Combinatoire algébrique liée aux ordres sur les permutations

Viviane Pons

Thèse de doctorat effectuée à l'université Paris-Est Marne-la-Vallée, sous la direction de
Jean-Christophe Novelli et **Jean-Yves Thibon**

7 octobre 2013

Définition

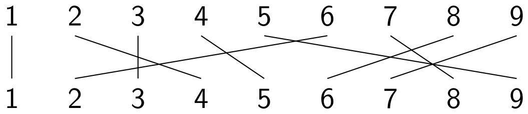
Une permutation est un mot de taille n sur l'alphabet $\{1, \ldots, n\}$ *où chaque lettre apparaît exactement une fois.*

Exemple : 143592867

Définition

Une permutation est un mot de taille n sur l'alphabet $\{1, \ldots, n\}$ où chaque lettre apparaît exactement une fois.

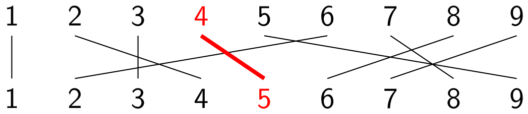
Exemple : 143592867



Définition

Une permutation est un mot de taille n sur l'alphabet $\{1, \ldots, n\}$ *où chaque lettre apparaît exactement une fois.*

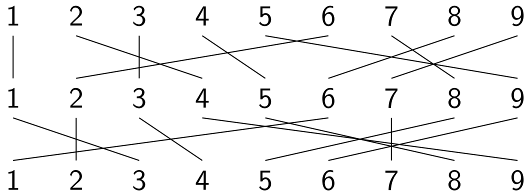
Exemple : 143592867



Structure de groupe

$\sigma = 143592867$

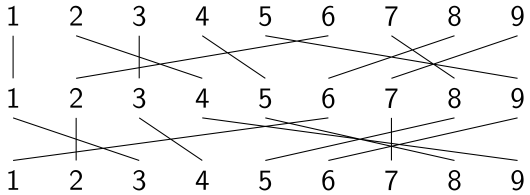
$\mu = 324981756$



Structure de groupe

$\sigma = 143592867$

$\mu = 324981756$

