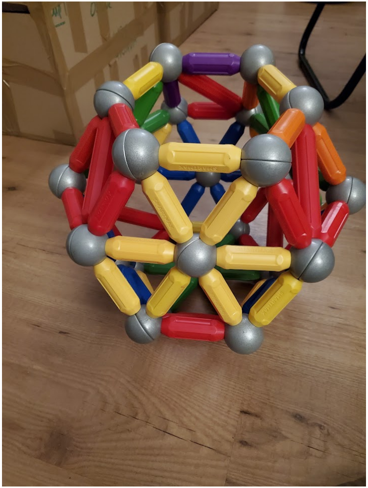
Viviane Pons

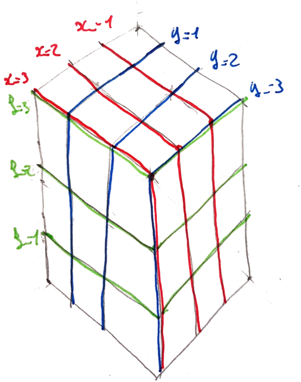
Habilitation à Diriger des recherches, 20 octobre 2023
LISN, Univ. Paris-Saclay

Combinatorics of the Permutahedra, Associahedra, and Friends

Meet the permutahedron



Meet the permutahedron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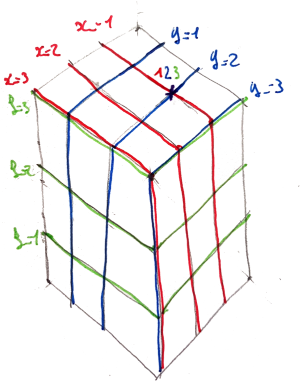
x=2
x=3
y=1
y=2
y=3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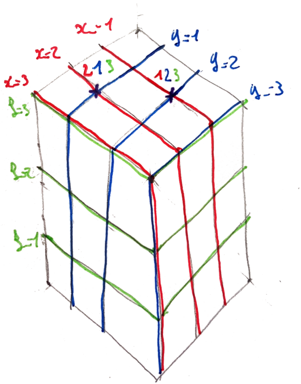
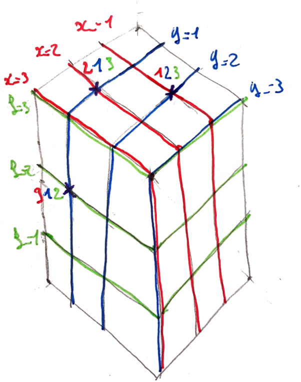
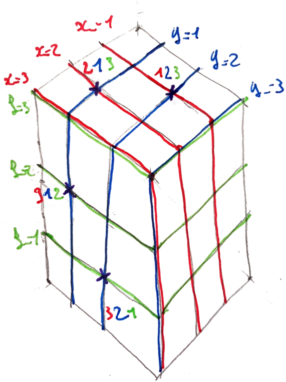
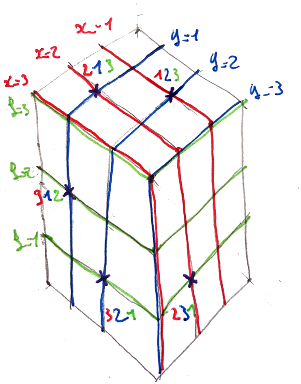
x=2
x=3
y=1
y=2
y=3
123
z=3
z=2
z=1


x=1
x=2
x=3
y=1
y=2
y=3
213
123


x=1
x=2
x=3
y=1
y=2
y=3
213
123
z=3
z=2
312
z=1


x=1
x=2
x=3
y=1
y=2
y=3
213
123
312
321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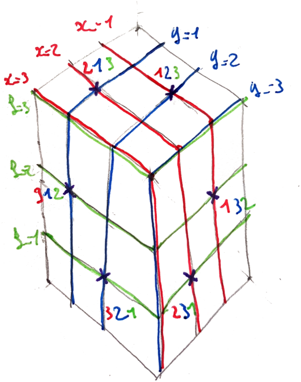
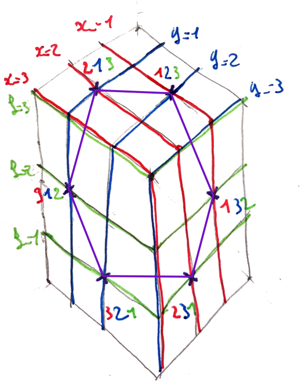
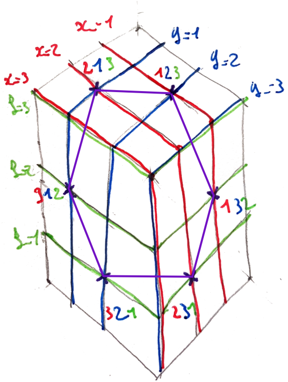
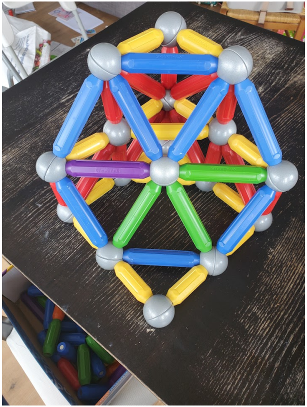
x=2
x=3
y=1
y=2
y=3
213
123
312
321
231


x=2
x=3
y=1
y=2
213
123
312
132
321
231


x=1
x=2
x=3
y=1
y=2
y=3
213
123
312
132
321
231





$$x + y + z = 6$$

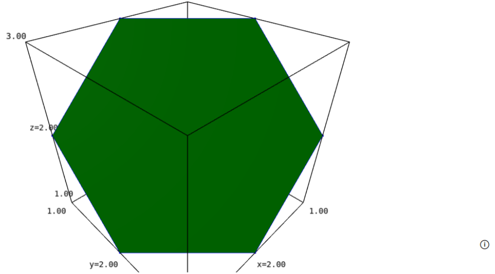
Using SageMath

Entrée [1]:
```
P = Polyhedron(list(Permutations(3)))
P
```

Out[1]: A 2-dimensional polyhedron in ZZ^3 defined as the convex hull of 6 vertices (use the .plot() method to plot)

Entrée [2]:
```
P.plot()
```

Out[2]:



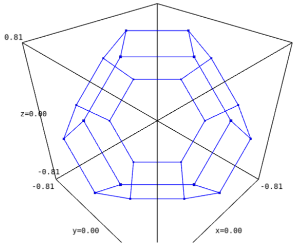
For size 4

Entrée [3]:
```
P = Polyhedron(list(Permutations(4)))
P
```

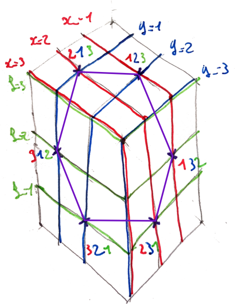
Out[3]: A 3-dimensional polyhedron in ZZ^4 defined as the convex hull of 24 vertices (use the .plot() method to plot)

Entrée [4]:
```
P.plot()
```

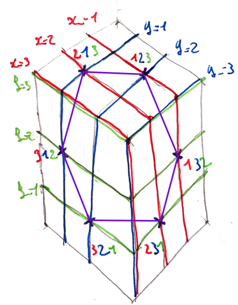
Out[4]:

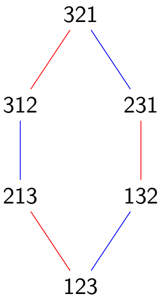


Looking at the “skeleton” of the polytope”

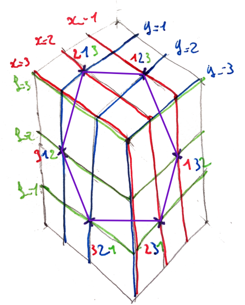


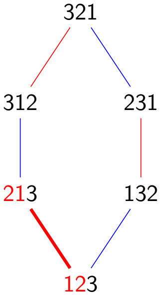
Looking at the “skeleton” of the polytope”



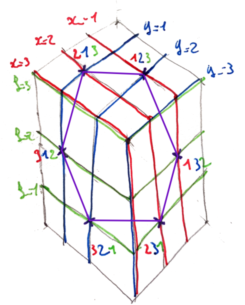


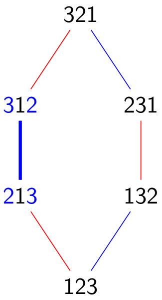
Looking at the “skeleton” of the polytope”



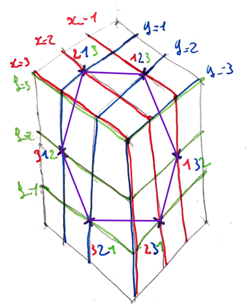


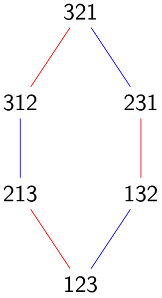
Looking at the “skeleton” of the polytope”





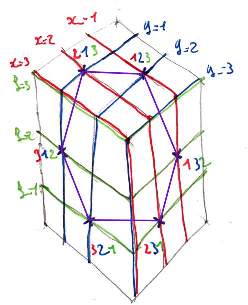
Looking at the “skeleton” of the polytope”

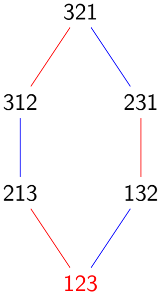




left weak order

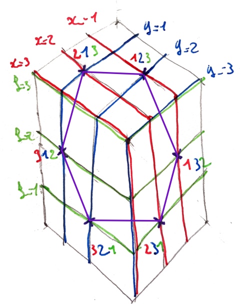
Looking at the “skeleton” of the polytope”

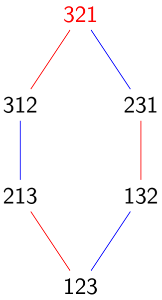




left weak order

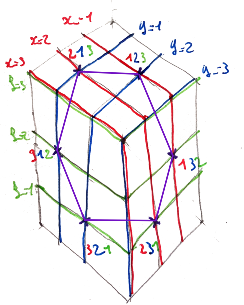
Looking at the “skeleton” of the polytope”

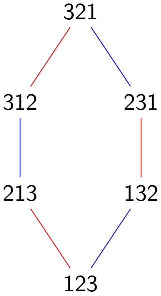




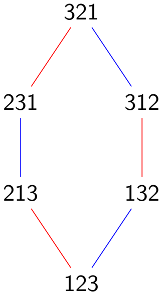
left weak order

Looking at the "skeleton" of the polytope"



