# Intervals of the Tamari lattice

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# Tamari lattice

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- 1962, Tamari : poset of formal bracketing
- 1972, Huang, Tamari : lattice structure
- 2007, Chapoton : number of intervals

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$$

## $m$-Tamari lattices

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- Bergeron, Préville-Ratelle : $m$-Tamari posets
- Bousquet-Mélou, Fusy, Préville-Ratelle : lattice structure and number of intervals

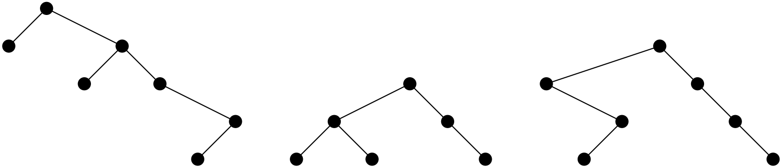
$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n + m}{n-1}$$

## Binary trees

Recursive definition :

- the empty tree or
- a left subtree and a right subtree grafted to a root node

## Examples



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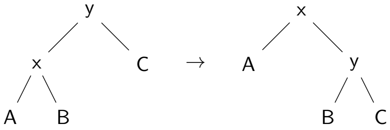


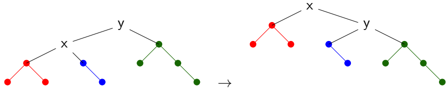

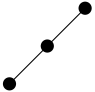

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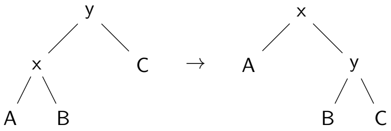


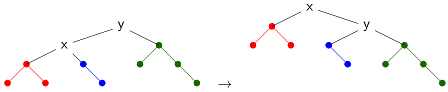

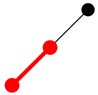

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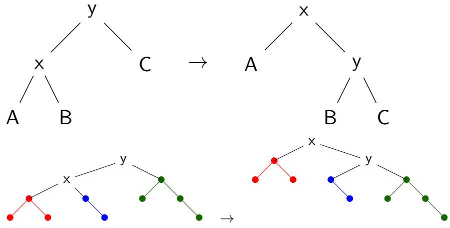

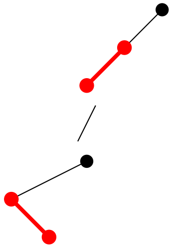

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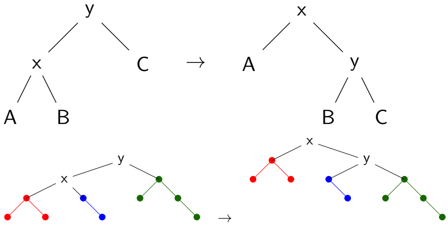

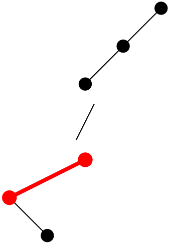

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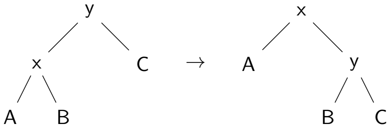


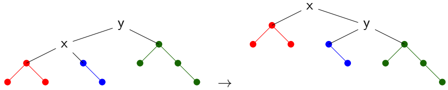

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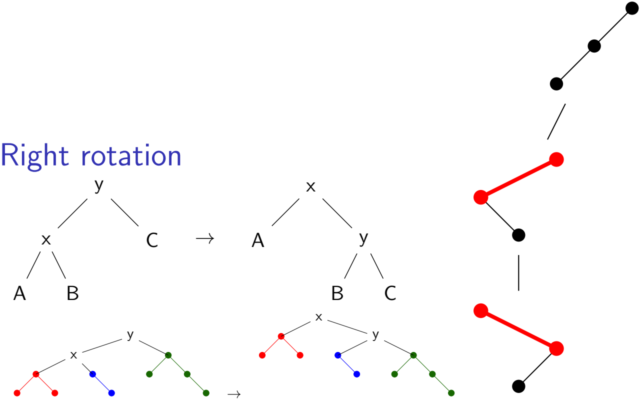

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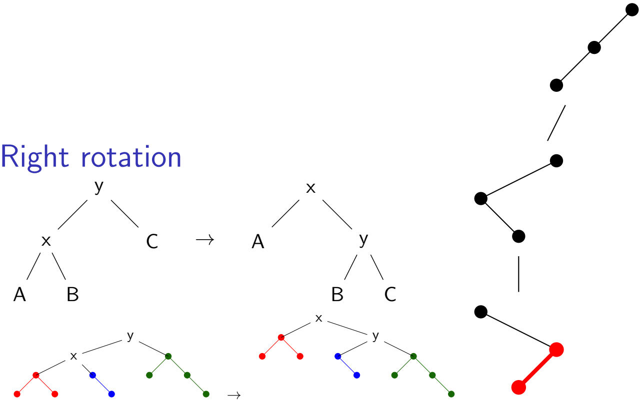

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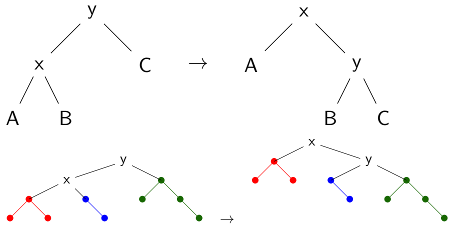


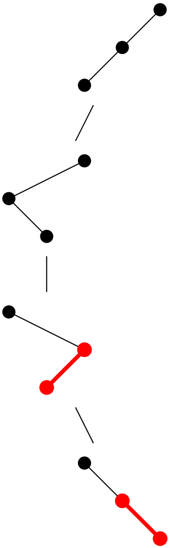

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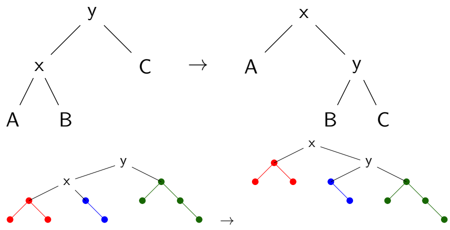


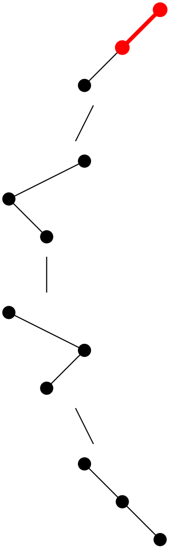

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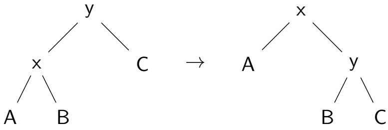