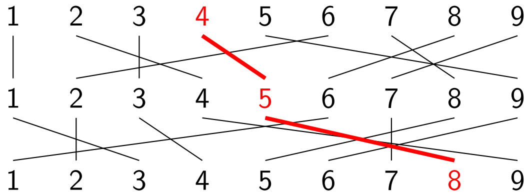


$$\mu.\sigma = 394862517$$

Structure de groupe

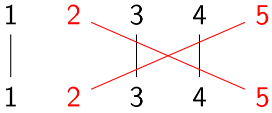
$\sigma = 143592867$

$\mu = 324981756$



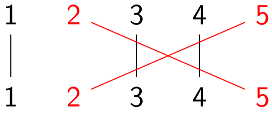
$\mu.\sigma = 394862517$

Transpositions



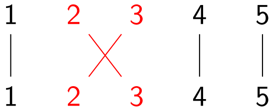
$(2, 5)$

Transpositions



$$(2,5)$$

Transpositions simples



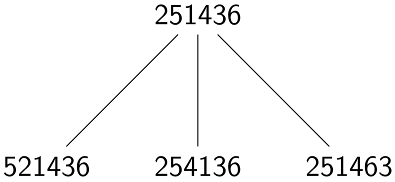
$$(2,3) = s_2$$

Ordre faible droit

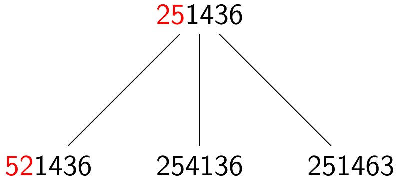
$$\ell(\sigma s_i) = \ell(\sigma) + 1$$

Ordre faible droit

$$\ell(\sigma s_i) = \ell(\sigma) + 1$$



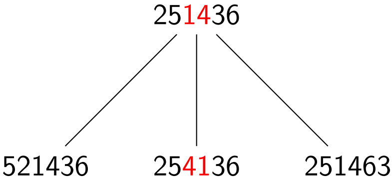
Ordre faible droit



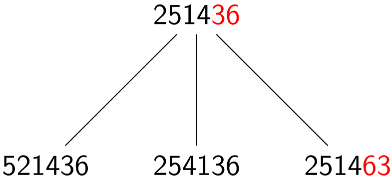
$$\ell(\sigma s_i) = \ell(\sigma) + 1$$

Ordre faible droit

$$\ell(\sigma s_i) = \ell(\sigma) + 1$$

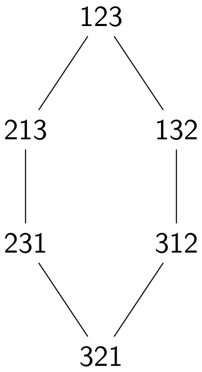


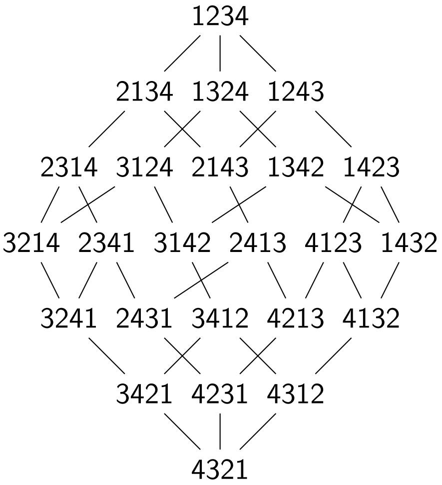
Ordre faible droit



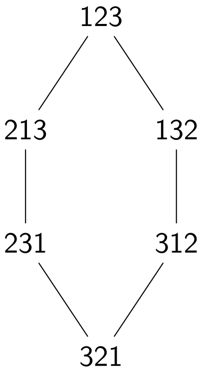
$$\ell(\sigma s_i) = \ell(\sigma) + 1$$