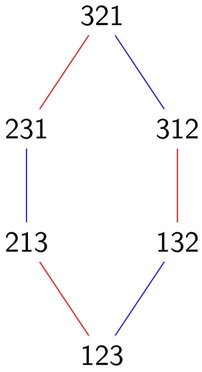


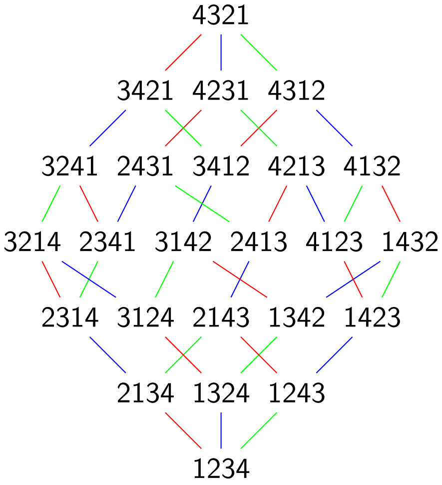
left weak order



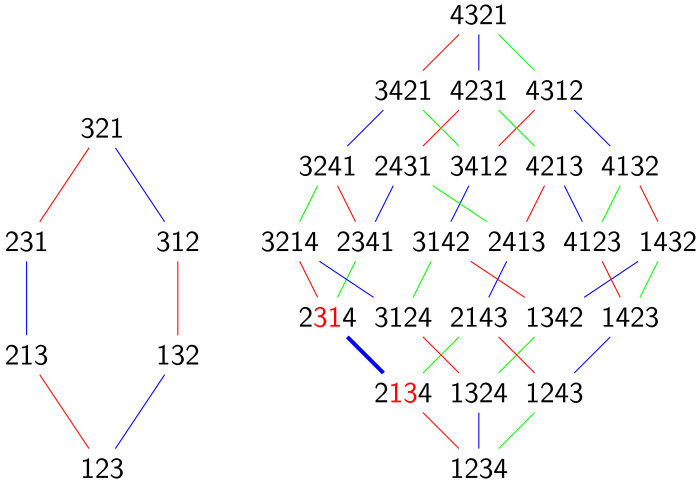
right weak order

Right weak Order

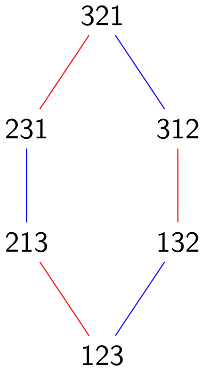


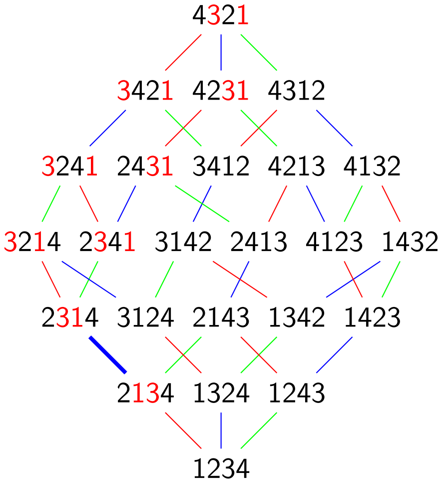


Right weak Order

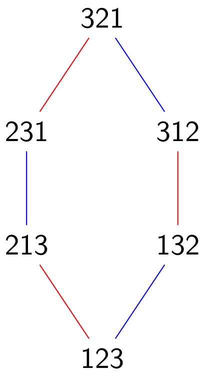


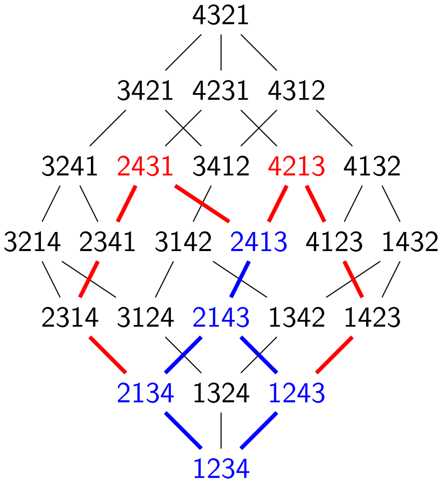
Right weak Order





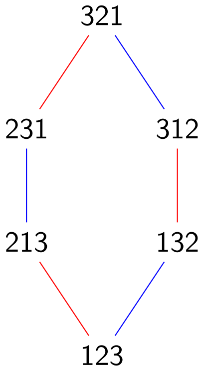
Right weak Order

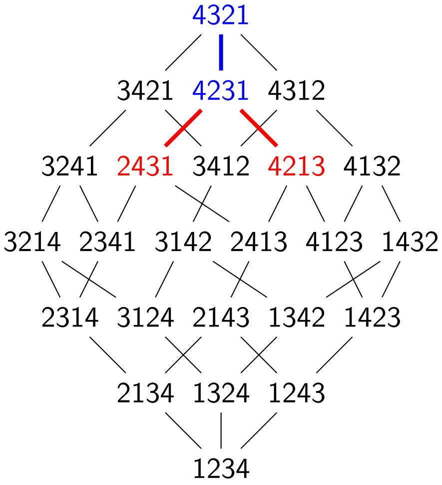




$2413 \wedge 4213 = 2413$ $\qquad$ $2413 \vee 4213 = 4231$

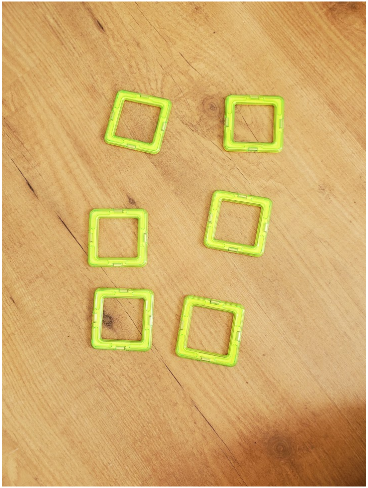
Right weak Order

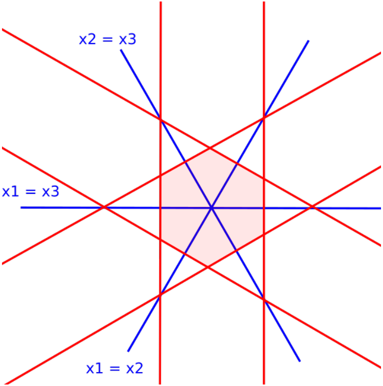




$2413 \wedge 4213 = 2413$ $\qquad$ $2413 \vee 4213 = 4231$

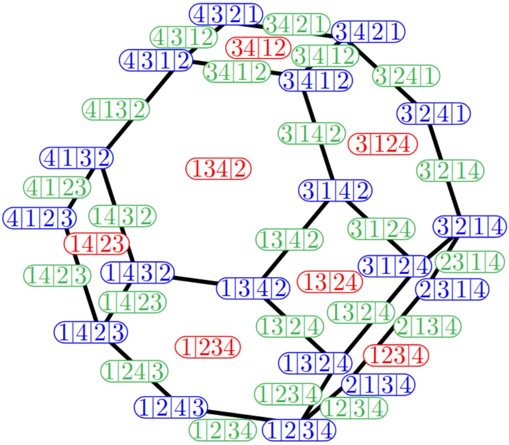
Combinatorics of faces





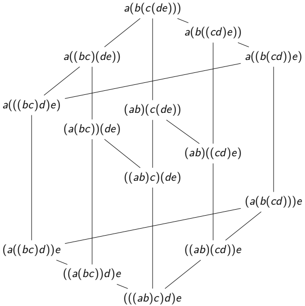
$$J \subseteq [n] \rightarrow \sum_{j \in J} x_j \geq \binom{|J|+1}{2}$$

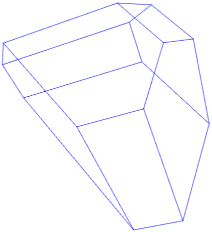
$12\|3$	$x_1 + x_2 \geq 3$
$2\|13$	$x_2 \geq 1$
$23\|1$	$x_2 + x_3 \geq 3$
$3\|12$	$x_3 \geq 1$
$13\|2$	$x_1 + x_3 \geq 3$
$1\|23$	$x_1 \geq 1$



(image from V. Pilaud's talk "The Associahedron and its friends")

Meet the Tamari lattice and associahedron

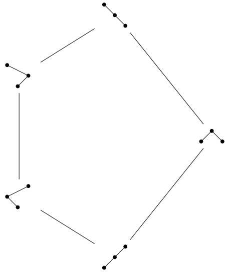


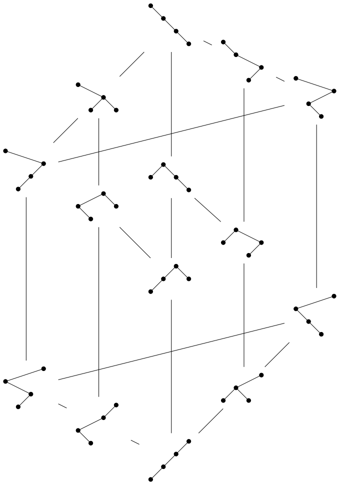


Tamari 51, Loday 04

The Tamari lattice

A structure on Catalan objects: $\frac{1}{n+1}\binom{n}{2n}$.

