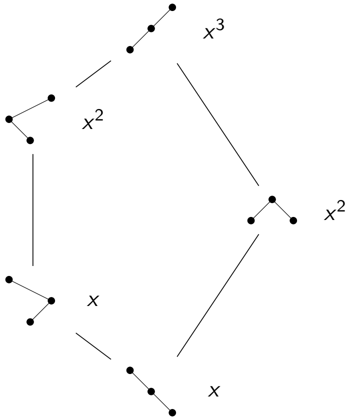
$$5 + 3 + 2 + 2 + 1$$

Number of intervals



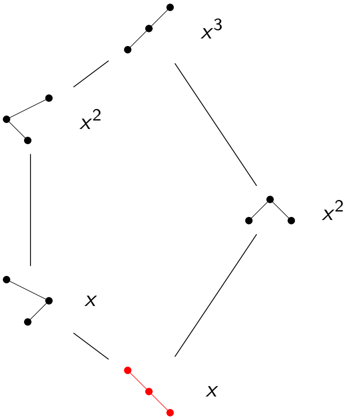
$$5 + 3 + 2 + 2 + 1 = 13$$

Number of intervals



$$5 + 3 + 2 + 2 + 1 = 13$$

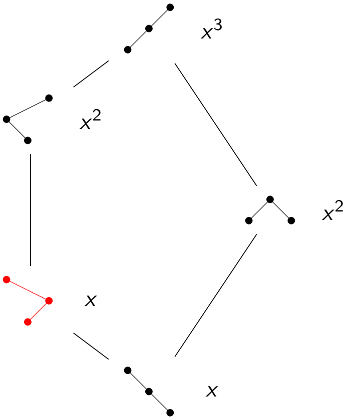
Number of intervals



$$5 + 3 + 2 + 2 + 1 = 13$$

$$(2x + 2x^2 + x^3)$$

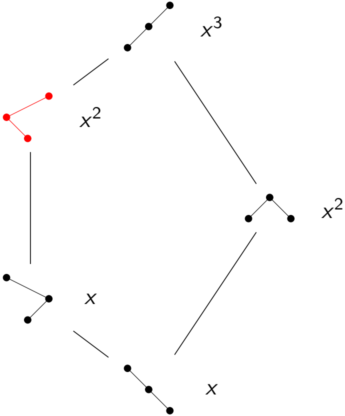
Number of intervals



$$5 + 3 + 2 + 2 + 1 = 13$$

$$(2x + 2x^2 + x^3)$$
$$+ (x + x^2 + x^3)$$

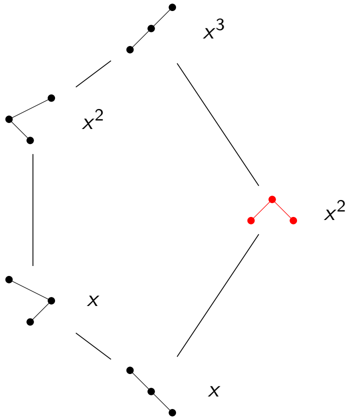
Number of intervals



$$5 + 3 + 2 + 2 + 1 = 13$$

$$(2x + 2x^2 + x^3) + (x + x^2 + x^3) + (x^2 + x^3)$$

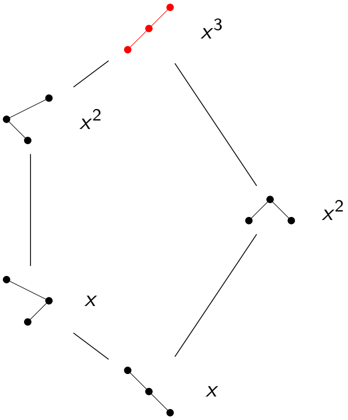
Number of intervals



$$5 + 3 + 2 + 2 + 1 = 13$$

$$(2x + 2x^2 + x^3)$$
$$+ (x + x^2 + x^3)$$
$$+ (x^2 + x^3)$$
$$+ (x^2 + x^3)$$

Number of intervals



$$5+3+2+2+1=13$$

$$(2x+2x^2+x^3)$$
$$+(x+x^2+x^3)$$
$$+(x^2+x^3)$$
$$+(x^2+x^3)$$
$$+x^3$$

Theorem (Chapoton)

The generating functions of Tamari intervals satisfy the functional equation

$$\Phi(x,y) = \mathrm{B}(\Phi,\Phi) + 1$$

where

$$\mathrm{B}(f,g) = xyf(x,y)\frac{xg(x,y) - g(1,y)}{x-1}$$

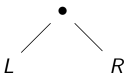
Tamari Polynomials

$\mathcal{B}_T$ is recursively defined by

$$\mathcal{B}_{\emptyset} := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

avec T =



Theorem (Châtel, P.)

$\mathcal{B}_T$ counts the number of trees smaller than or equal to T in the Tamari lattice according to the number of nodes on their left border.

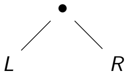
Tamari Polynomials

$\mathcal{B}_T$ is recursively defined by

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

avec T $=$



Theorem (Châtel, P.)

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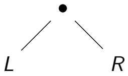
Tamari Polynomials

$\mathcal{B}_T$ is recursively defined by

$$\mathcal{B}_{\emptyset} := 1$$

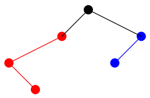
$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$

avec T =



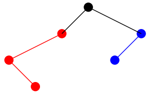
Theorem (Châtel, P.)

$\mathcal{B}_T$ counts the number of trees smaller than or equal to T in the Tamari lattice according to the number of nodes on their left border.



$$\mathcal{B}_\emptyset := 1$$

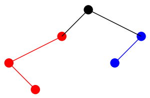
$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$



$$\mathcal{B}_{\emptyset} := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

$$\mathcal{B}_L(x) = x^3 + x^2$$

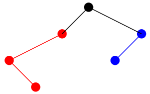


$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

$$\mathcal{B}_L(x) = x^3 + x^2$$

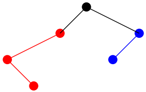
$$\mathcal{B}_R(x) = x^2$$



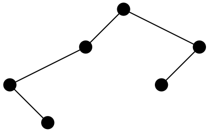
$$\mathcal{B}_\emptyset := 1$$

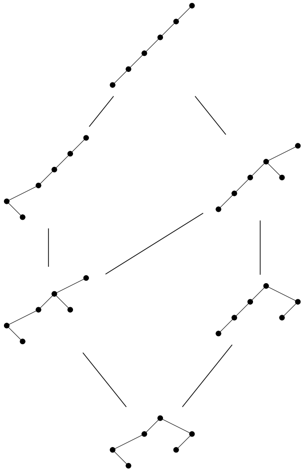
$$\mathcal{B}_T(x) := x(x^3 + x^2)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

$$\mathcal{B}_R(x) = x^2$$

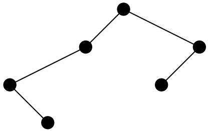


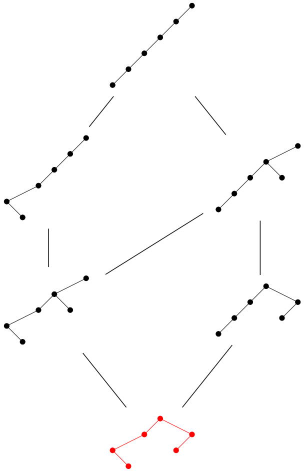
$$\mathcal{B}_\emptyset := 1$$
$$\mathcal{B}_T(x) := x(x^3 + x^2)(1 + x + x^2)$$