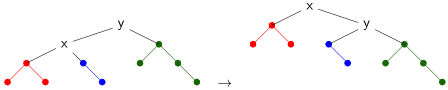


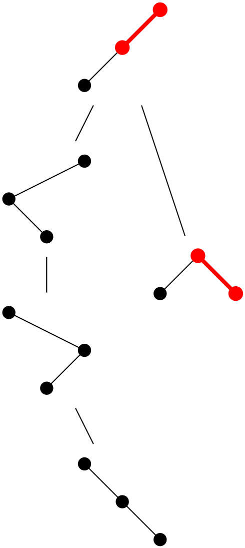

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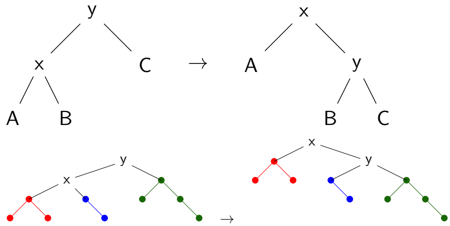


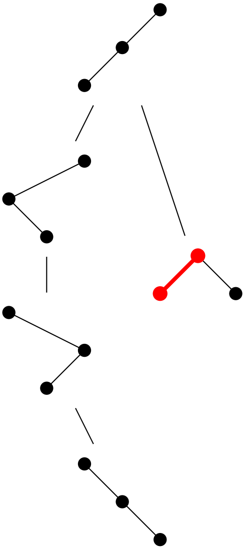

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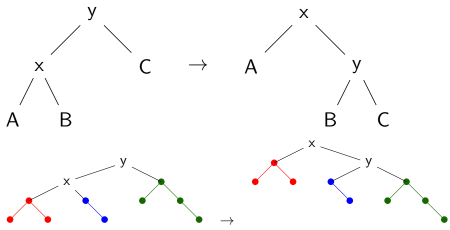


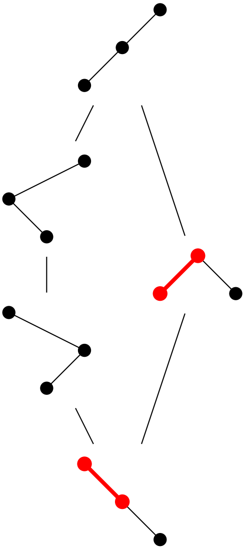

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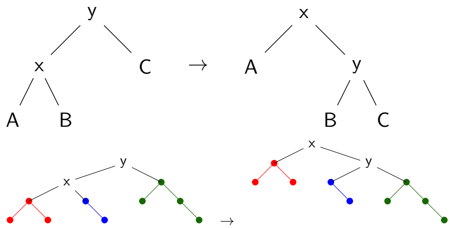
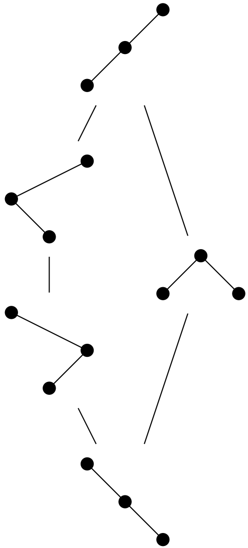
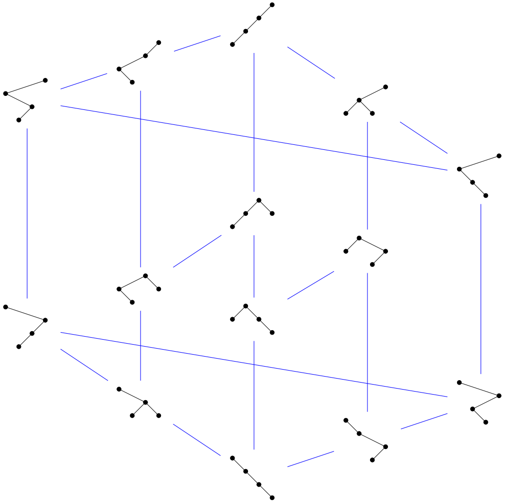
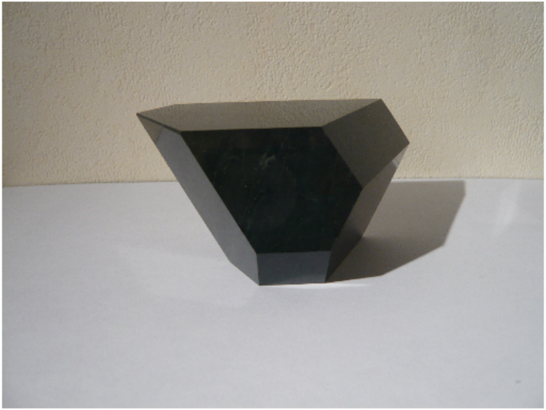
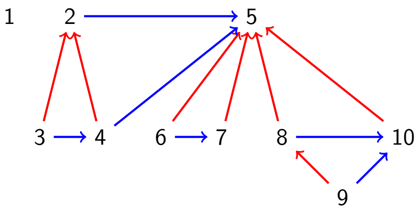




# Associahedron



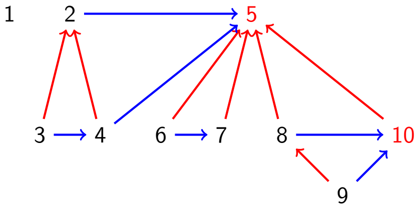



## Definition

An interval-poset is a poset of size $n$, labelled with $1, \ldots, n$ such that

- if $a < c$ and $c \lhd a$ then $b \lhd a$ for all $a < b < c$,
- if $a < c$ and $a \lhd c$ then $b \lhd c$ for all $a < b < c$.

We write $a \lhd b$ for $a$ lower than $b$ in the poset.

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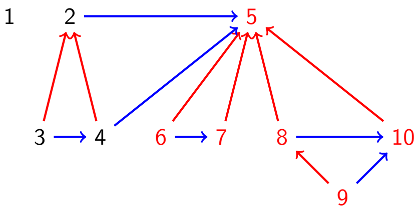


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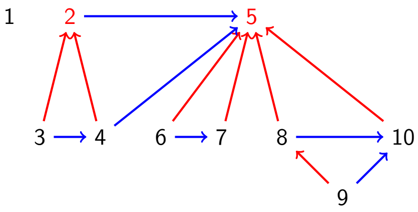


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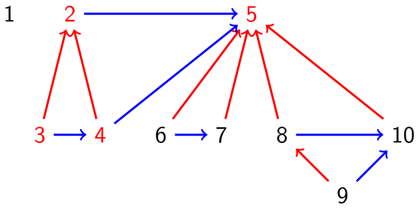