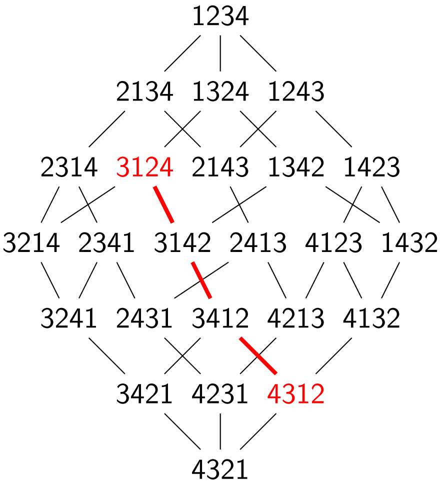
Ordre faible droit



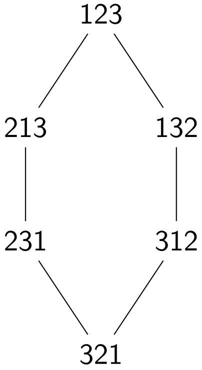


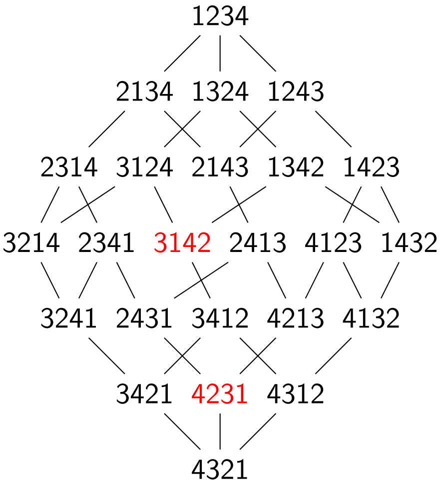
Ordre faible droit



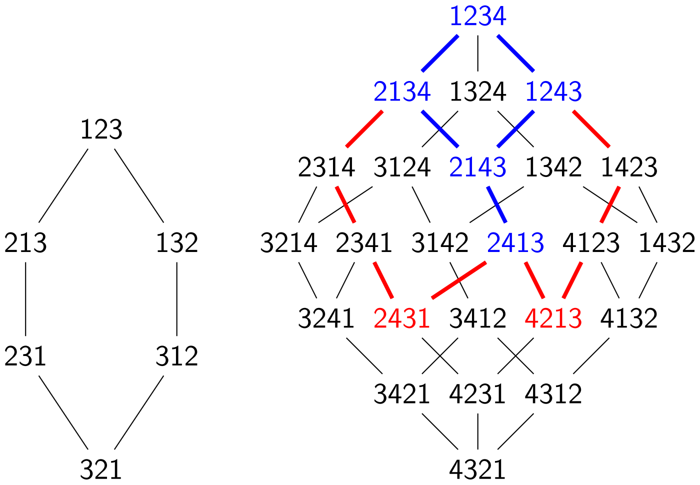


Ordre faible droit

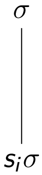




Ordre faible droit

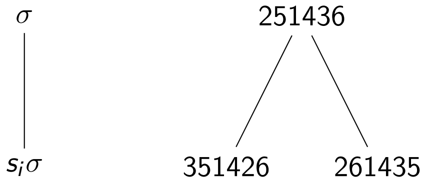


Ordre faible gauche



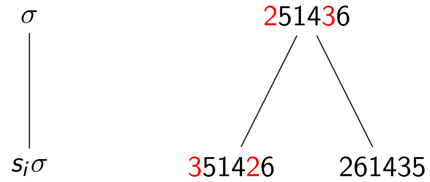
$$\ell(s_i\sigma) = \ell(\sigma) + 1$$

Ordre faible gauche



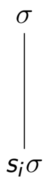
$$\ell(s_i\sigma) = \ell(\sigma) + 1$$

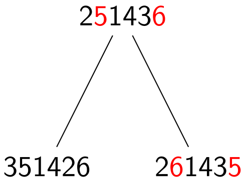
Ordre faible gauche



$$\ell(s_i\sigma) = \ell(\sigma) + 1$$

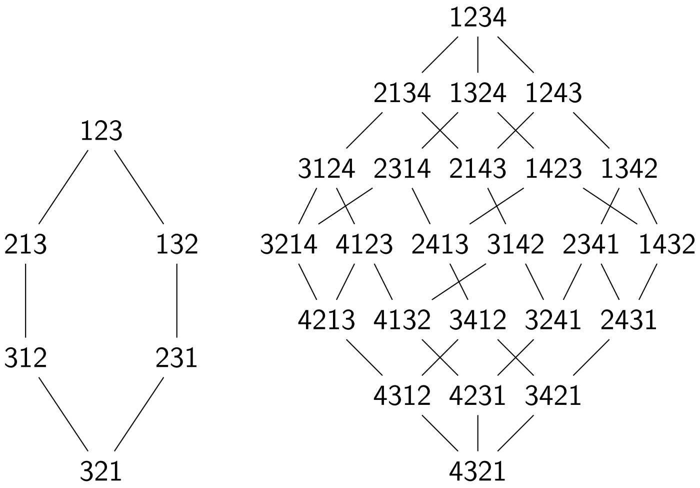
Ordre faible gauche



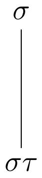


$$\ell(s_i\sigma) = \ell(\sigma) + 1$$

Ordre faible gauche

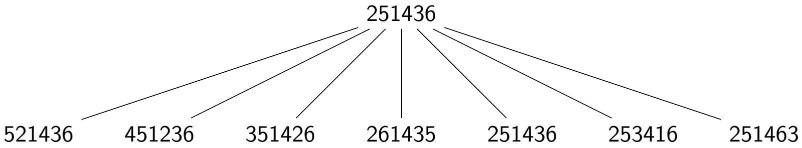


Ordre de Bruhat

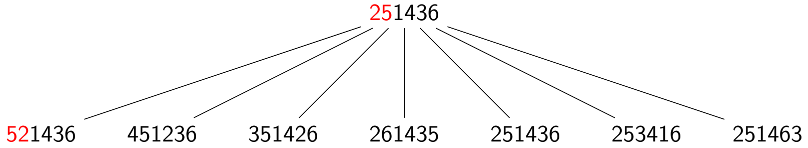


$$\ell(\sigma\tau) = \ell(\sigma) + 1$$

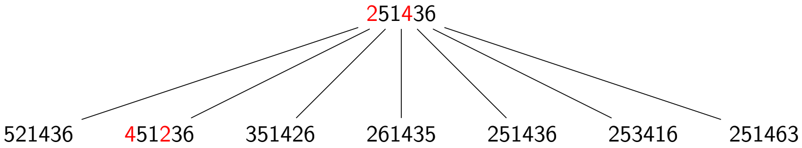
Ordre de Bruhat