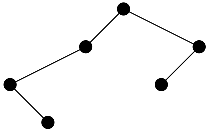


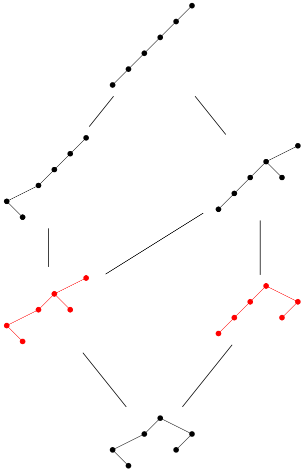
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



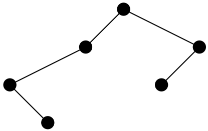


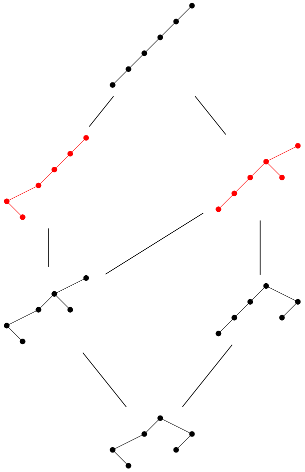
$$\mathcal{B}_T(x) = {\color{red}x^3} + 2x^4 + 2x^5 + x^6$$



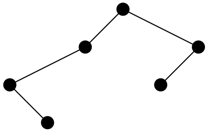


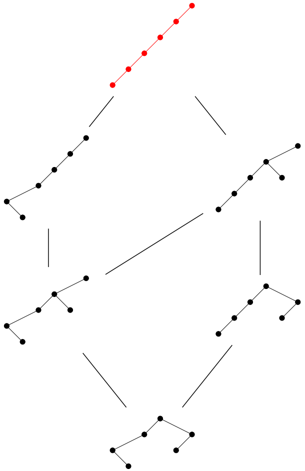
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



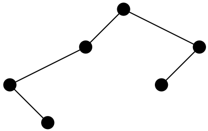


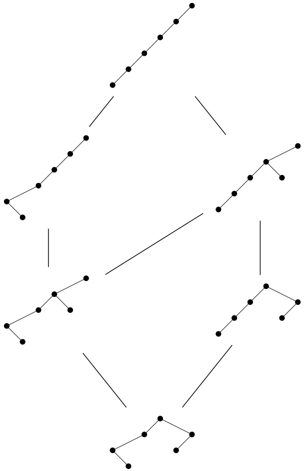
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



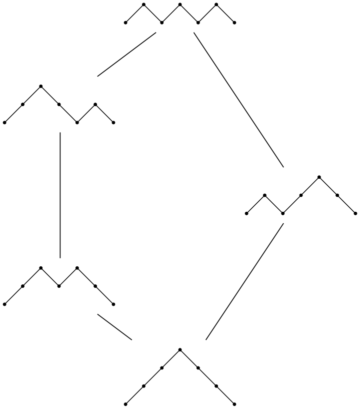


$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$

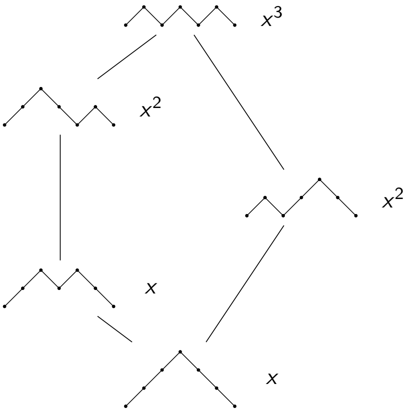




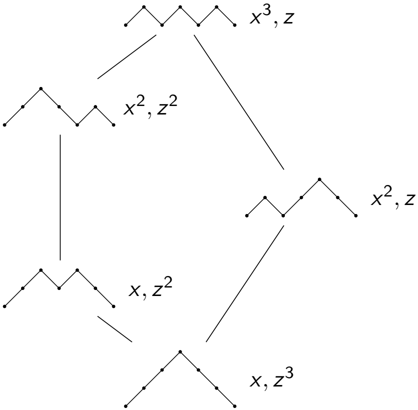
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$
$$\mathcal{B}_T(1) = 6$$



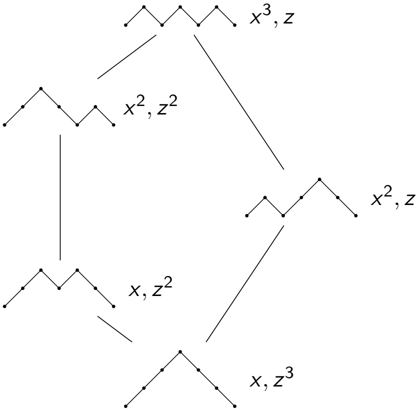
Symmetric statistic distribution



Symmetric statistic distribution

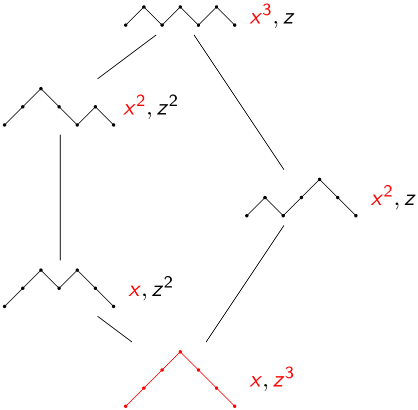


Symmetric statistic distribution



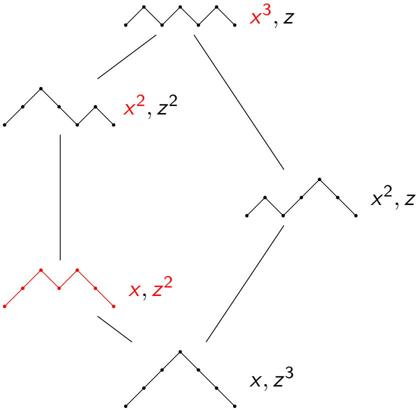
Symmetric statistic distribution

$$
\begin{aligned}
& (2x + 2x^2 + x^3) \\
+ \quad & (x + x^2 + x^3) \\
+ \quad & (x^2 + x^3) \\
+ \quad & (x^2 + x^3) \\
+ \quad & x^3
\end{aligned}
$$