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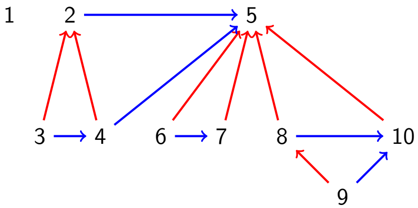
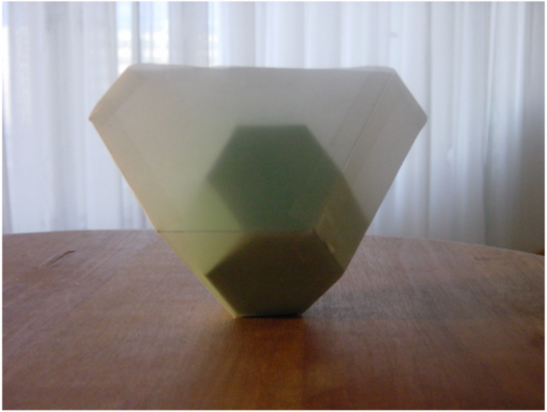



## Theorem (Châtel, P.)

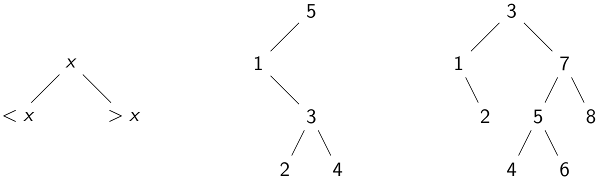
*Interval-posets are in bijections with intervals of the Tamari lattice.*

## Link with the weak order



(image from Jean-Louis Loday)

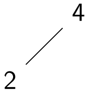
## Binary search tree



## Binary search tree insertion

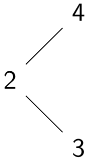
$$15324 \quad \rightarrow \quad 4$$

## Binary search tree insertion

15324 $\rightarrow$ 

# Binary search tree insertion

15324 $\rightarrow$



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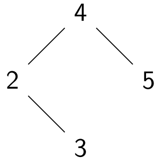

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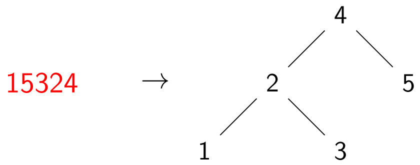

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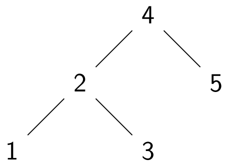
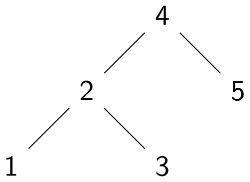
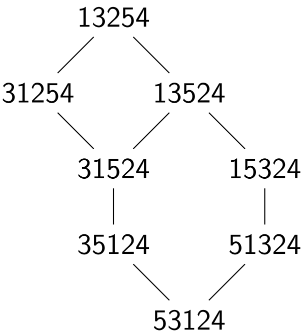


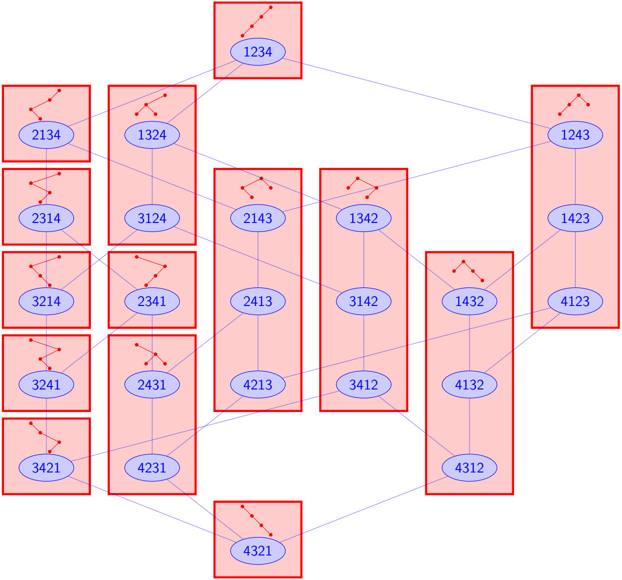

Characterization : the permutations sent to a given tree are its linear extensions

$15324, 31254, 35124, 51324, \ldots$

## Binary search tree insertion

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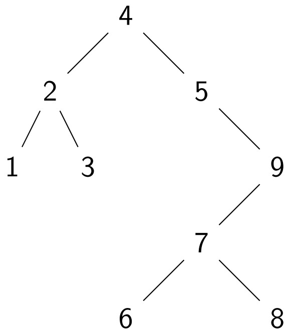
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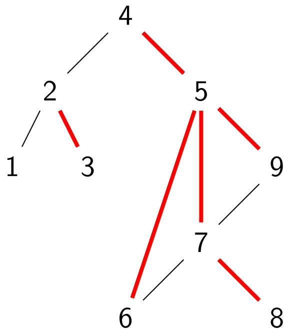
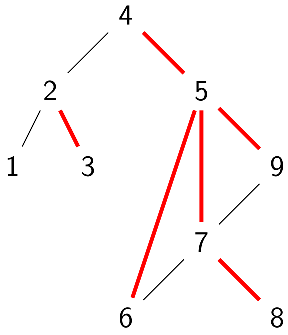
final forest $F_{\geq}(T)$



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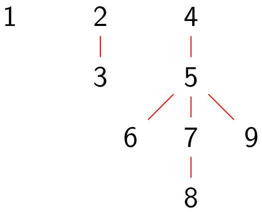

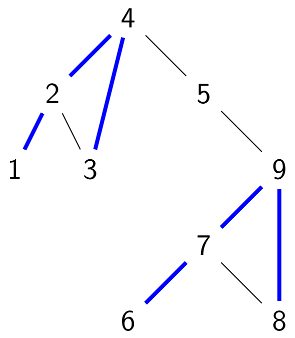


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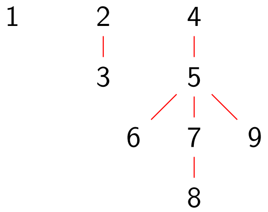


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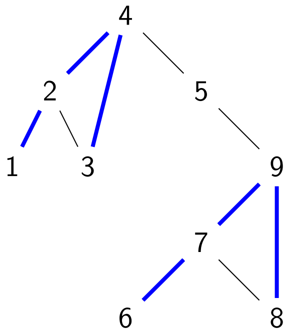


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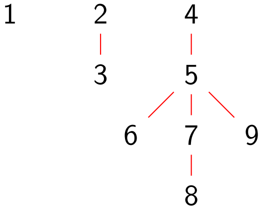


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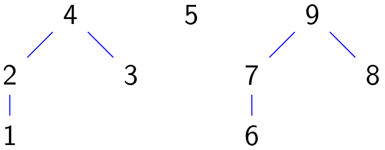