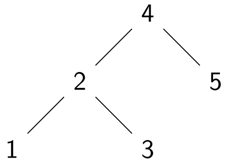


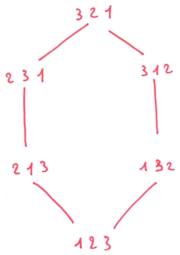


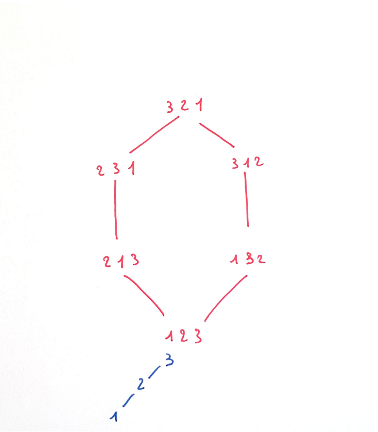
From permutations to binary trees

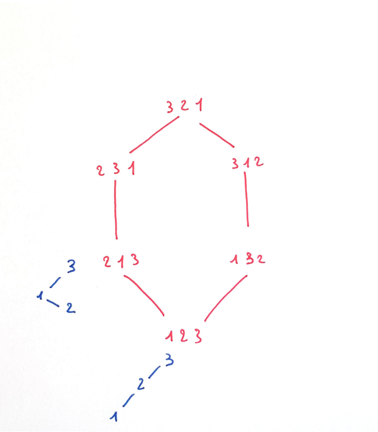
Binary search tree insertion

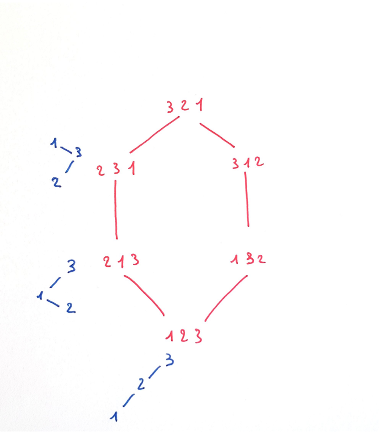
15324 $\rightarrow$

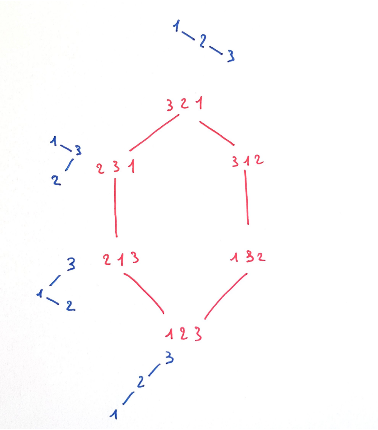


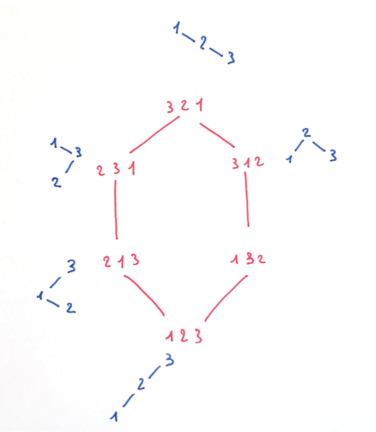

3 2 1
2 3 1
3 1 2
2 1 3
1 3 2
1 2 3

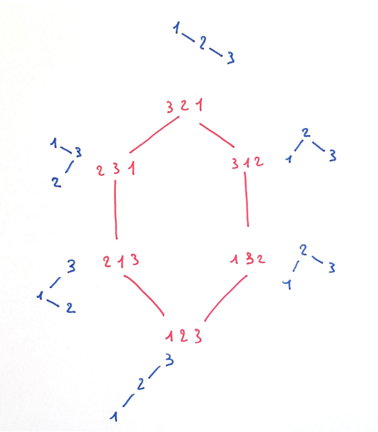

3 2 1
2 3 1
3 1 2
2 1 3
1 3 2
1 2 3
3
2
1

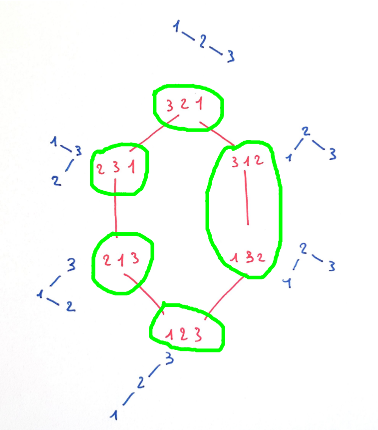

3 2 1
2 3 1
3 1 2
2 1 3
1 3 2
1 2 3
3
1
2
3
2
1

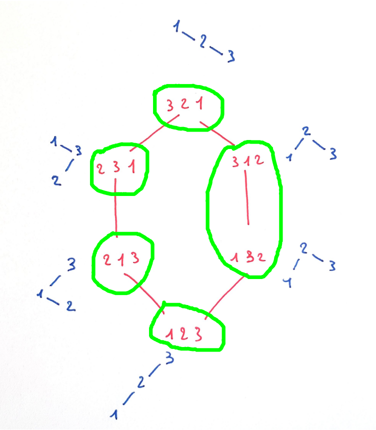

3 2 1
2 3 1
3 1 2
2 1 3
1 3 2
1 2 3


3 2 1
2 3 1
3 1 2
2 1 3
1 3 2
1 2 3

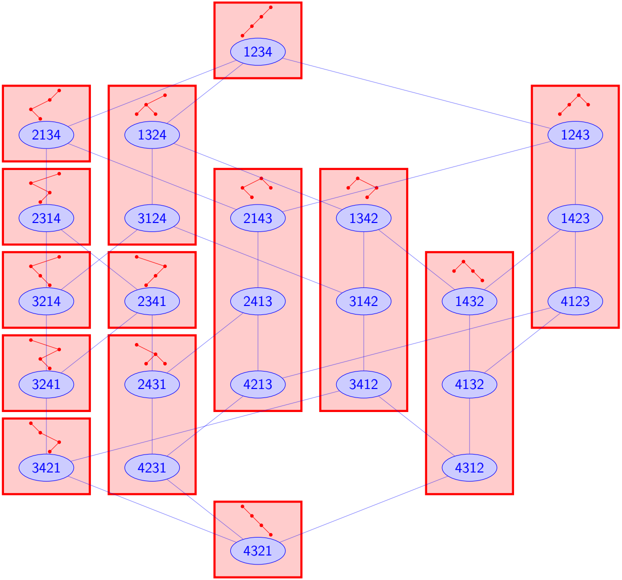

1 2 3
3 2 1
2 3 1
3 1 2
2 1 3
1 3 2
1 2 3

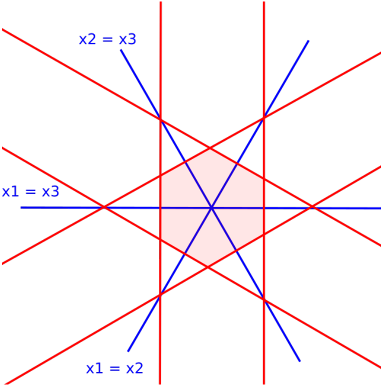

321
231
312
213
132
123


321
231
312
213
132
123

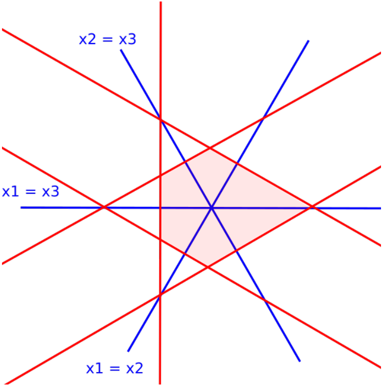


The **Tamari lattice** forms a **quotient lattice** of the weak order.


1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321



12|3	$x_1 + x_2 \geq 3$
2|13	$x_2 \geq 1$
23|1	$x_2 + x_3 \geq 3$
3|12	$x_3 \geq 1$
13|2	$x_1 + x_3 \geq 3$
1|23	$x_1 \geq 1$



$12\|3$	$x_1 + x_2 \geq 3$
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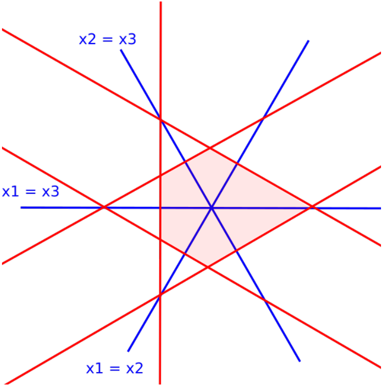
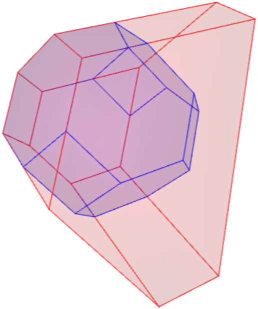
$23\|1$	$x_2 + x_3 \geq 3$
$3\|12$	$x_3 \geq 1$
$13\|2$	
$1\|23$	$x_1 \geq 1$


x2 = x3
x1 = x3
x1 = x2



Why is it interesting?

Nice interplay between algebra, combinatorics and geometry

Tamari Memorial Festschrift, 2012

Celebrating Loday's associahedron by Pilaud, Santos and Ziegler, 2023

Why is it interesting?

The Tamari lattice appears everywhere

In algorithmic

- AVL trees

  Adelson-Velsky, Landis 1962

- map enumeration

  Bernadi, Bonichon, Fang, Fusy, Nadeau

- flip distance (open problem)
- diameter of associahedra

  Sleator, Tarjan, Thurston, 1988

  Pournin, 2014

- diagonal of associahedra

  Loday, Masuda, Thomas, Tonks, Vallette, Costan, Chyzak, Pilaud

Why is it interesting?

The Tamari lattice appears everywhere

In algebra

- representation theory

  Bergeron, Préville-Ratelle, Bousquet-Mélou, Chapuy

- Hopf algebras

  Loday, Ronco, Hivert, Novelli, Thibon

- cluster algebras

  Fomin, Zelevinsky

Why is it interesting?