Symmetric statistic distribution

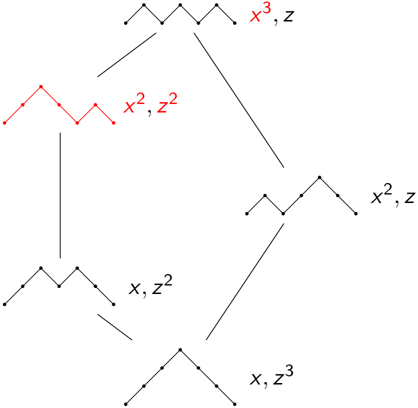
$$z^3(2x + 2x^2 + x^3)$$
$$+ \quad (x + x^2 + x^3)$$
$$+ \quad (x^2 + x^3)$$
$$+ \quad (x^2 + x^3)$$
$$+ \quad x^3$$



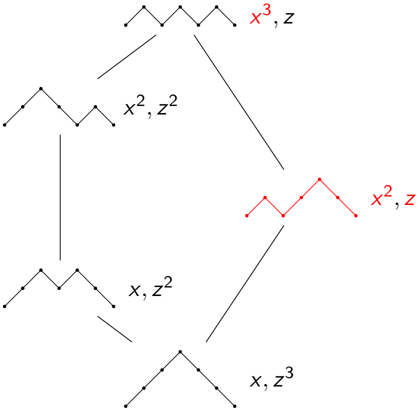
Symmetric statistic distribution

$$
\begin{aligned}
& z^3(2x + 2x^2 + x^3) \\
& + z^2(x + x^2 + x^3) \\
& + \quad (x^2 + x^3) \\
& + \ (x^2 + x^3) \\
& + \ x^3
\end{aligned}
$$

Symmetric statistic distribution

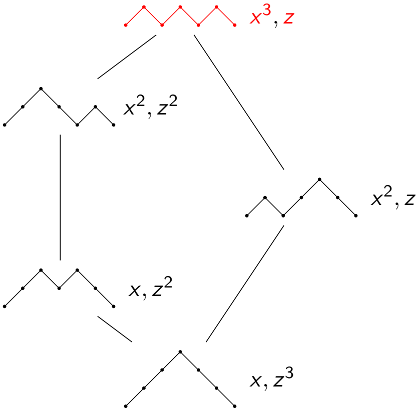


$$z^3(2x + 2x^2 + x^3)$$
$$+ z^2(x + x^2 + x^3)$$
$$+ z^2(x^2 + x^3)$$
$$+ \ (x^2 + x^3)$$
$$+ \ x^3$$



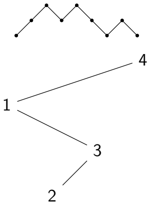
Symmetric statistic distribution

$$z^3(2x + 2x^2 + x^3) + z^2(x + x^2 + x^3) + z^2(x^2 + x^3) + z(x^2 + x^3) + x^3$$

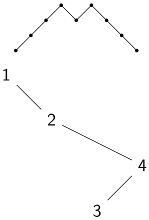


Symmetric statistic distribution

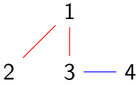
$$z^3(2x + 2x^2 + x^3)$$
$$+ z^2(x + x^2 + x^3)$$
$$+ z^2(x^2 + x^3)$$
$$+ z(x^2 + x^3)$$
$$+ zx^3$$

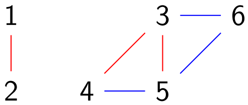


x^2



z^3

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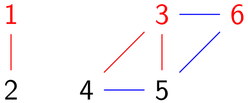
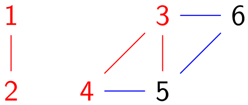
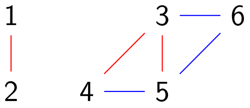


x^3

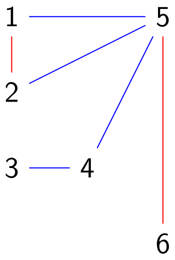


$x^3 z^4$

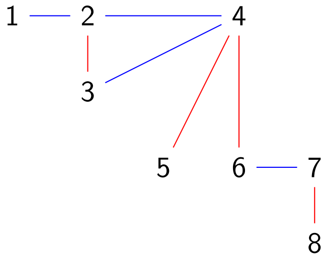


$x^3 z^4$

$\leftrightarrow$



$x^4 z^3$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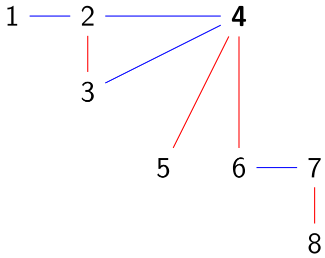
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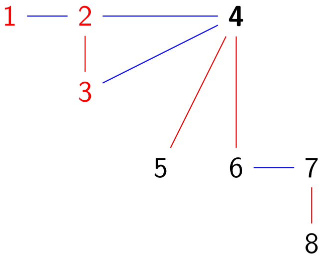
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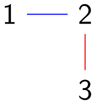
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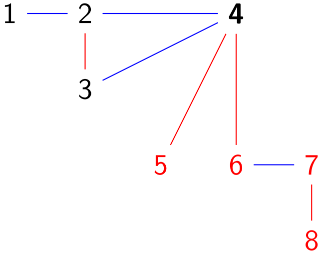
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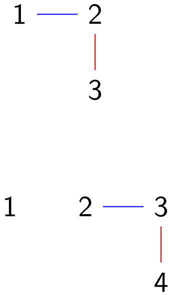
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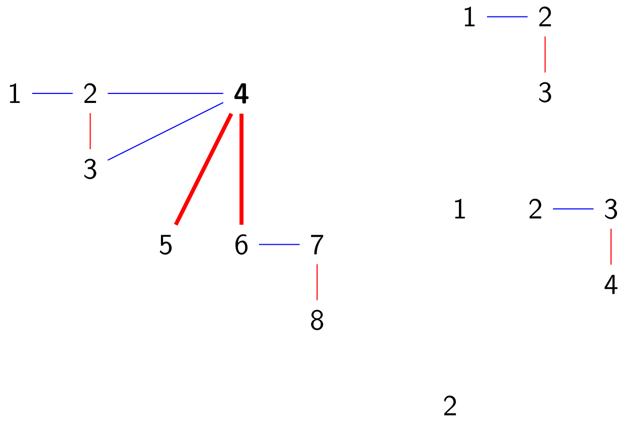
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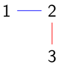
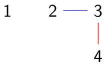
$\mathbb{B}($ 