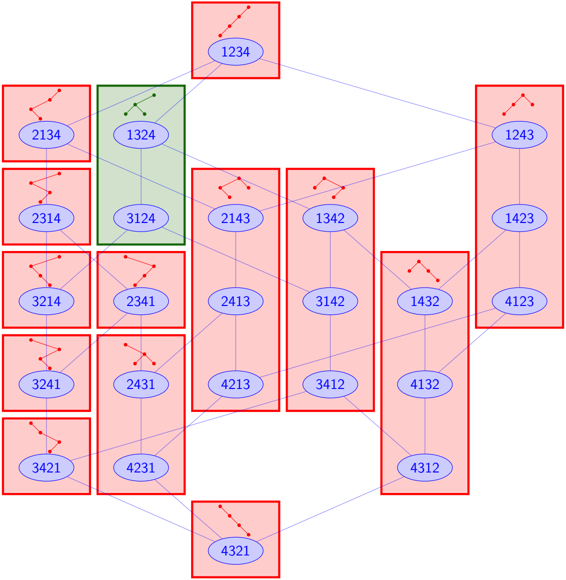
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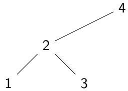
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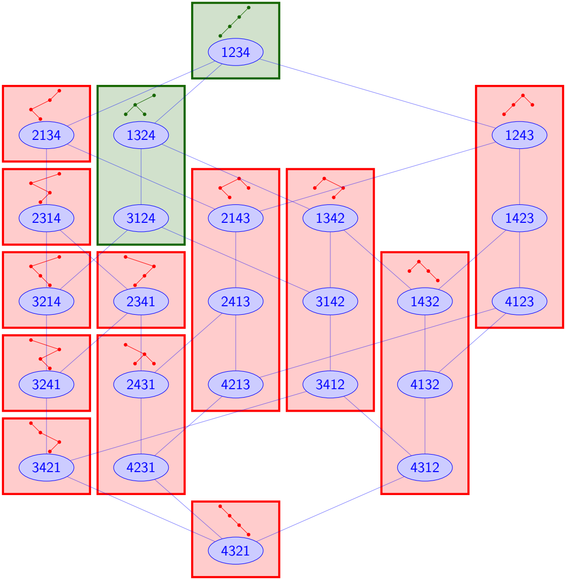
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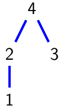
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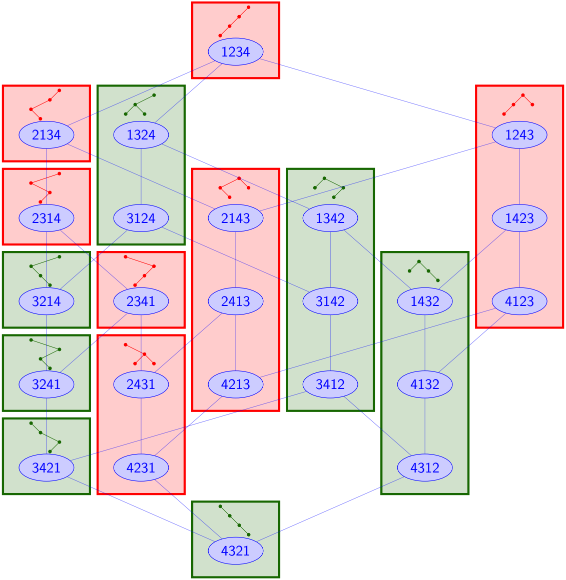
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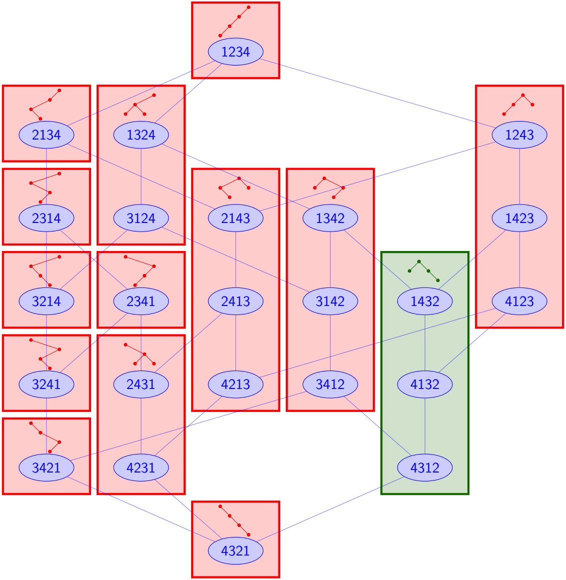
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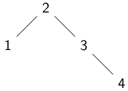
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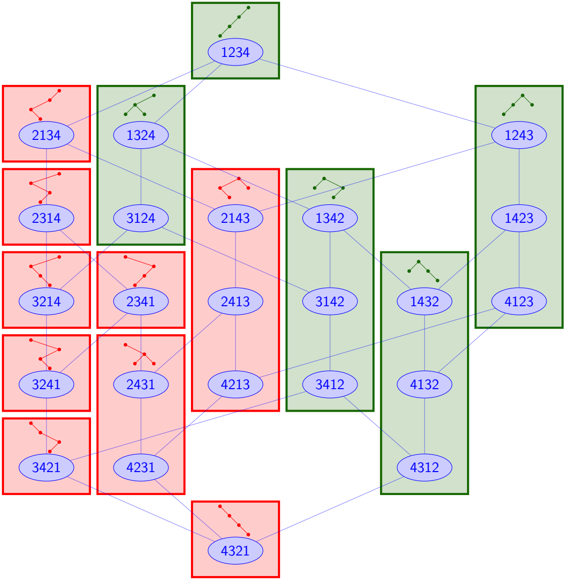
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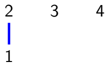
$F_{\leq}(T')$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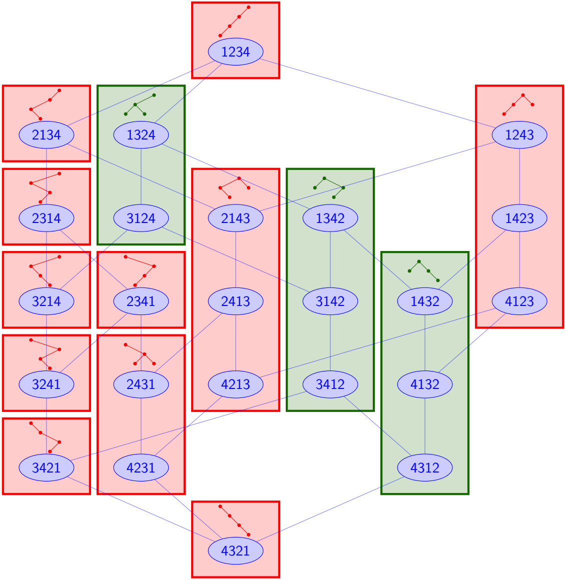
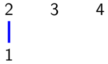