And has many generalizations

- m-Tamari lattices

  Bergeron, Préville-Ratelle, 2012

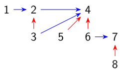
- ν-Tamari lattices

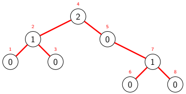
  Viennot, Préville-Ratelle, 2017

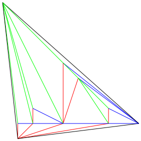
- Cambrian lattices

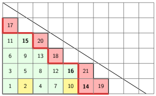
  Reading, 2006

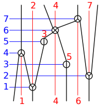
Friends of permutahedra and associahedra

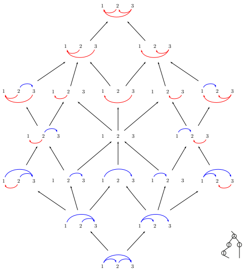


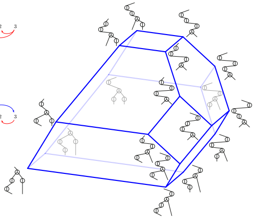


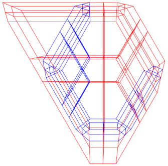


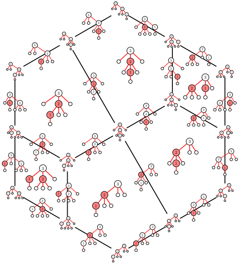




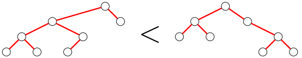


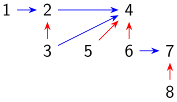


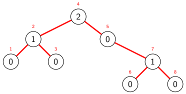


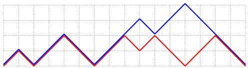


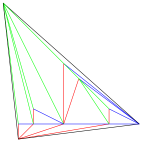
Bijections and enumerations: Tamari intervals

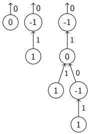




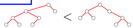








couples of Binary trees



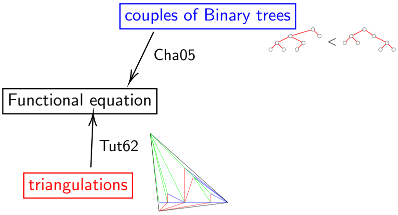
Cha05

Functional equation

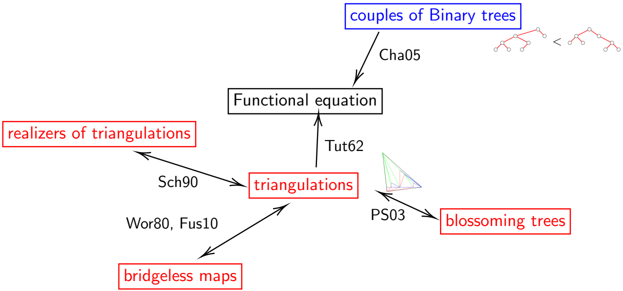
$\Phi(z,x) = 1 + zx\Phi(z,x)\frac{x\Phi(z,x)-\Phi(z,1)}{x-1}$

$I_n = \frac{2}{n(n+1)}\binom{4n+1}{n-1}$

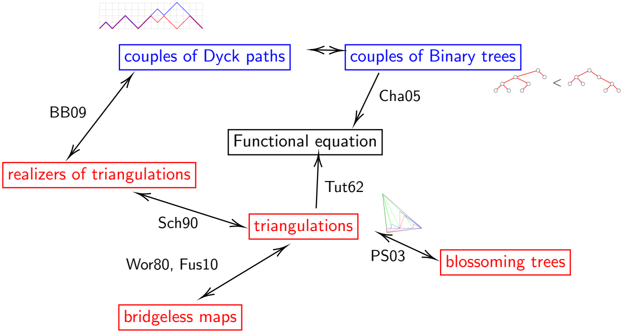
Chapoton



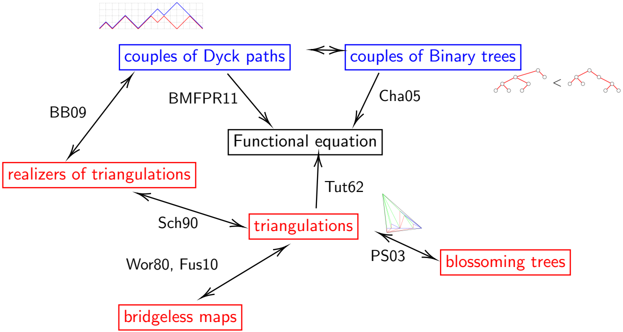
Chapoton, Tutte



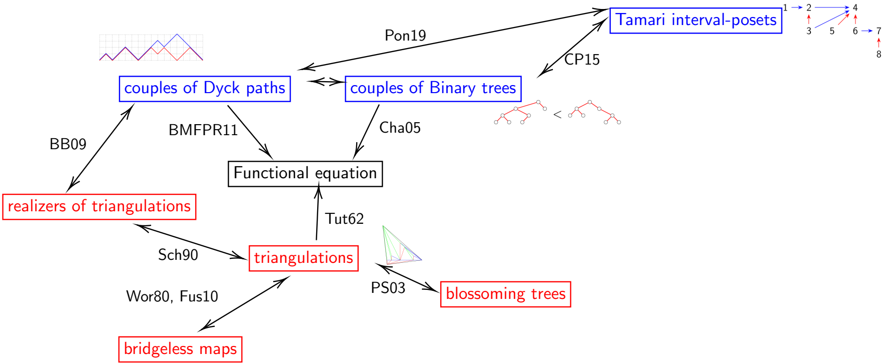
Chapoton, Tutte, Schnyder, Wormald, Poulahon, Schaeffer, Fusy



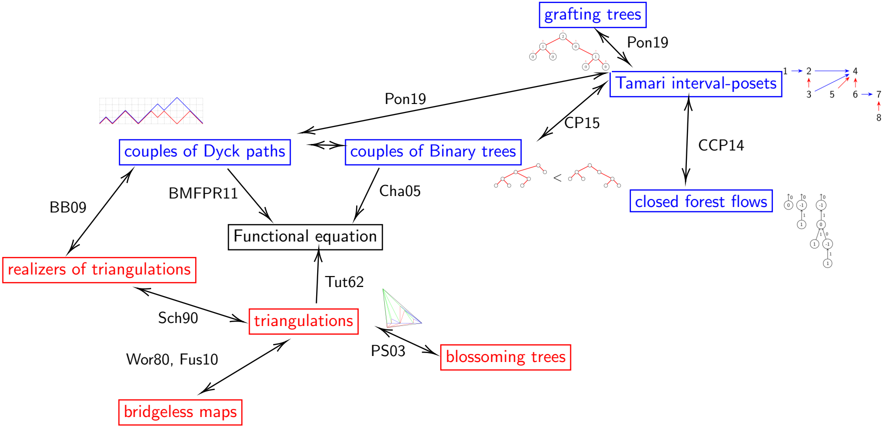
Chapoton, Tutte, Schnyder, Wormald, Poulahon, Schaeffer, Fusy, Bernadi, Bonichon



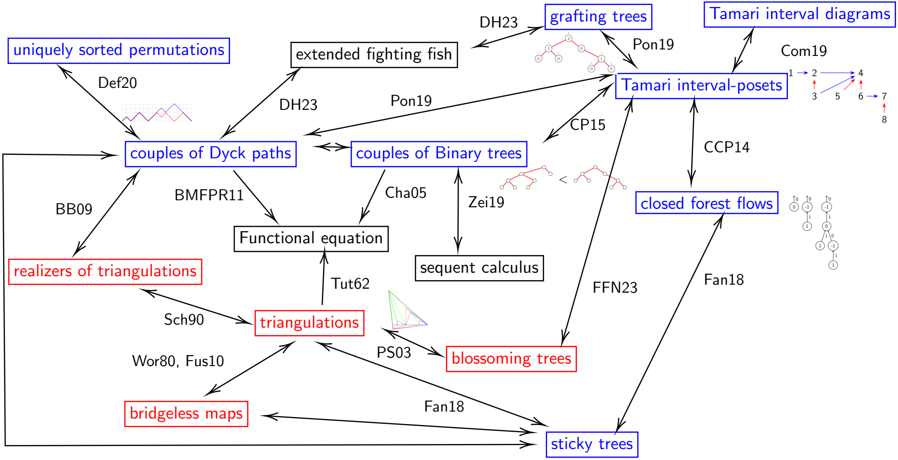
Chapoton, Tutte, Schnyder, Wormald, Poulahon, Schaeffer, Fusy, Bernadi, Bonichon, Bousquet-Mélou, Préville-Ratelle



Chapoton, Tutte, Schnyder, Wormald, Poulahon, Schaeffer, Fusy, Bernadi, Bonichon, Bousquet-Mélou, Préville-Ratelle, Châtel, P.

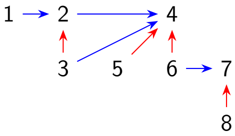


Chapoton, Tutte, Schnyder, Wormald, Poulahon, Schaeffer, Fusy, Bernadi, Bonichon, Bousquet-Mélou, Préville-Ratelle, Châtel, P.



Chapoton, Tutte, Schnyder, Wormald, Poulahon, Schaeffer, Fusy, Bernadi, Bonichon, Bousquet-Mélou, Préville-Ratelle, Châtel, P., Fang, Combe, Zeilberger, Defant, Duchi, Henriet, Nadeau

Tamari interval-posets

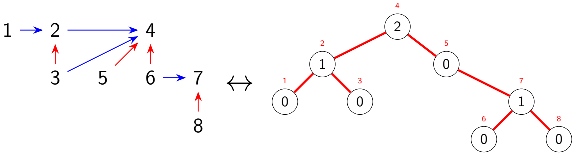


for all $a < b < c$, $a \lhd c \Rightarrow b \lhd c$ and $c \lhd a \Rightarrow b \lhd a$.

Theorem (Châtel, P., 2015)

Tamari interval-posets are in bijection with intervals of the Tamari lattice

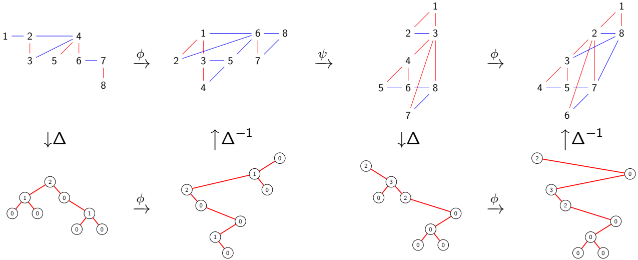
Grafting trees



Theorem (P., 2019)

Grafting trees are in bijection with intervals of the Tamari lattice

Rise-Contact involution (P. 2019)



Combinatorial proof of the rise-contact symmetry found by Bousquet-Mélou, Fusy and Préville-Ratelle

Why are we interested in statistics on intervals?

Motivation: diagonal harmonic polynomials. Compute the dimension of the alternating component of an $\mathfrak{S}_n$-module.