, 

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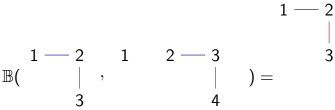
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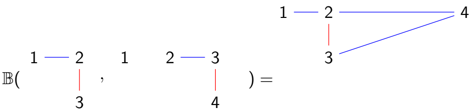
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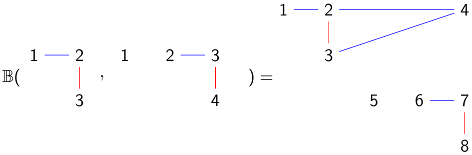
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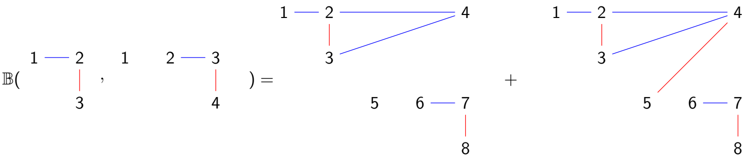
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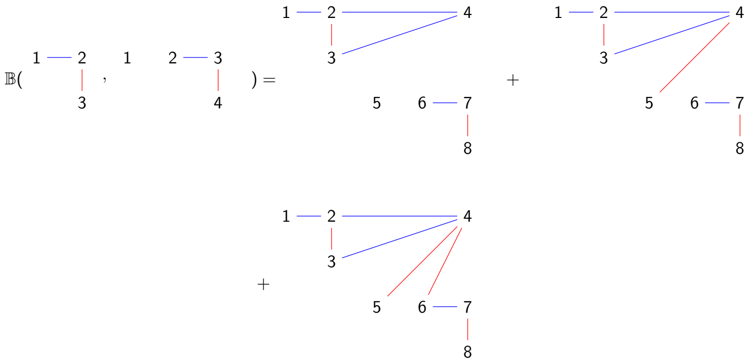
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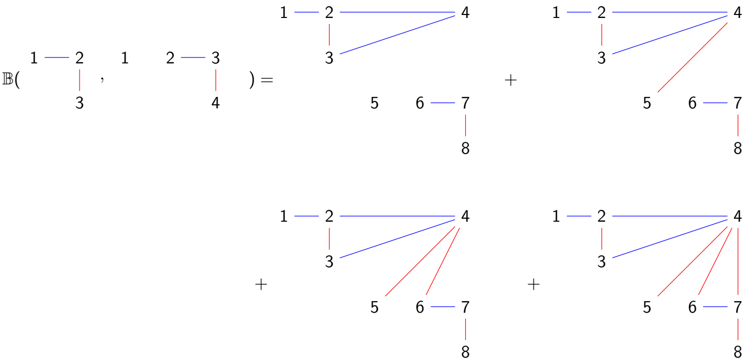

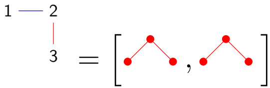
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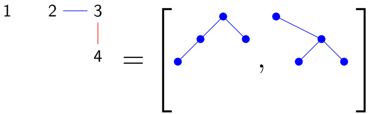
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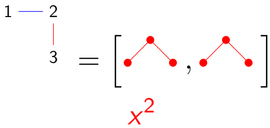
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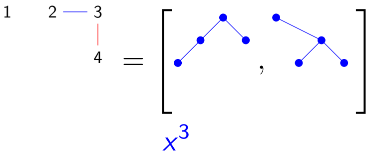
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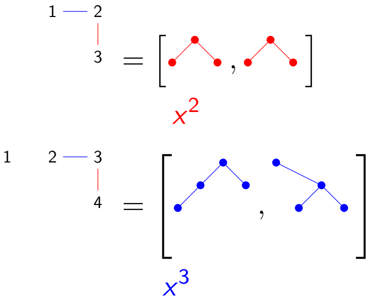
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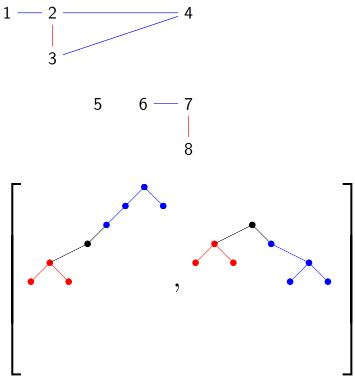
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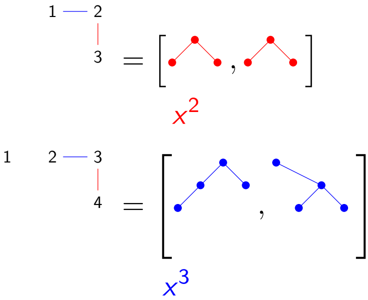
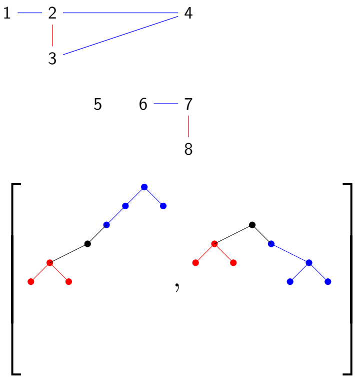
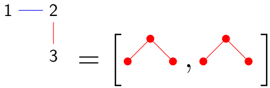
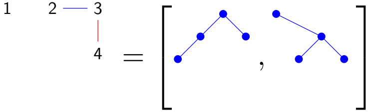




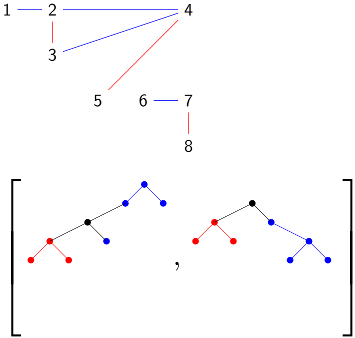
$$x^2.x.x^3$$



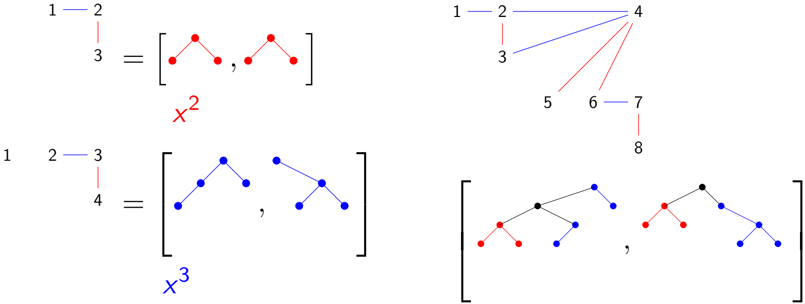
x^2



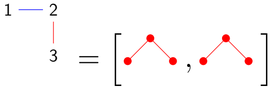
x^3



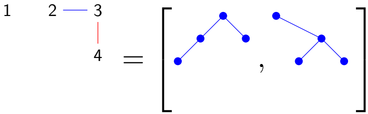
$$x^2.x.x^3 + x^2.x.x^2$$



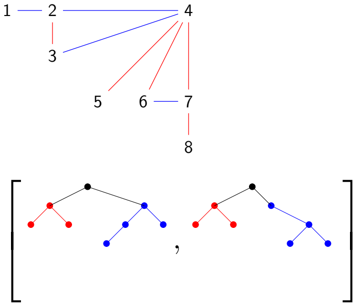
$$x^2.x.x^3 + x^2.x.x^2 + x^2.x.x$$



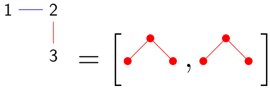
x^2



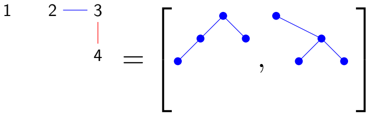
x^3



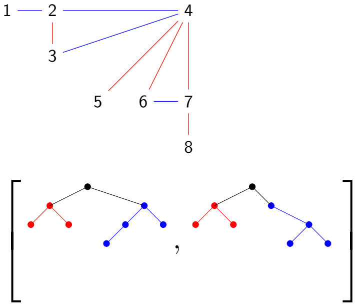
$$x^2.x.x^3 + x^2.x.x^2 + x^2.x.x + x^2.x$$