$F_{\geq}(T)$


$F_{\leq}(T')$

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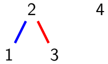

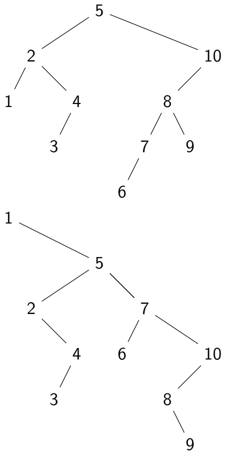
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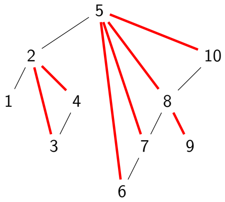
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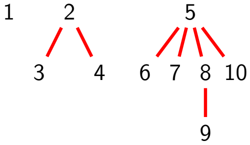
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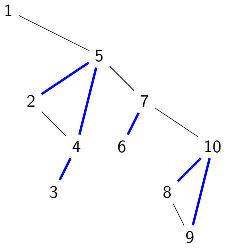
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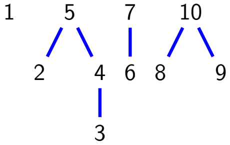
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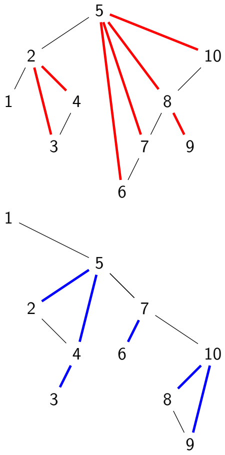
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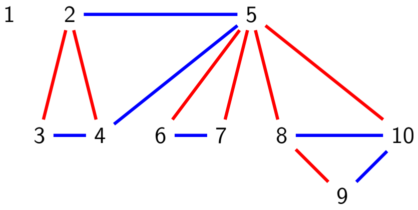
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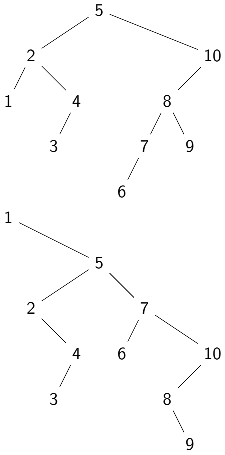
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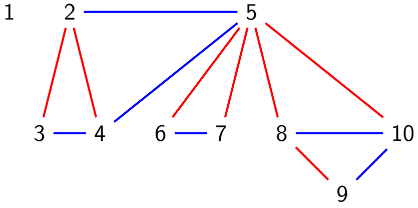
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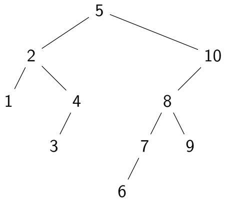
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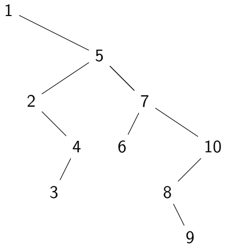
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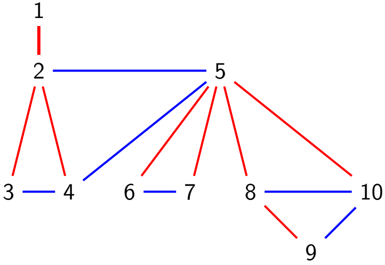
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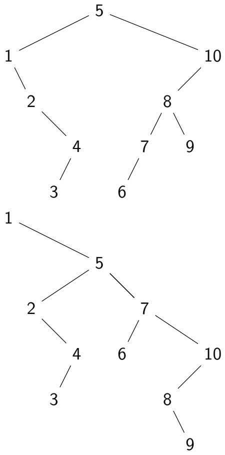
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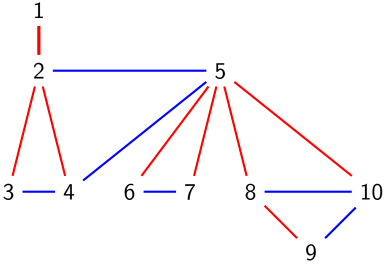
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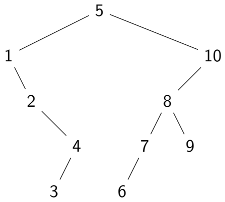
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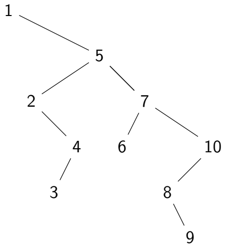
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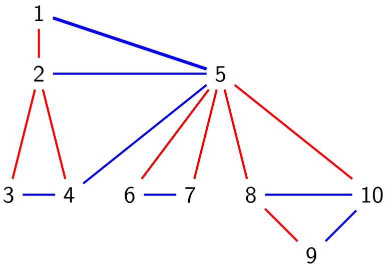
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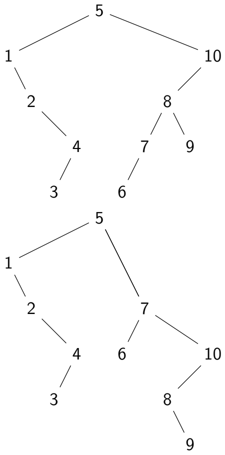
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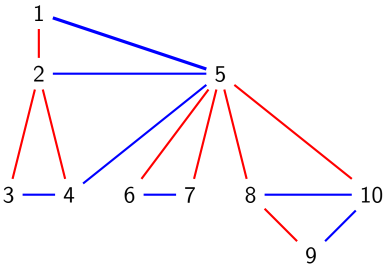
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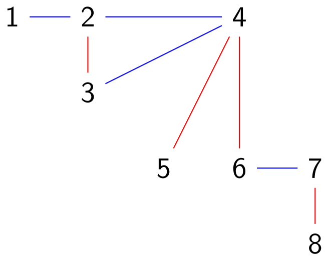
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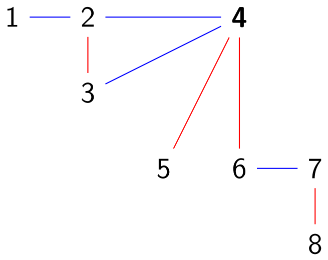
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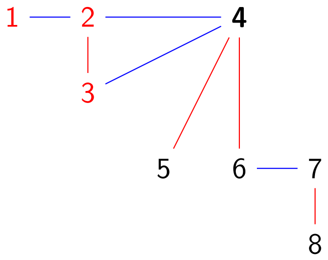
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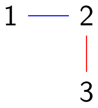
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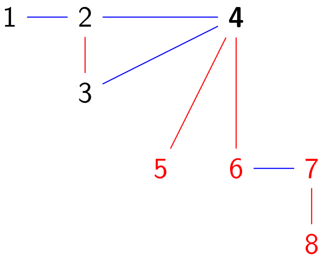
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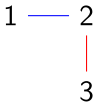
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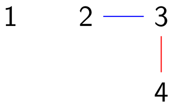
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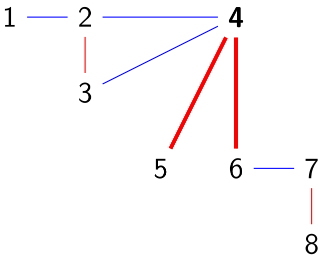