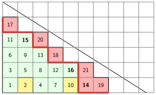
bivariate polynomials: q, t-Catalan

Bergeron, Garsia, Haiman, Haglund

trivariate polynomials: (conjecture) q, t, r enumeration of Tamari intervals

Bergeron, Préville-Ratelle

Current project and perspectives (with Loic Le Mogne)

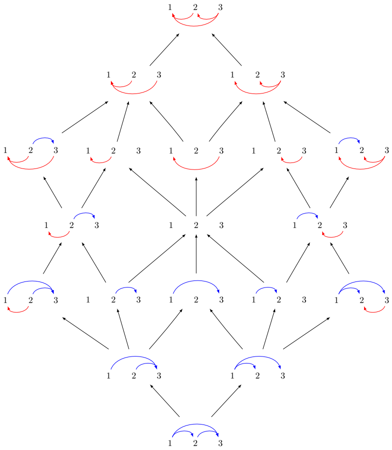


- study the "triangular" case

  Bergeron, Mazin, 2022

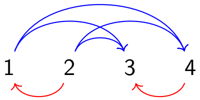
- results and conjectures with connection to ν-Tamari lattice
- try to generalize the Catalan ecosystem to the triangular case

New friends: Integer posets



Combinatorial objects as integer posets

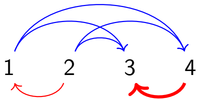
$\sigma = 2143 \longrightarrow$



Permutations

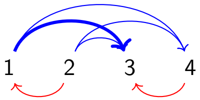
Combinatorial objects as integer posets

$\sigma = 2143 \longrightarrow$



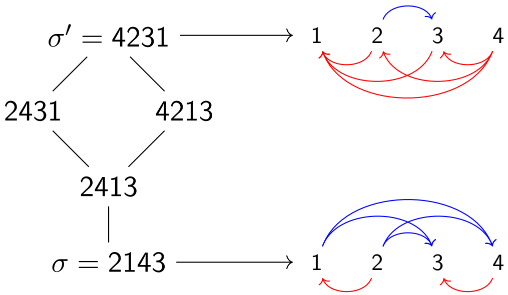
Permutations

Combinatorial objects as integer posets

$\sigma = 2143 \longrightarrow$ 

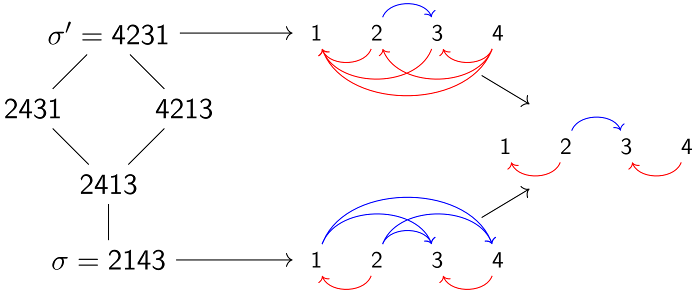
Permutations

Combinatorial objects as integer posets



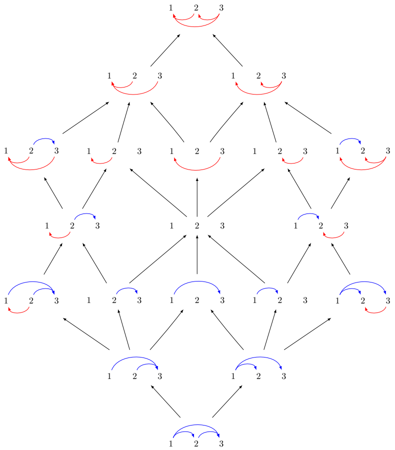
Permutations

Combinatorial objects as integer posets



Permutations interval $\rightarrow$ weak order integer poset

All integer posets

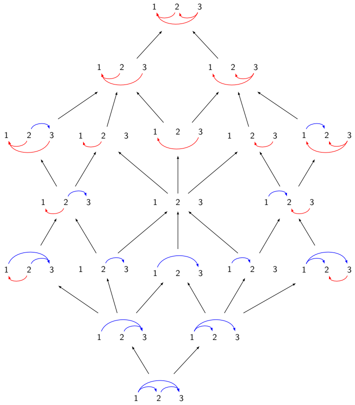


Theorem (Châtel, Pilaud, P. 2019)

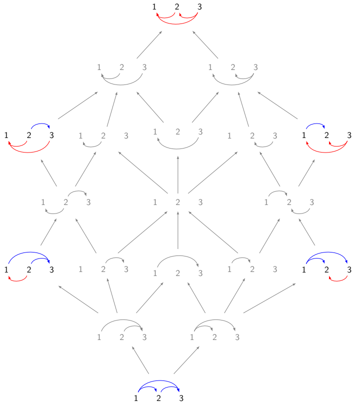
The weak order on integer posets is a lattice.

Theorem (Pilaud, P. 2020)

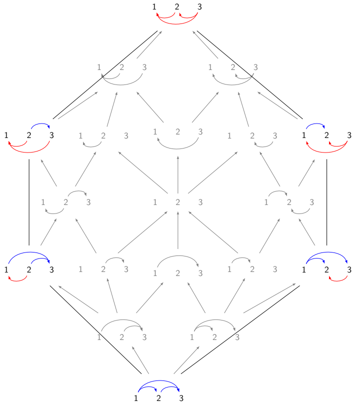
The vector space of integer posets can be endowed with product and co-product operations to form a Hopf algebra.



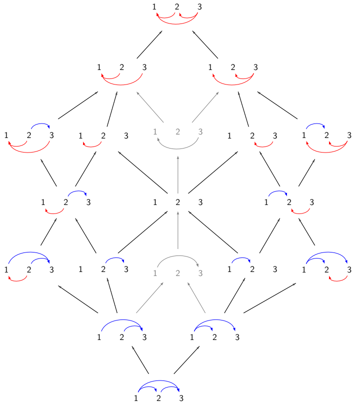
integer posets



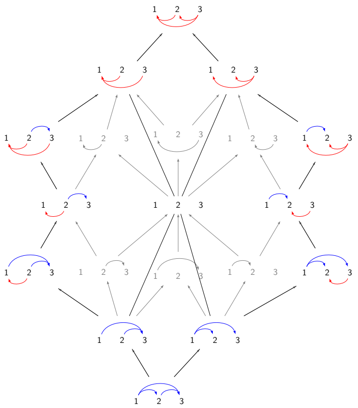
permutations (classical weak order)



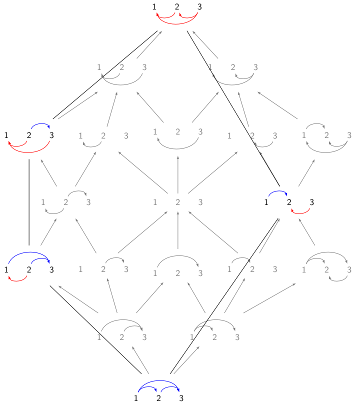
permutations (classical weak order)



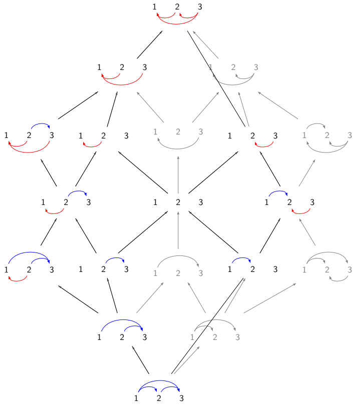
permutations intervals



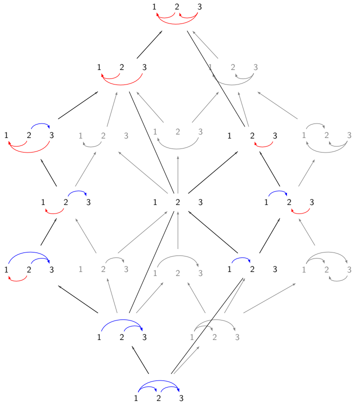
faces of the permutahedron (facial weak order)



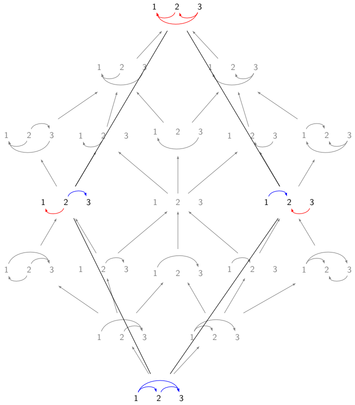
binary trees (Tamari lattice)



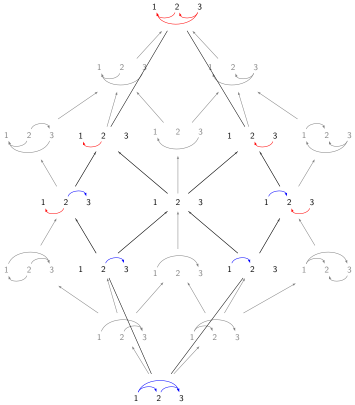
Tamari interval-posets



faces of the associahedron

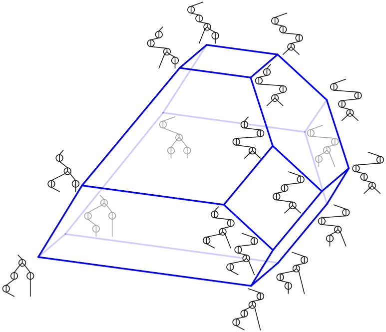


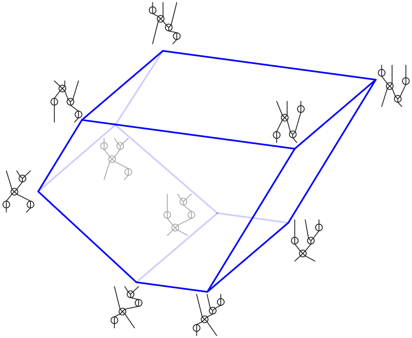
binary sequences (boolean lattice)



faces of the cube

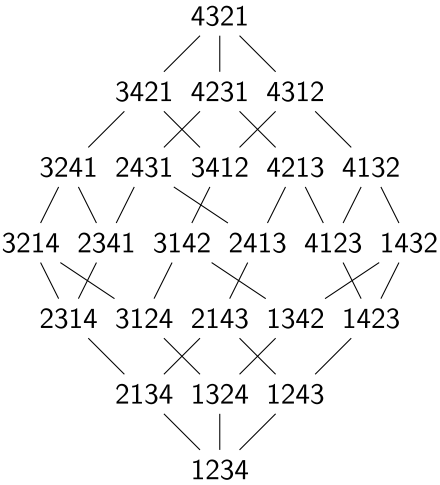
New friends: permutrees

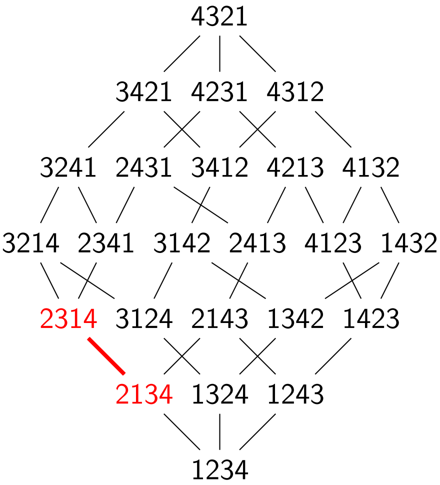




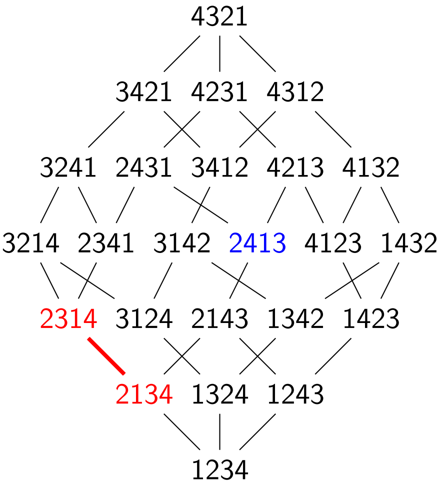
Preamble: lattice congruences

Lattice Congruences of the Weak Order, Nathan Reading, Order, 2005


4321
3421 4231 4312
3241 2431 3412 4213 4132
3214 2341 3142 2413 4123 1432
2314 3124 2143 1342 1423
2134 1324 1243
1234