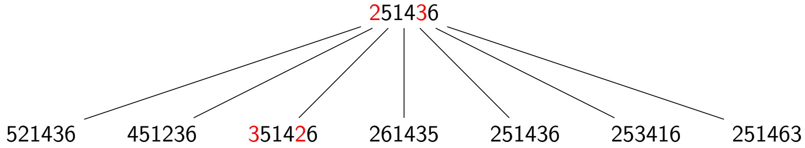
Ordre de Bruhat



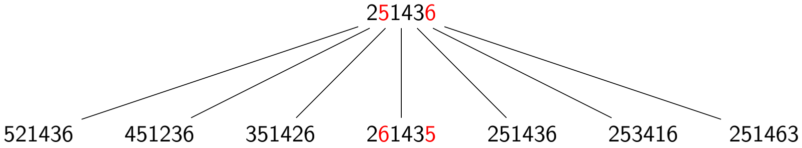
Ordre de Bruhat



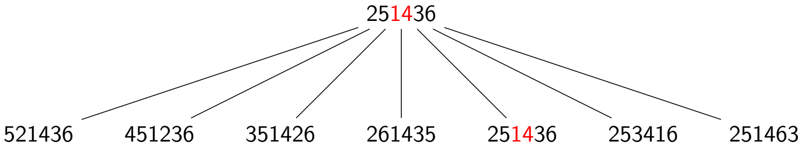
Ordre de Bruhat



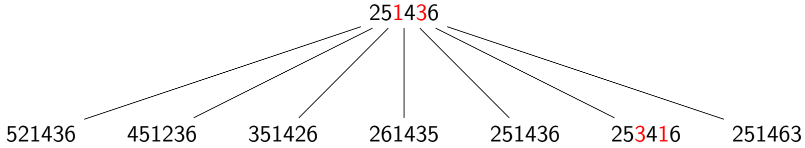
Ordre de Bruhat



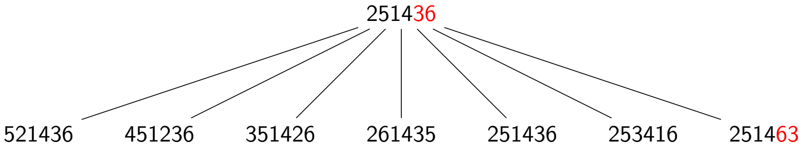
Ordre de Bruhat



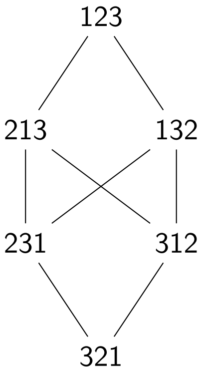
Ordre de Bruhat

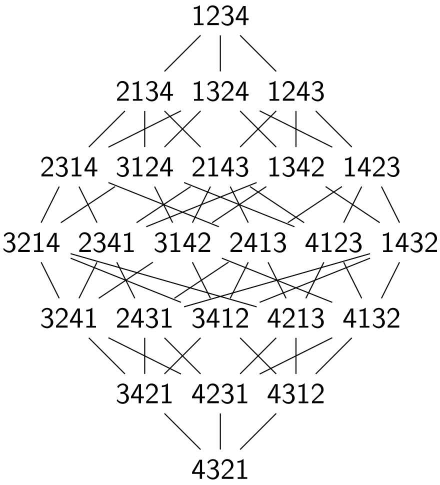


Ordre de Bruhat

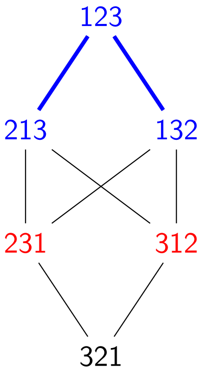


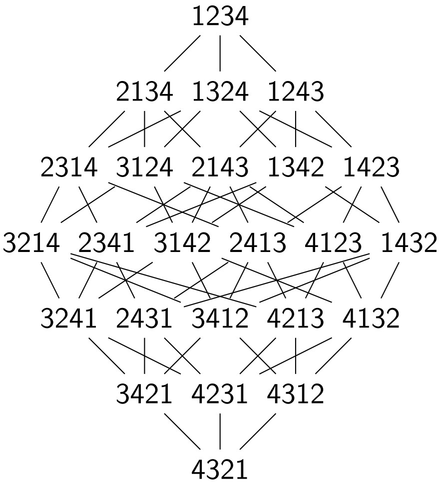
Ordre de Bruhat





Ordre de Bruhat





Ordre de Bruhat et polynômes multivariés

Ordre de Bruhat et polynômes multivariés

- Produit des polynômes de Grothendieck

Ordre de Bruhat et polynômes multivariés

- Produit des polynômes de Grothendieck

Théorème (P.)

Le produit $G_\sigma G_{s_k}$ se développe dans la base des G comme une somme alternée sur un intervalle de l'ordre de Bruhat.

Ordre de Bruhat et polynômes multivariés

- Produit des polynômes de Grothendieck

Théorème (P.)