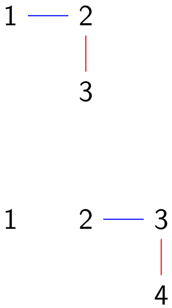


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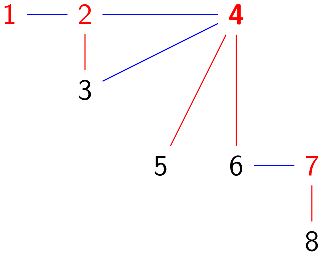
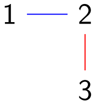
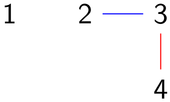



$x^4$

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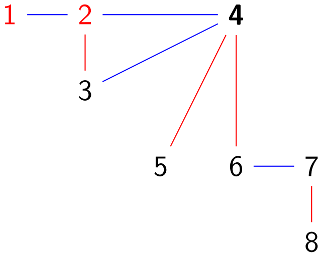
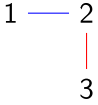
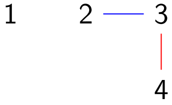



$$x^4 = {\color{red}x^2}.x.\frac{x^3}{x^2}$$




${\color{red}x^2}$




$x^3$

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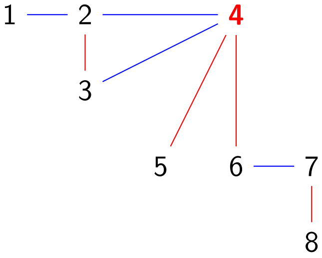


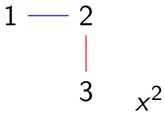


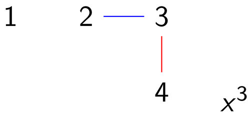


$$x^4 = x^2 . {\color{red}x} . \frac{x^3}{x^2}$$

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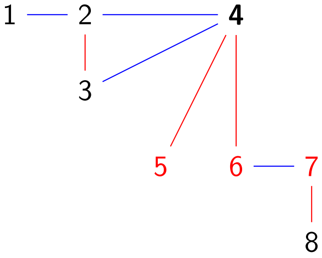
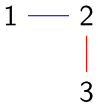
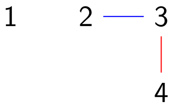



$$x^4 = x^2.x.\frac{x^3}{x^2}$$




$x^2$




$x^3$

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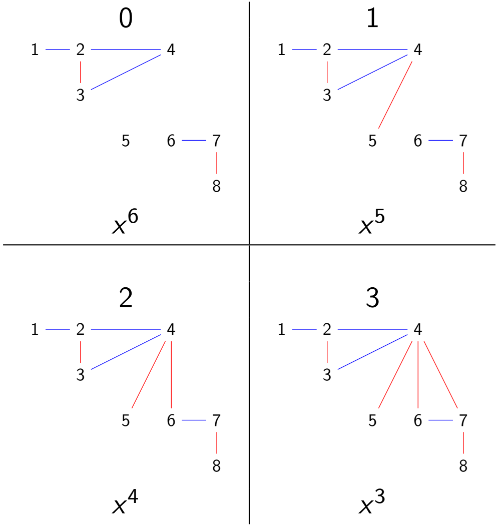
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$x^3$

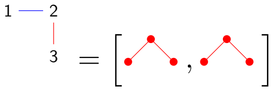
## Theorem (Chapoton)

*The generating functions of Tamari intervals satisfy the functional equation*

$$\Phi(x, y) = \mathrm{B}(\Phi, \Phi) + 1$$

*where*

$$\mathrm{B}(f, g) = xyf(x, y)\frac{xg(x, y) - g(1, y)}{x - 1}$$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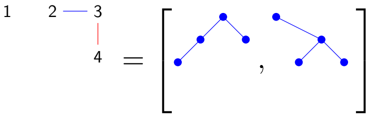
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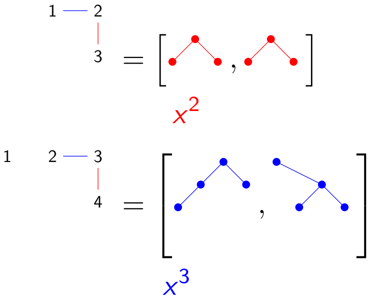
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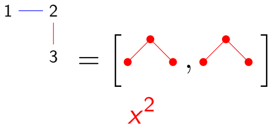
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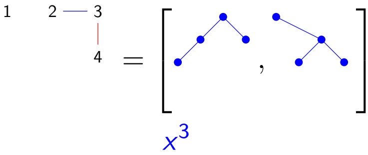
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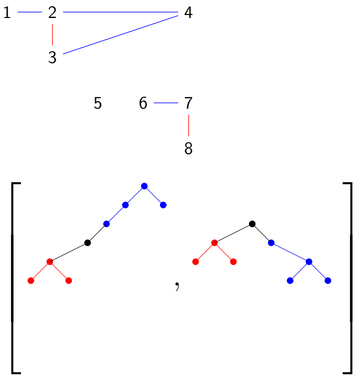
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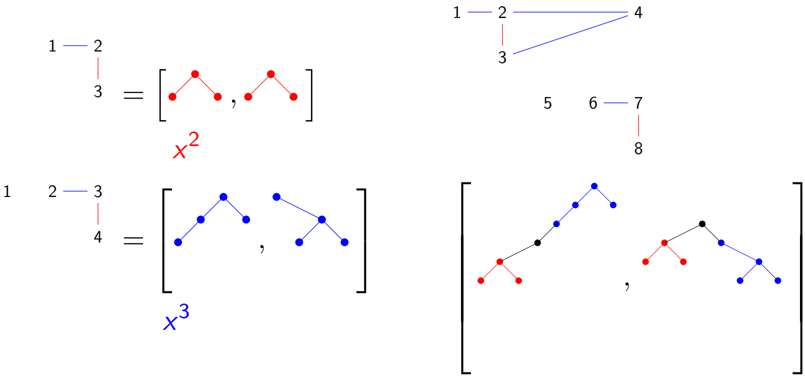
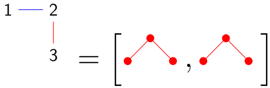
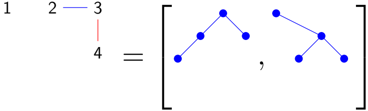