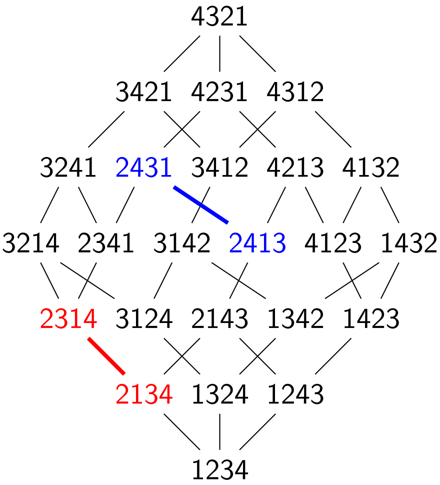


$2134 \equiv 2314$



$$2134 \equiv 2314$$

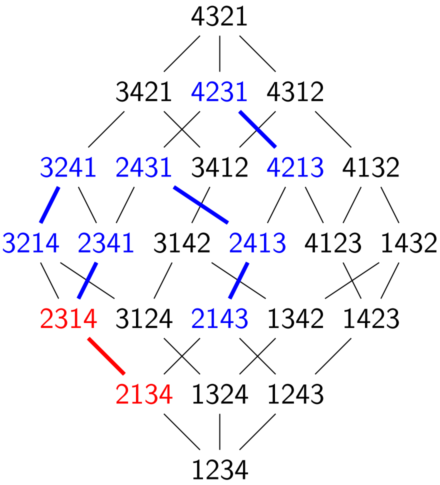
$$2134 \vee 2413 \equiv 2314 \vee 2413$$



$$2134 \equiv 2314$$

$$2134 \vee 2413 \equiv 2314 \vee 2413$$

$$2431 \equiv 2413$$

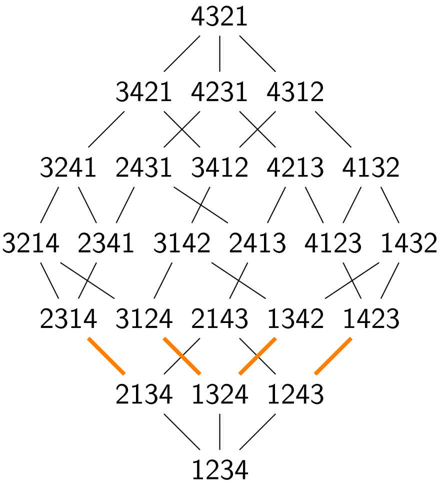


$2134 \equiv 2314 \equiv 2341$

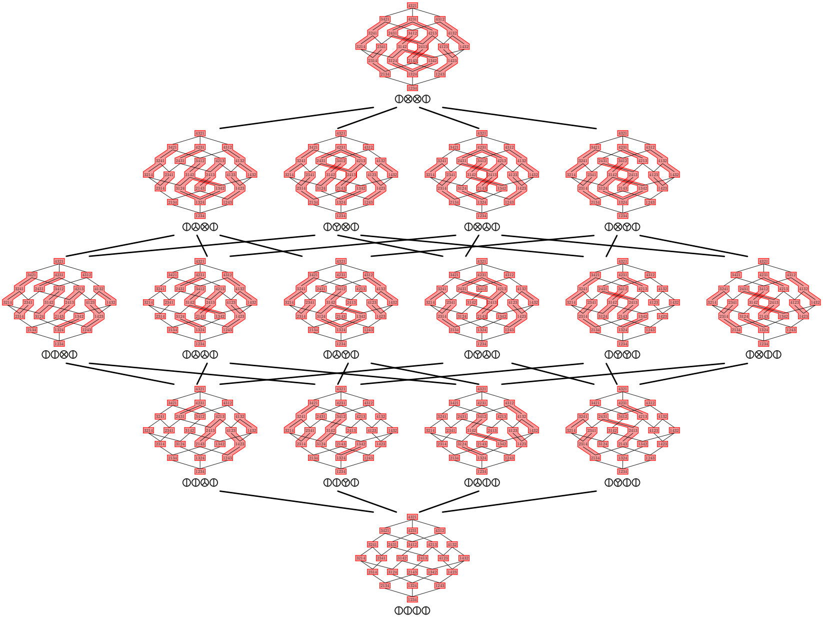
$2143 \equiv 2413 \equiv 2431$

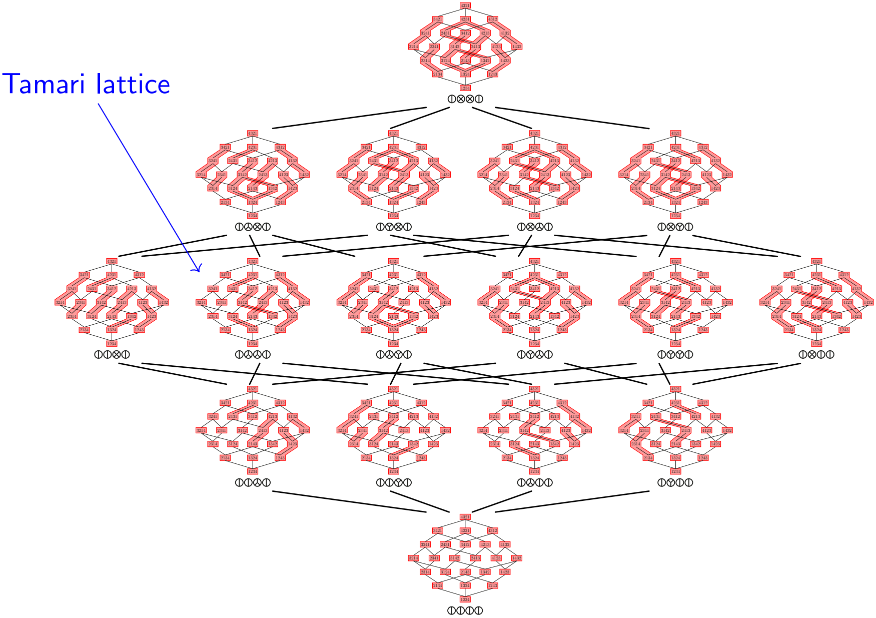
$3214 \equiv 3241$

$4213 \equiv 4231$



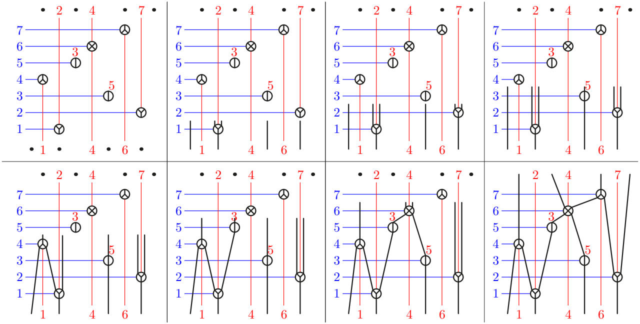
Permutree congruences



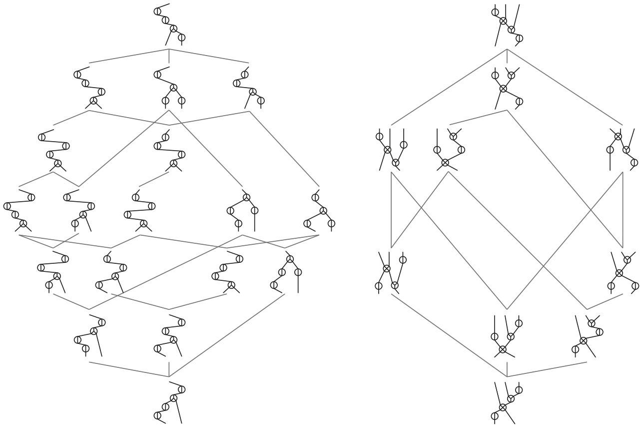

Tamari lattice

Permutree insertion (Pilaud, P. 2018)