Le produit $G_\sigma G_{s_k}$ se développe dans la base des G comme une somme alternée sur un intervalle de l'ordre de Bruhat.

- Implantation en Sage des bases des polynômes (Schubert, Grothendieck, etc.)

Treillis de Tamari

Treillis de Tamari

- 1962, Tamari : ordre sur les parenthésages formels

Treillis de Tamari

- 1962, Tamari : ordre sur les parenthésages formels
- 1972, Huang, Tamari : structure de treillis

Treillis de Tamari

- 1962, Tamari : ordre sur les parenthésages formels
- 1972, Huang, Tamari : structure de treillis
- 2007, Chapoton : nombre d'intervalles

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$$

Treillis de m-Tamari

Treillis de m-Tamari

- Bergeron, Préville-Ratelle : posets de m-Tamari

Treillis de m-Tamari

- Bergeron, Préville-Ratelle : posets de m-Tamari
- Bousquet-Mélou, Fusy, Préville-Ratelle : nombre d'intervalles

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1}$$

Arbres binaires

Définition récursive :

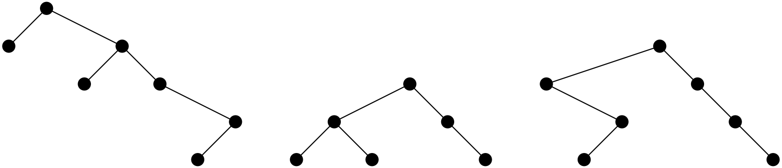
- l'arbre vide ou
- une racine possédant 2 sous-arbres (gauche et droit)

Arbres binaires

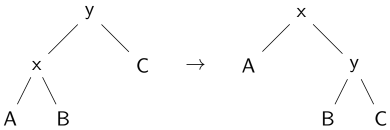
Définition récursive :

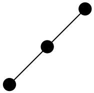
- l'arbre vide ou
- une racine possédant 2 sous-arbres (gauche et droit)