x^2



x^3

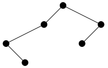


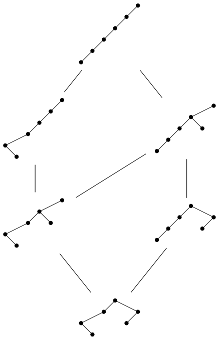
$$x^2.x.(x^3 + x^2 + x + 1)$$

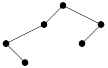
$$S_T := \sum_{T' \leq T} [T', T]$$

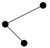
$$S_T = \mathbb{B}(S_L, S_R)$$

$$\rightarrow \mathcal{B}_T(x) = x\,\mathcal{B}_L(x)\frac{\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$









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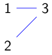
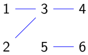


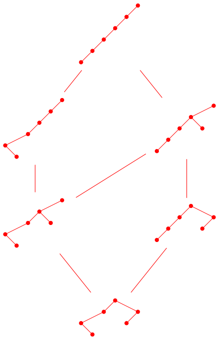
x^3

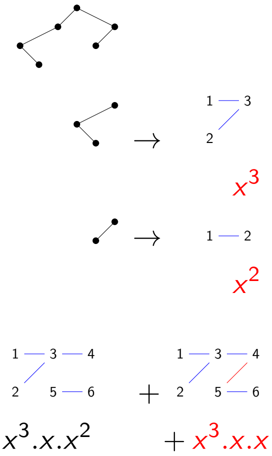
$\longrightarrow$

x^2



$x^3.x.x^2$

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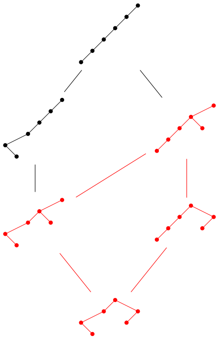
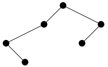
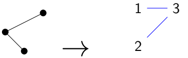
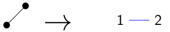
$x^3.x.x^2$
$+ x^3.x.x$



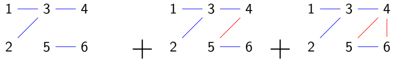




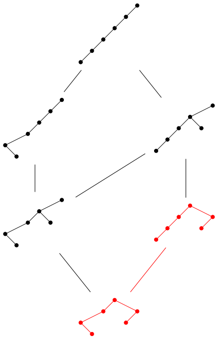
x^3

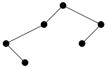


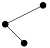
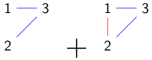
x^2



$x^3.x.x^2 \quad + x^3.x.x + x^3.x$



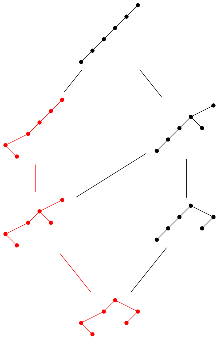


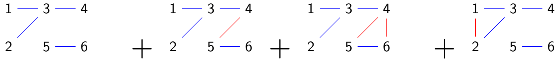
 $\longrightarrow$ 

$$x^3 + {\color{red}x^2}$$

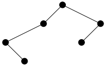
 $\longrightarrow$ 

$${\color{red}x^2}$$

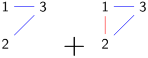




$$x^3.x.x^2 \quad + x^3.x.x + x^3.x \quad + {\color{red}x^2.x.x^2}$$



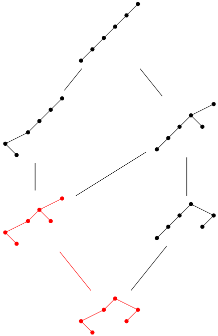
$\longrightarrow$

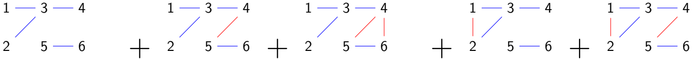


$$x^3 + x^2$$

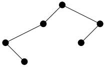
$\longrightarrow$

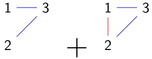
$$x^2$$





$$x^3.x.x^2 \quad + x^3.x.x + x^3.x \quad + x^2.x.x^2 + x^2.x.x$$

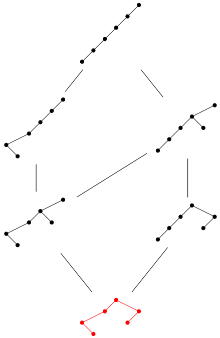


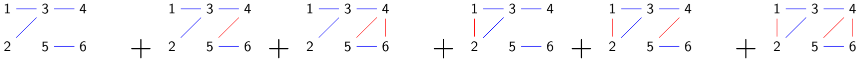

$\rightarrow$


$$x^3 + x^2$$

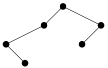

$\rightarrow$

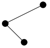
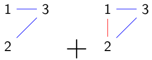

$$x^2$$





$$x^3.x.x^2 \quad + x^3.x.x + x^3.x \quad + x^2.x.x^2 + x^2.x.x \quad + x^2.x$$

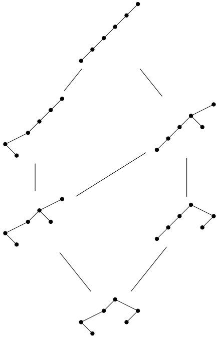


 $\rightarrow$ 

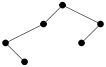
$$x^3 + x^2$$

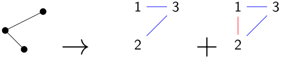
 $\rightarrow$ 

$$x^2$$



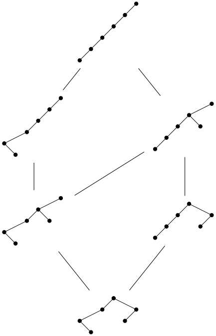
$$(x^3 + x^2).x.(x^2 + x + 1) =$$





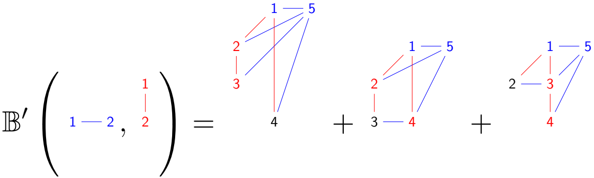
$$x^3 + x^2$$

$$x^2$$

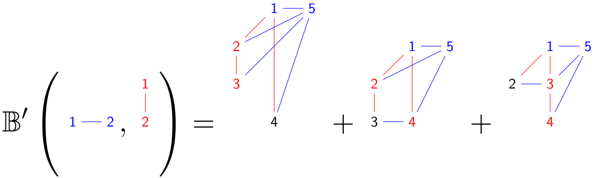


$$(x^3 + x^2).x.(x^2 + x + 1) = x\,\mathcal{B}_L(x)\frac{\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$