Generalizing the binary search tree insertion



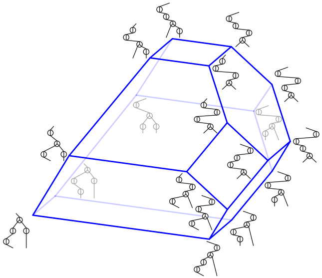
Permutree lattices



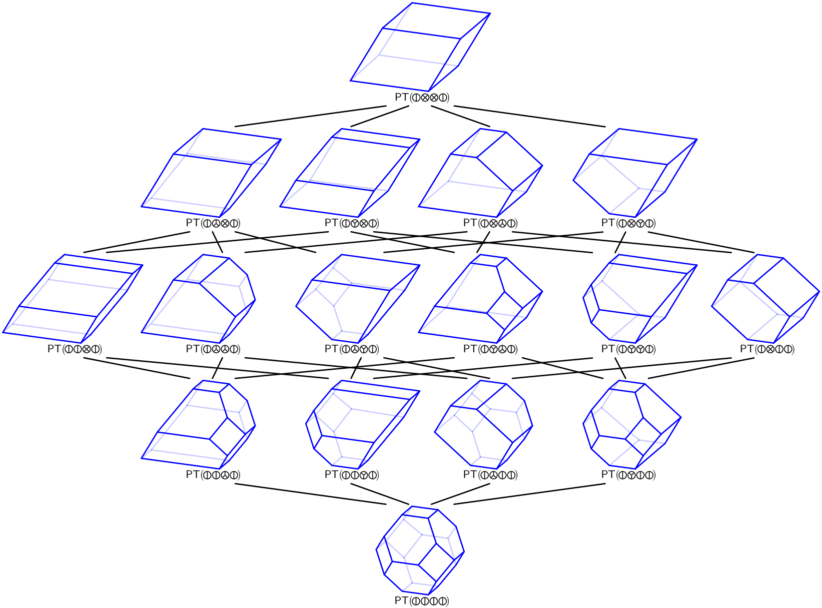
Permutreehedra

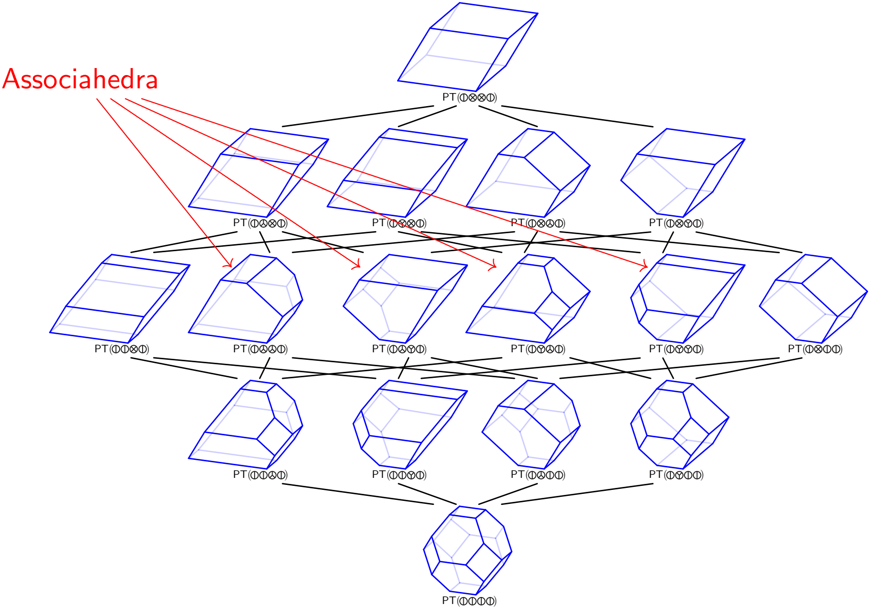
Theorem (Pilaud, P. 2018)

For each decoration, the permutree congruence can be realized as a polytope called the permutreehedron, obtained by **removing** *facets of the permutahedron.*



Albertin, Pilaud, Ritter 2021 → they are the **only** lattice congruences to have this property

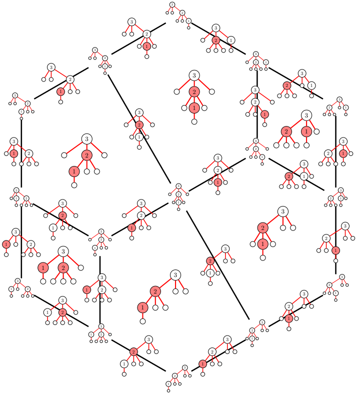


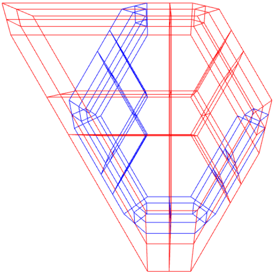

Associahedra

Recent work and perspectives

- thesis of Daniel Tamayo
  - permutrees via automata (Pilaud, P., Tamayo 2023)
  - cubic realizations
  - realizations via flow polytopes
- Perspective: other types of finite Coxeter groups

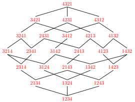
New friends: the *s*-weak order and *s*-Permutahedra



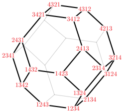


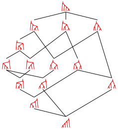
Motivation

Weak order

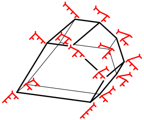


Permutahedron

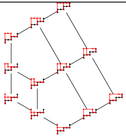




Tamari lattice

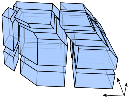


Associahedron



ν-Tamari

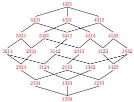
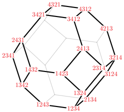
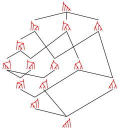
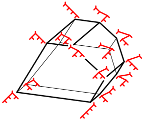
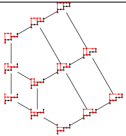
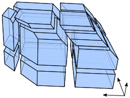
Préville-Ratelle, Viennot



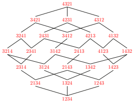
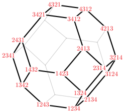
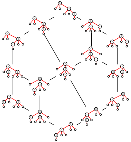
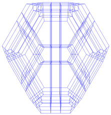
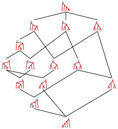
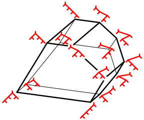
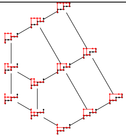
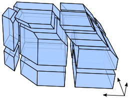
ν-Associahedron

Ceballos, Padrol, Sarmiento

Motivation

Weak order	Permutahedron	?	?
			
			
Tamari lattice	Associahedron	ν-Tamari	ν-Associahedron
		Préville-Ratelle, Viennot	Ceballos, Padrol, Sarmiento

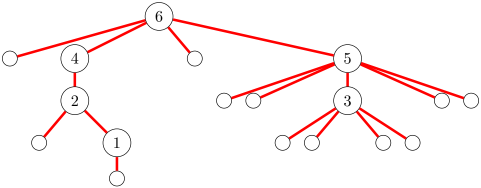
Motivation

Weak order	Permutahedron	s-Weak order	s-Permutahedron
			
			
Tamari lattice	Associahedron	ν-Tamari Préville-Ratelle, Viennot	ν-Associahedron Ceballos, Padrol, Sarmiento

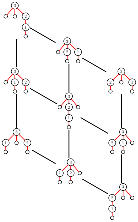
s-decreasing trees

- s is a sequence of non-negative integers
- The node i has $s(i)+1$ children
- Nodes labels decreasing from root to leaves

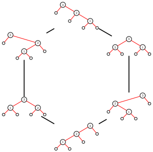
$$s = (0, 1, 3, 0, 4, 3)$$



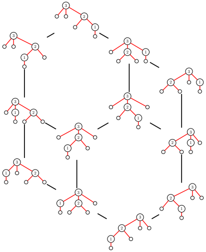
The *s*-weak order



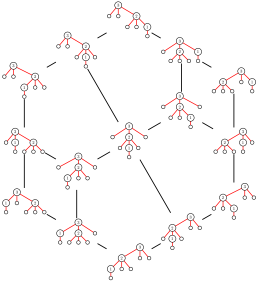
(0, 0, 2)



(1, 1, 1)



(0, 1, 2)



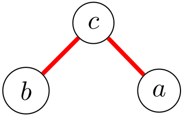
(0, 2, 2)

Theorem (Ceballos, P.)

The s-weak order is a lattice.

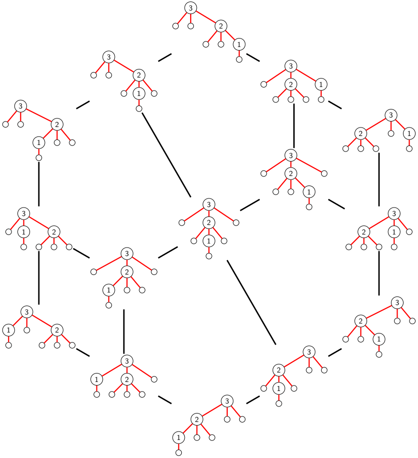
s-Tamari lattice

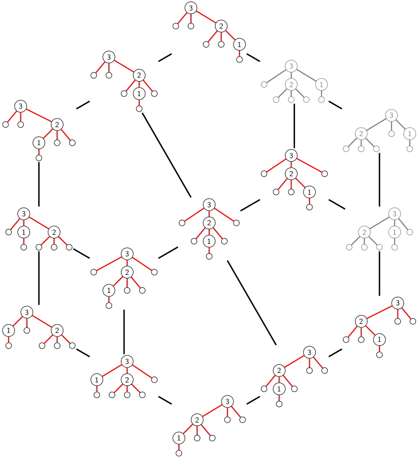
Select trees which avoid "pattern 231": $a < b < c$

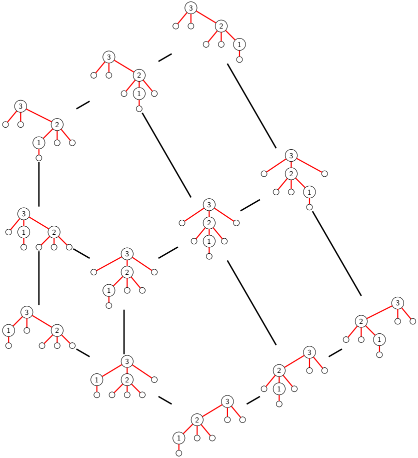


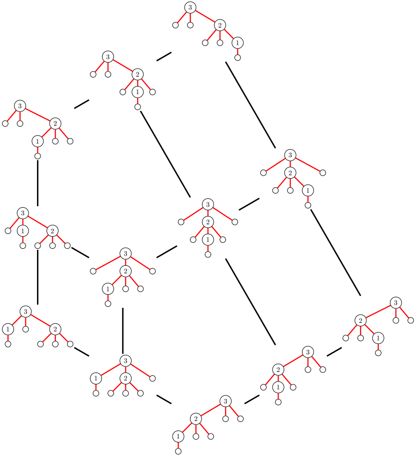
Theorem (Ceballos, P.)

The set of 231-avoiding s-decreasing trees form a sublattice, the s-Tamari lattice, isomorphic to the ν-Tamari lattice.