$$x^2 . x . x^3$$




$x^2$

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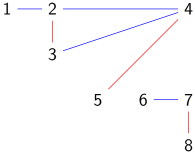
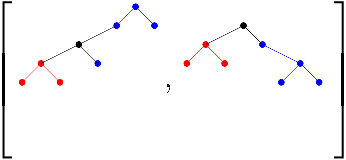
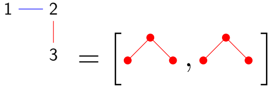
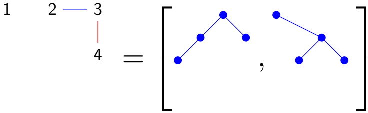


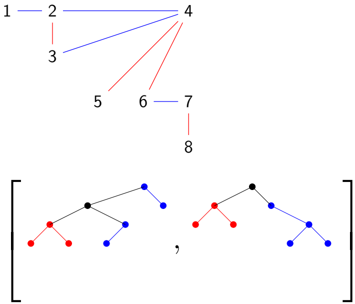
$$x^2.x.x^3 + x^2.x.x^2$$



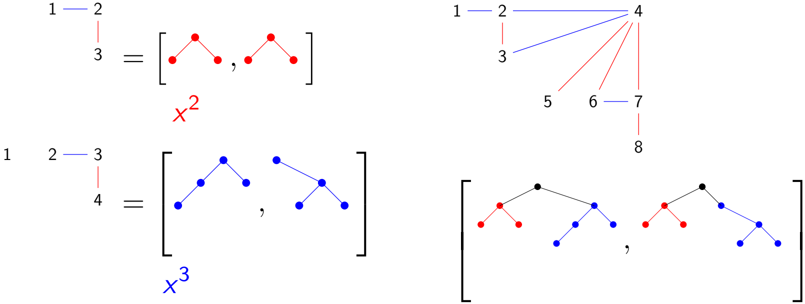

$x^2$



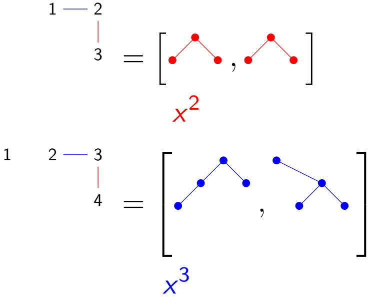

$x^3$

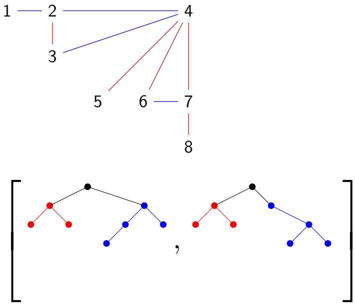



$$x^2.x.x^3 + x^2.x.x^2 + x^2.x.x$$




$$\color{red}{x^2}.x.\color{blue}{x^3} + \color{red}{x^2}.x.\color{blue}{x^2} + \color{red}{x^2}.x.\color{blue}{x} + \color{red}{x^2}.x$$




$$x^2.x.(x^3 + x^2 + x + 1)$$

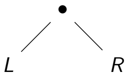
## Tamari Polynomials

$\mathcal{B}_T$ is recursively defined by

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

avec $T$ $=$




## Theorem (Châtel, P.)

*$\mathcal{B}_T$ counts the number of trees smaller than or equal to $T$ in the Tamari lattice according to the number of nodes on their left border.*

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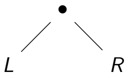


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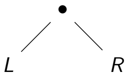


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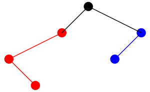

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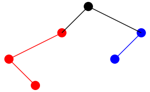

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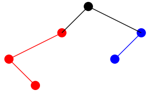
$$\mathcal{B}_L(x) = x^3 + x^2$$



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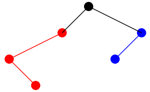
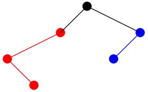


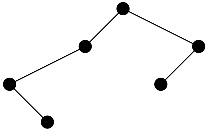
$$\mathcal{B}_\emptyset := 1$$

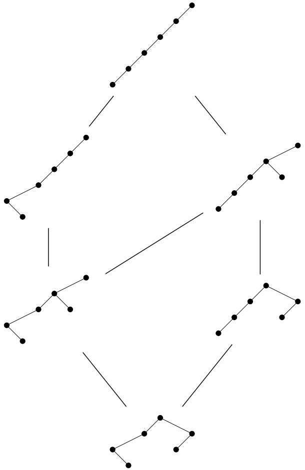
$$\mathcal{B}_T(x) := x(x^3 + x^2)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$

$$\mathcal{B}_R(x) = x^2$$

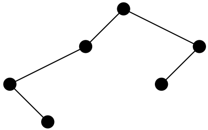


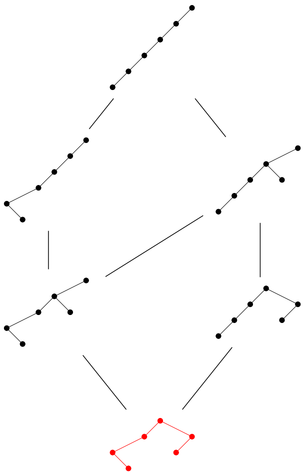
$$\mathcal{B}_{\emptyset} := 1$$
$$\mathcal{B}_T(x) := x(x^3 + x^2)(1 + x + x^2)$$





$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$





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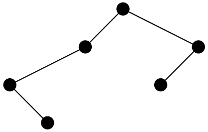

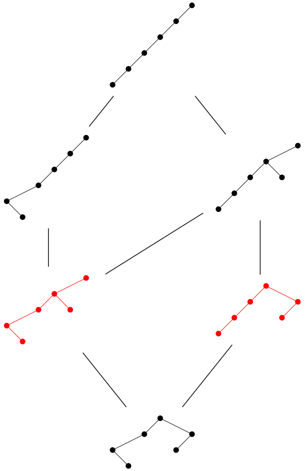

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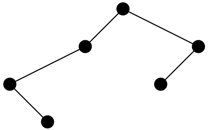
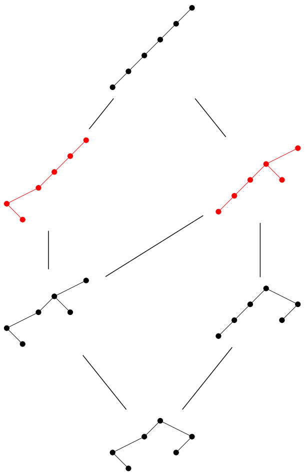
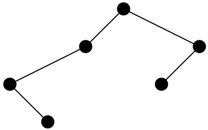
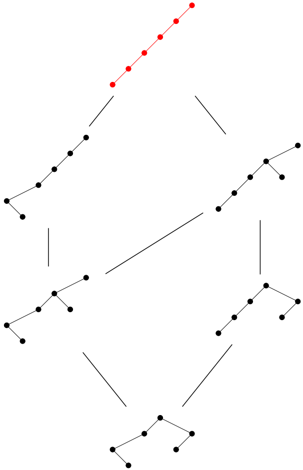