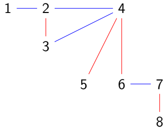
A second composition



A second composition



$$B'(z, z^2) = z^4 + z^3 + z^2$$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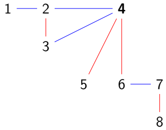
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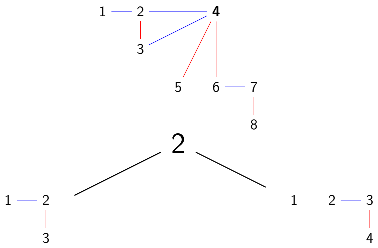
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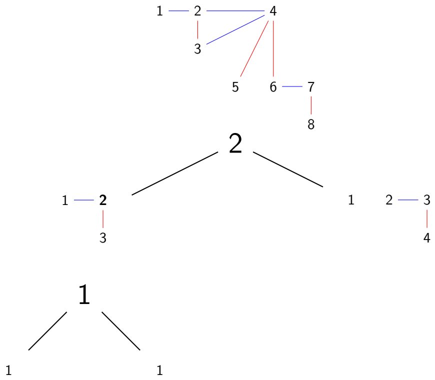
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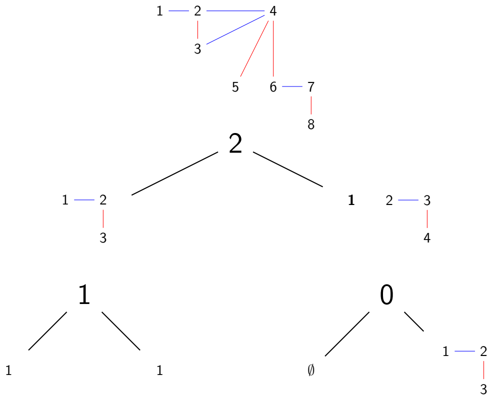
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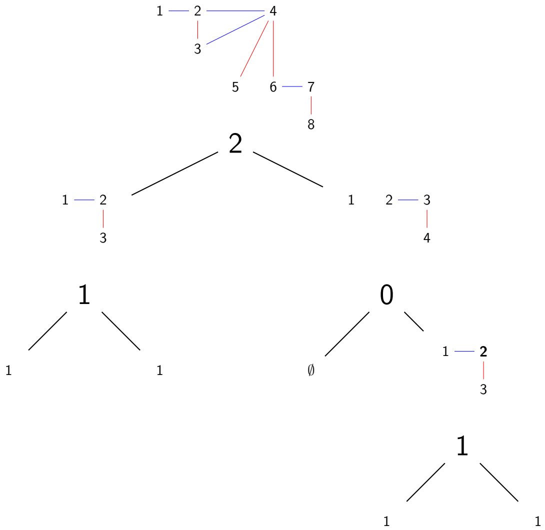
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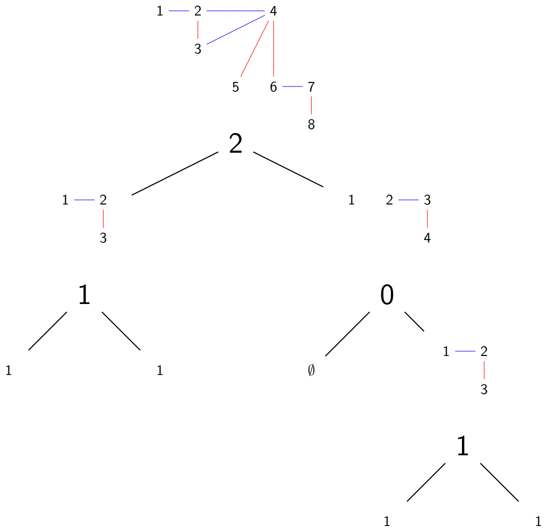
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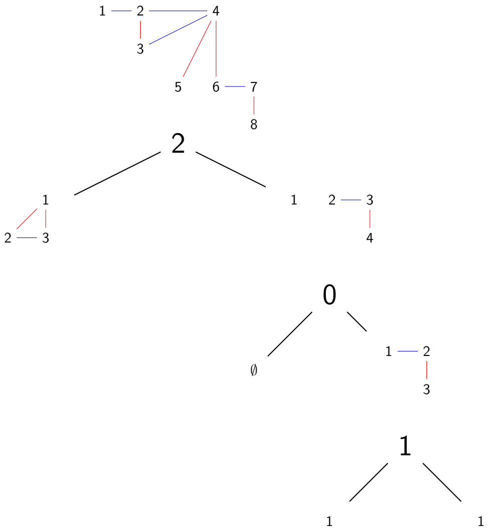
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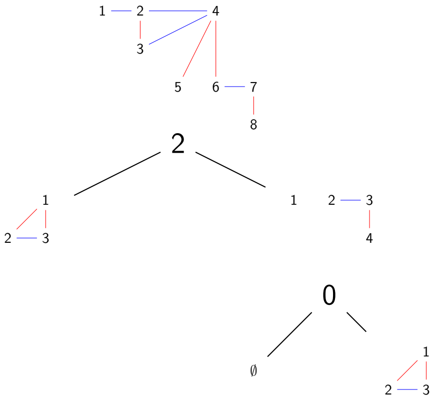
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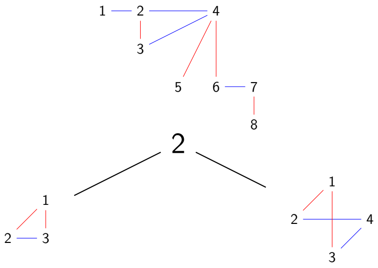
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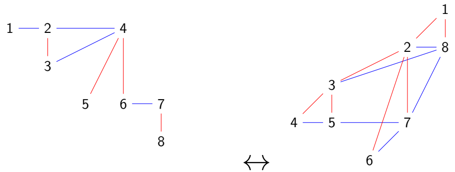
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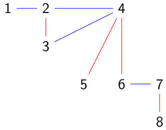

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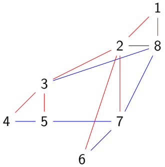


$x^4 z$ $\quad$ xz^4

m-Tamari posets

(2011-2012) Bergeron, Préville-Ratelle, *Higher trivariate diagonal harmonics via generalized Tamari posets.*

m-Tamari posets

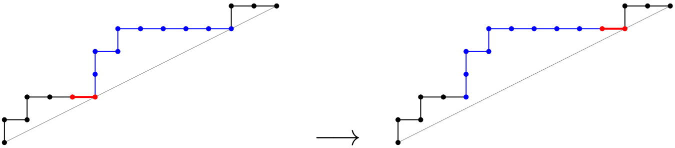
(2011-2012) Bergeron, Préville-Ratelle, *Higher trivariate diagonal harmonics via generalized Tamari posets.*