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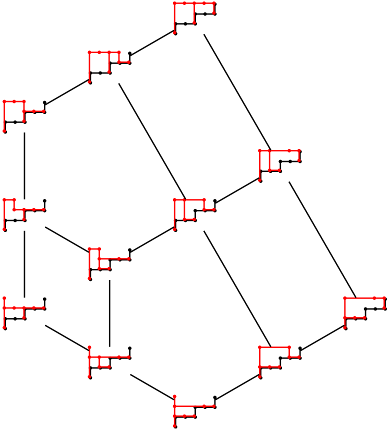
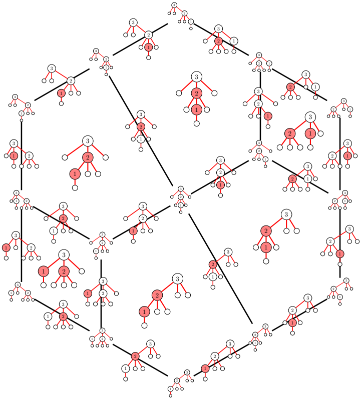
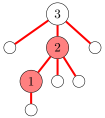


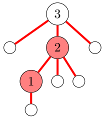
Geometry: the *s*-Permutahedron

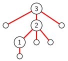


Faces : pure intervals

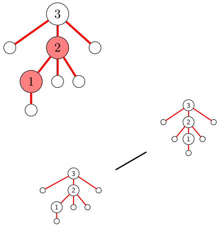


Faces : pure intervals





Faces : pure intervals



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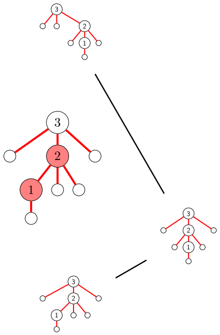

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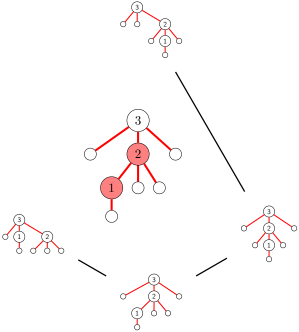

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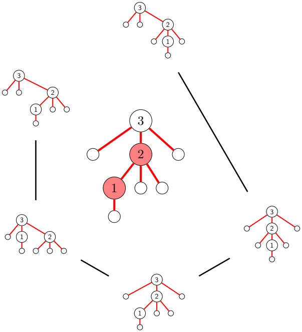

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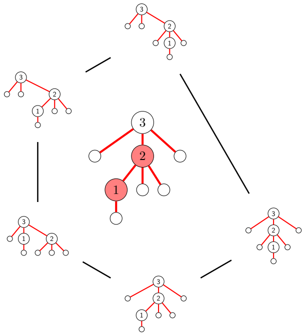

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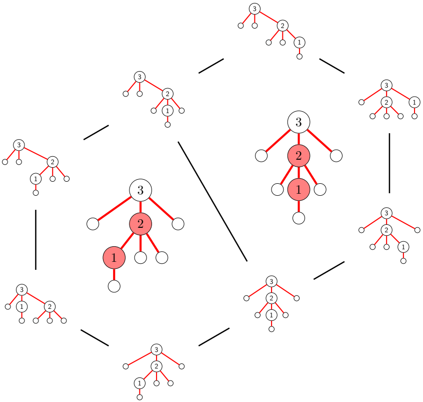

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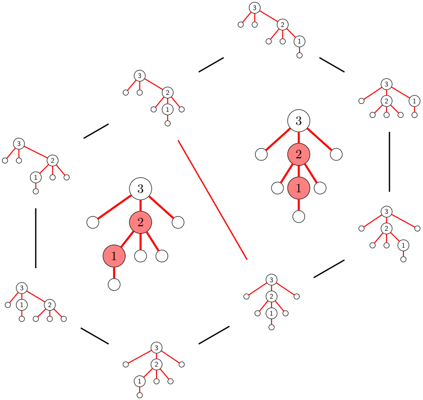

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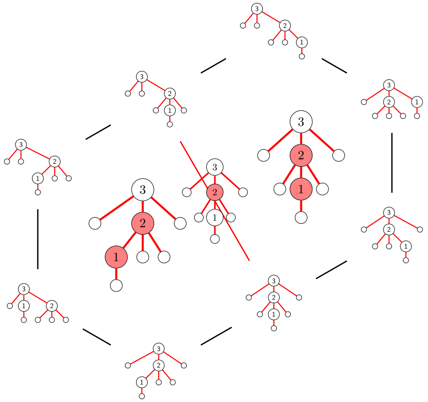
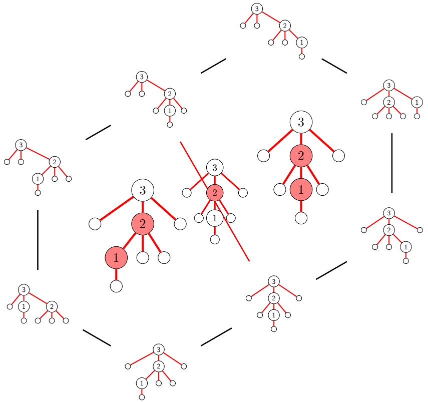


Faces : pure intervals



Theorem (Ceballos, P.)

The intersection of two pure intervals is a pure interval.

Conjecture (Ceballos, P.)

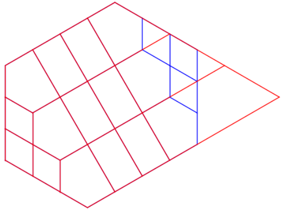
The s-Permutahedron is a polytopal complex

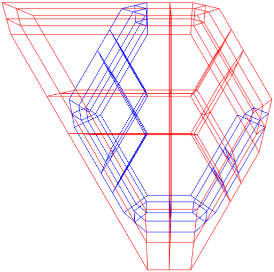
Conjecture (Ceballos, P.)

It can be realized as a polytopal subdivision of the classical permutahedron.

Conjecture (Ceballos, P.)

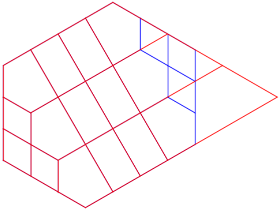
A realization of the ν-associahedron can be obtained by "removing" some facets of the s-permutahedron.





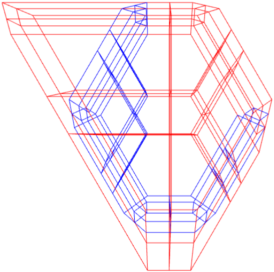
Conjecture (Ceballos, P.)

The s-Permutahedron is a polytopal complex



Conjecture (Ceballos, P.)

It can be realized as a polytopal subdivision of the classical permutahedron.



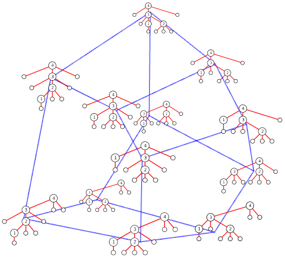
Conjecture (Ceballos, P.)

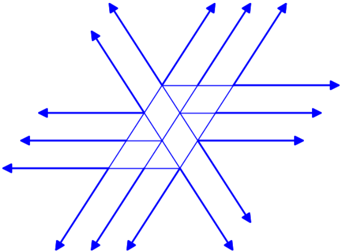
A realization of the ν-associahedron can be obtained by “removing” some facets of the s-permutahedron.

Solved for s without 0 (González D’Leon, Morales, Philippe, Tamayo and Yip)

Recent work and perspective

- Recent [GDMPTY23]: partially solved the conjectures via flow polytopes (see Tamayo's thesis)
- Perspectives:
  - can it be generalized to all s?
  - a direct realization without tropical geometry?
  - what about Conjecture 3 on ν-associahedron?
  - study the ascentopes: conjectural direct polytopal realizations of pure intervals
  - study the s-braid arrangement (the dual of s-permutahedra)
  - generalize to other Coxeter groups

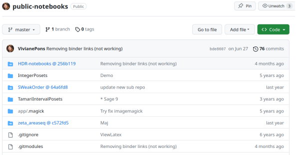


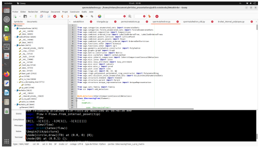


Closing remarks

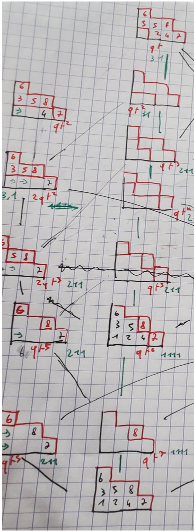
Methodology: computer exploration

- Computer exploration as an essential element of my research (using `SageMath`)
- Producing and sharing code
- using open licenses and advocating for open-source
- organizing community events, training other scientist, sharing good practices and knowledge





Perspectives and research philosophy



- Motivation: difficult open problems
  - q, t-Catalan symmetry
  - rotation distance in binary trees
  - chute moves lattice conjecture
  - ...
- Goal: solve? Maybe not... But construct new knowledge
- We create rich and inclusive research environments with active teams and open discussions
- 2 graduated PhD students, 1 in second year, 1 postdoc arriving next months: hoping to have many more

Thank you! Merci !

Tamari intervals and q, t-Catalan

- Châtel, P., Counting smaller elements in the Tamari and m-Tamari lattices. *JCTA*, 2015.
- P., The Rise-Contact involution on Tamari intervals. *Elec. J. of Comb.*, 2019.
- P., A description of the Zeta map on Dyck paths area sequences. arXiv:2205.06375, 2022.
- Le Mogne, P., Deficit and (q, t)-symmetry in triangular Dyck paths. *FPSAC*, 2023.

Integer posets and permutrees

- Pilaud, P., Permutrees. *ALCO*, 2018.
- Châtel, Pilaud, P., The weak order on integer posets. *ALCO*, 2019.
- Pilaud, P., The Hopf algebra of integer binary relations.*CARMA*, 2020.
- Pilaud, P., Tamayo, Permutree sorting.*ALCO*, 2023.

s-Weak order

- Ceballos, P., The s-weak order and s-permutahedra. *FPSAC*, 2019.
- Ceballos, P., The s-weak order and s-permutahedra I: combinatorics and lattice structure. arXiv:2212.11556, 2022.
- Ceballos, P., The s-weak order and s-permutahedra II: The combinatorial complex of pure intervals. arXiv:2309.14261, 2023.