$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$





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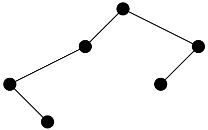
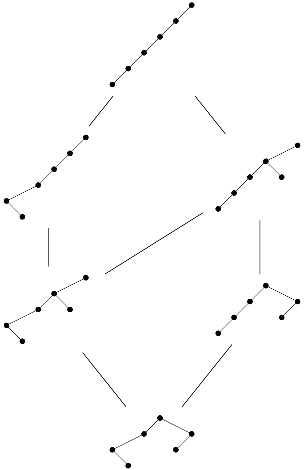
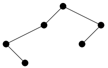
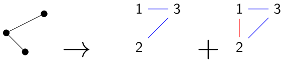




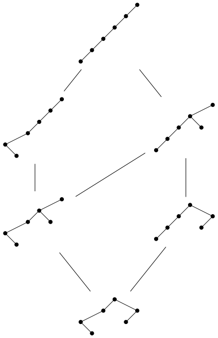
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$
$$\mathcal{B}_T(1) = 6$$

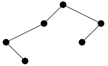


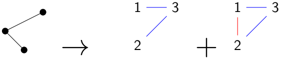



$$x^3 + x^2$$


$$x^2$$

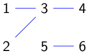




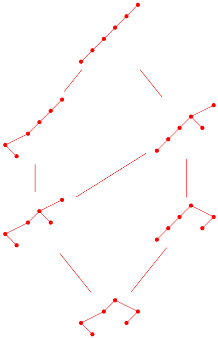


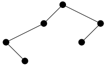

$x^3 + x^2$

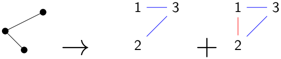

$x^2$




$x^3.x.x^2$



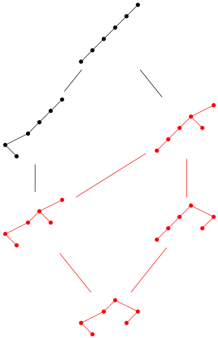


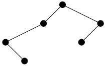


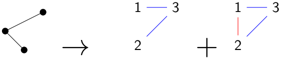

$x^3 + x^2$


$x^2$

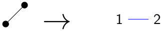

$x^3.x.x^2 \quad + x^3.x.x$



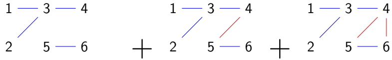




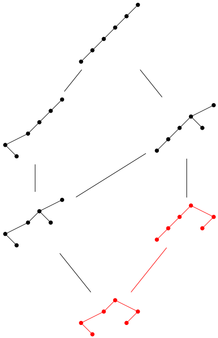

$$x^3 + x^2$$

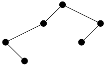


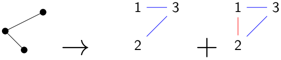

$$x^2$$




$$x^3.x.x^2 \qquad + x^3.x.x + x^3.x$$

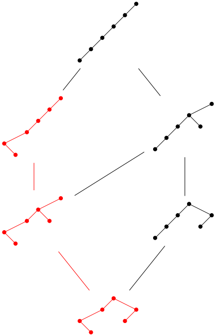


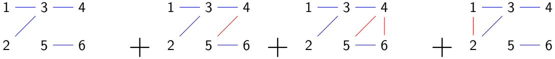




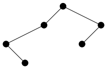

$$x^3 + x^2$$

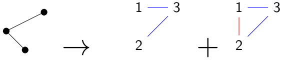

$$x^2$$





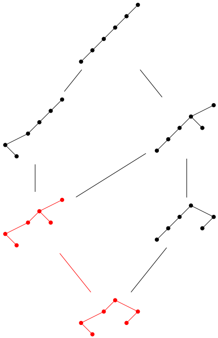

$$x^3.x.x^2 \quad + x^3.x.x + x^3.x \quad + x^2.x.x^2$$

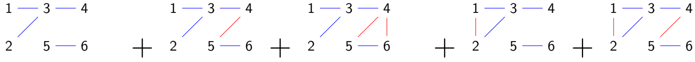




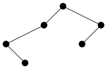

$$x^3 + x^2$$

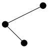
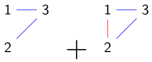

$$x^2$$






$$x^3.x.x^2 + x^3.x.x + x^3.x + x^2.x.x^2 + x^2.x.x$$

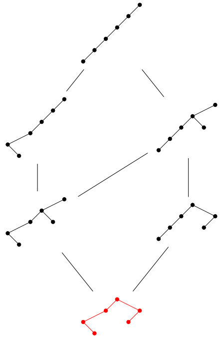


 $\rightarrow$ 

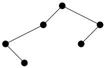
$x^3 + x^2$

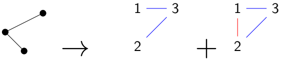
 $\rightarrow$ 

$x^2$



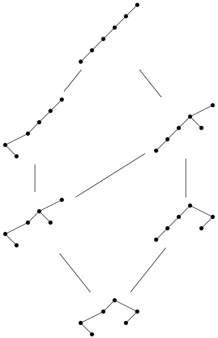
$x^3.x.x^2 + x^3.x.x + x^3.x + x^2.x.x^2 + x^2.x.x + x^2.x$



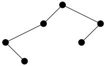


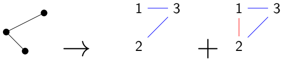

$$x^3 + x^2$$


$$x^2$$

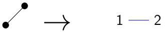


$$(x^3 + x^2).x.(x^2 + x + 1) =$$

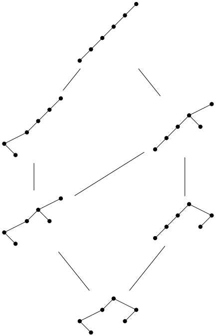





$$x^3 + x^2$$




$$x^2$$



$$(x^3 + x^2).x.(x^2 + x + 1) = x\,\mathcal{B}_L(x)\frac{\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$