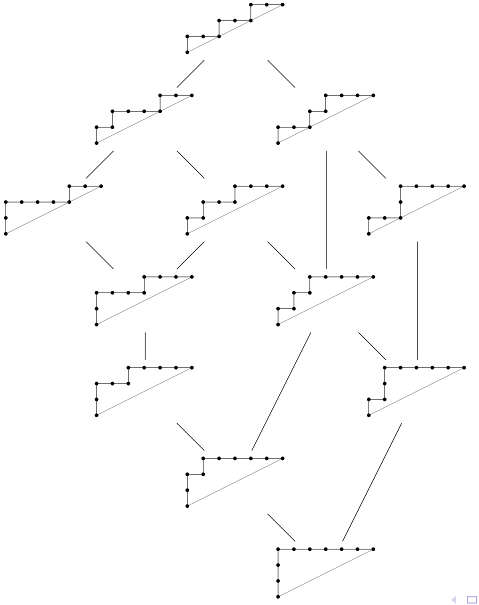
Structure de treillis, intervalles

(2011) Bousquet-Mélou, Fusy, Préville-Ratelle, *The number of intervals in the m-Tamari lattices.*

m-ballots paths

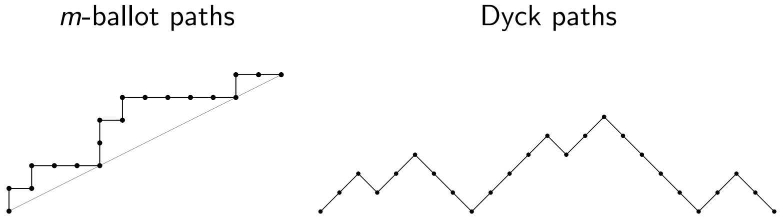
Example $m = 2$.

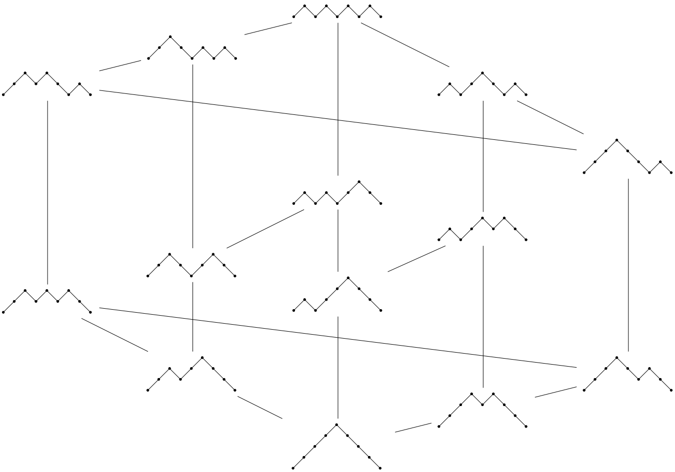


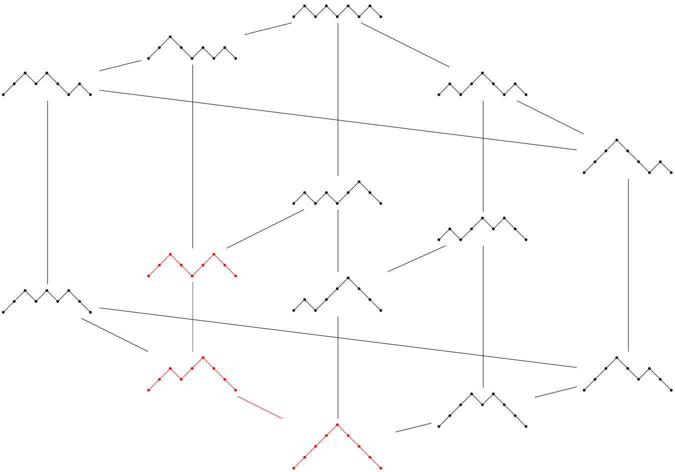


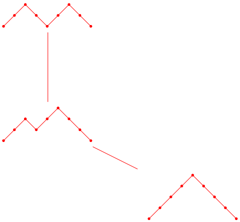
m-Tamari posets are lattices

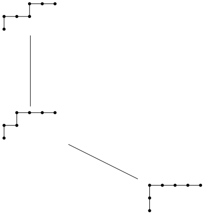
Bousquet-Mélou, Fusy, Préville-Ratelle : *m*-Tamari posets are ideals of the Tamari lattice $n \times m$.



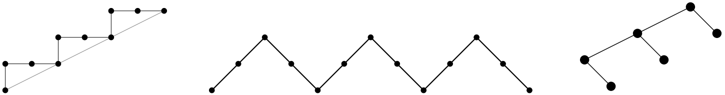


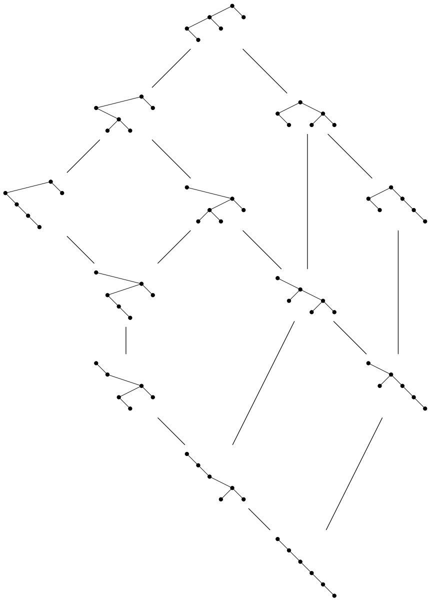




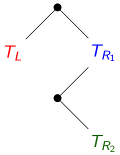


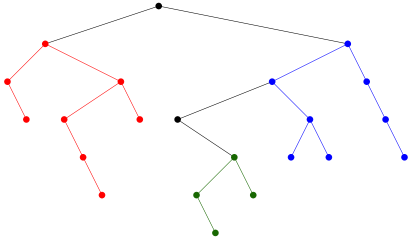
m-binary trees

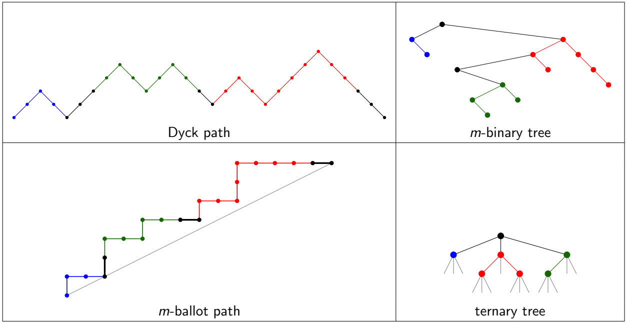




Ternary structure






Dyck path
m-binary tree
m-ballot path
ternary tree

Tamari lattice
(Chapoton)

$$\Phi = 1 + \mathrm{B}(\Phi, \Phi)$$

$$\mathrm{B}(f, g) = xf(x)\Delta(g)$$

$$\Delta(g) = \frac{xg(x) - g(1)}{x - 1}$$

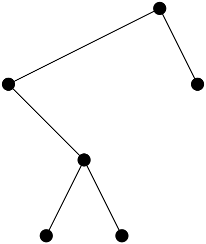
m-Tamari lattices

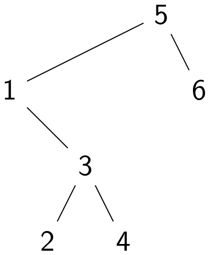
(Bousquet-Mélou, Fusy, Préville-Ratelle)

$$\Phi^{(2)} = 1 + \mathrm{B}^{(2)}(\Phi^{(2)}, \Phi^{(2)}, \Phi^{(2)})$$

$$\mathrm{B}^{(2)}(f, g_1, g_2) = xf(x)\Delta(g_1\Delta(g_2))$$

$$\Delta(g) = \frac{xg(x) - g(1)}{x - 1}$$

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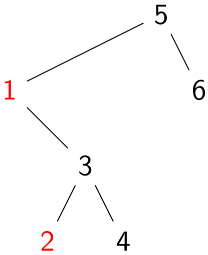
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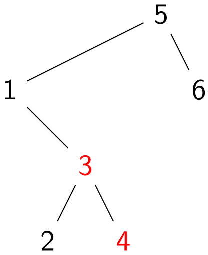
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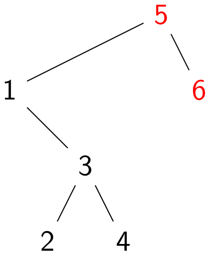
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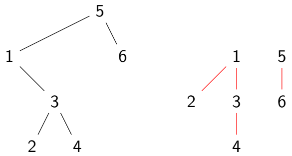
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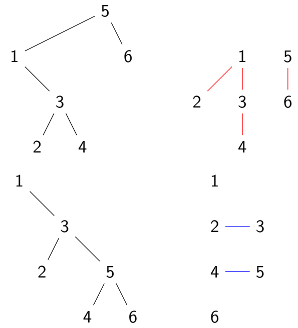
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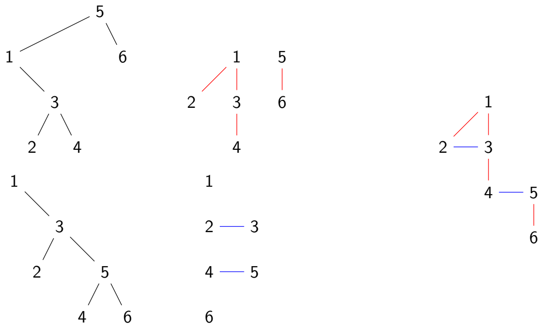
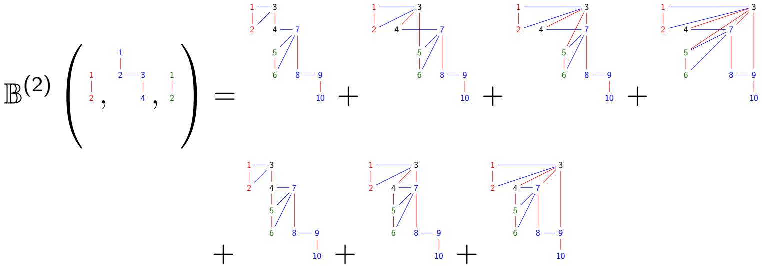


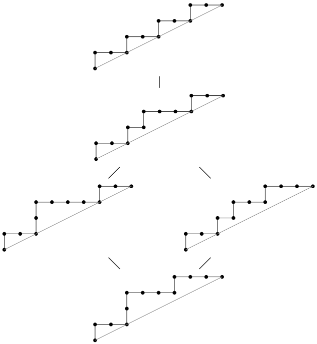


Theorem (Châtel, P.)

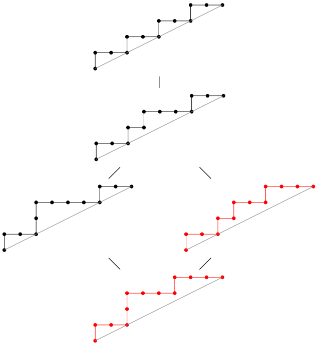
Let T be an element of m-Tamari composed from $L, R_1, \ldots, R_m$. $\mathcal{B}_T^{(m)}$ is defined recursively by

$$\mathcal{B}_\emptyset^{(m)} := 1$$
$$\mathcal{B}_T^{(m)}(x) := \mathrm{B}^{(\mathrm{m})}(L, R_1, \ldots, R_m)$$

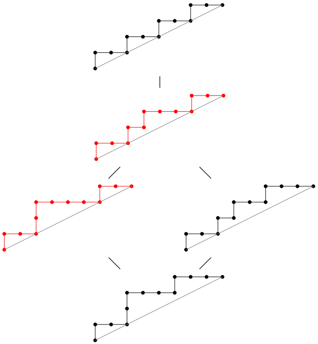
Then $\mathcal{B}_T^{(m)}$ counts the number of elements smaller than or equal to T in the m-Tamari lattice.



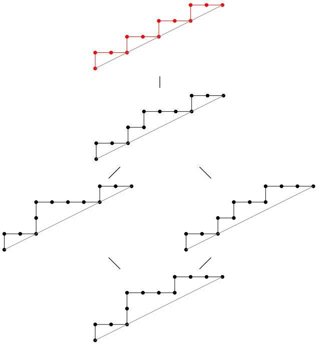
$$\mathcal{B}_T^{(2)} = \mathrm{B}^{(2)}(x,x,x)$$
$$= x^2\Delta(x\Delta(x))$$
$$= x^2\Delta(x(1+x))$$
$$= x^2(2+2x+x^2)$$
$$= 2x^2 + 2x^3 + x^4$$



$$
\begin{aligned}
\mathcal{B}_T^{(2)} &= \mathrm{B}^{(2)}(x,x,x) \\
&= x^2\Delta(x\Delta(x)) \\
&= x^2\Delta(x(1+x)) \\
&= x^2(2+2x+x^2) \\
&= {\color{red}2x^2} + 2x^3 + x^4
\end{aligned}
$$



$$\begin{aligned}\mathcal{B}_T^{(2)} &= \mathrm{B}^{(2)}(x,x,x) \\ &= x^2\Delta(x\Delta(x)) \\ &= x^2\Delta(x(1+x)) \\ &= x^2(2+2x+x^2) \\ &= 2x^2 + 2x^3 + x^4\end{aligned}$$



$$\mathcal{B}_T^{(2)} = \mathrm{B}^{(2)}(x,x,x) = x^2\Delta(x\Delta(x)) = x^2\Delta(x(1+x)) = x^2(2+2x+x^2) = 2x^2+2x^3+x^4$$