Exemples

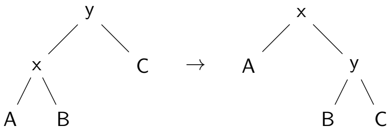


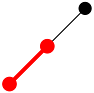
Rotation droite



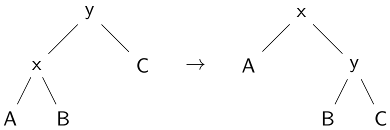


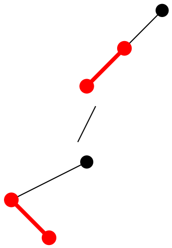
Rotation droite



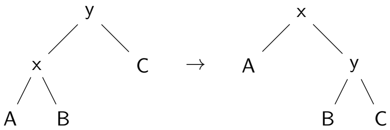


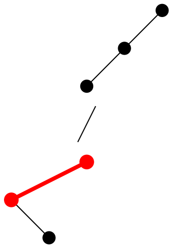
Rotation droite



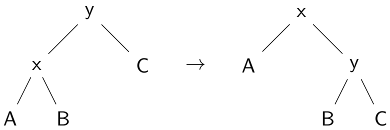


Rotation droite

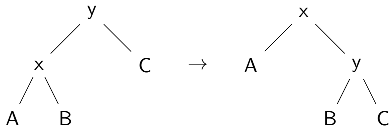


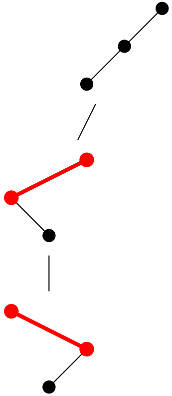


Rotation droite

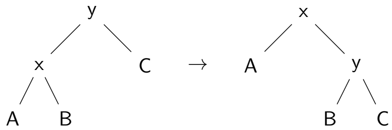


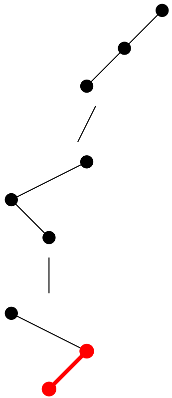
Rotation droite



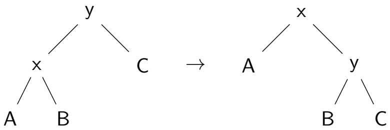


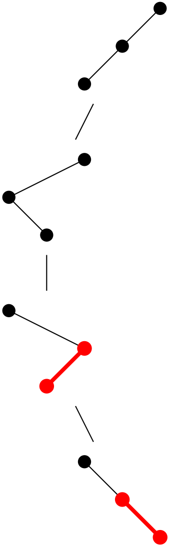
Rotation droite



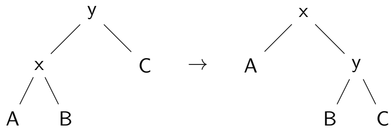


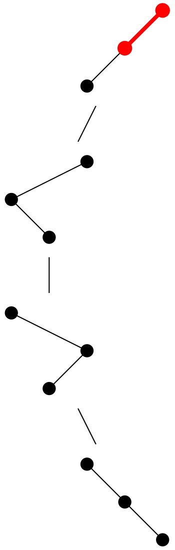
Rotation droite



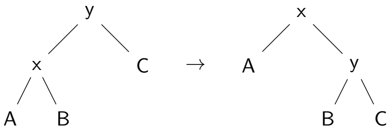


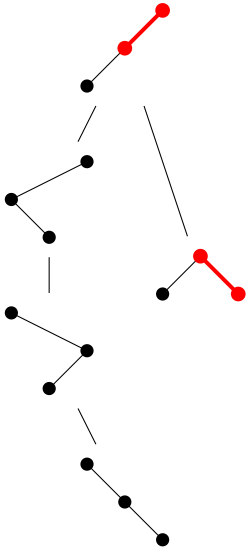
Rotation droite



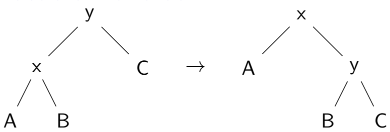


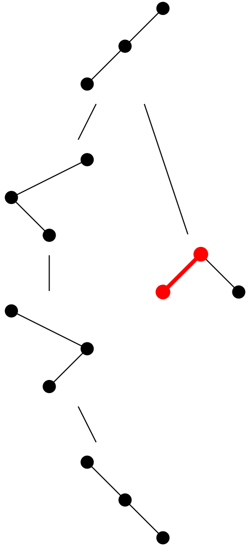
Rotation droite



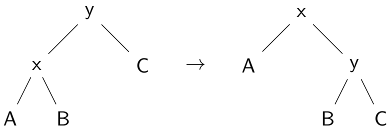


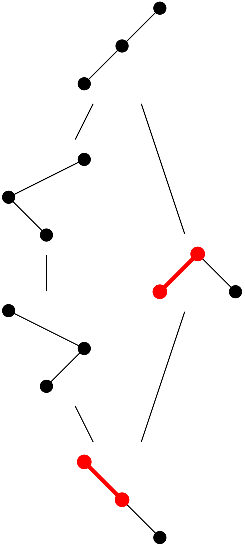
Rotation droite



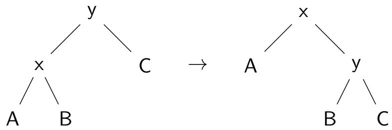


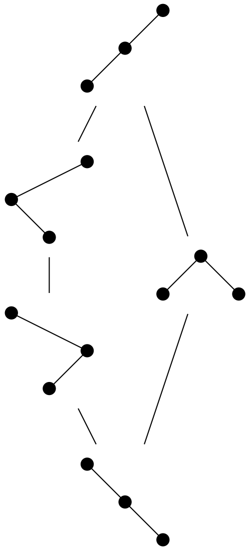
Rotation droite

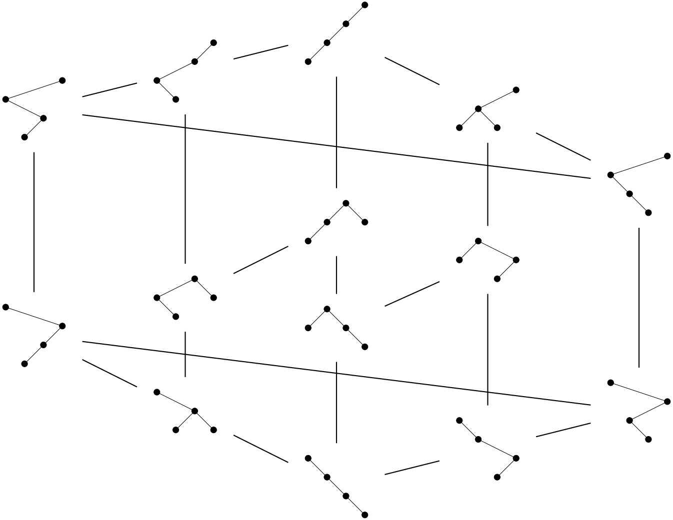




Rotation droite

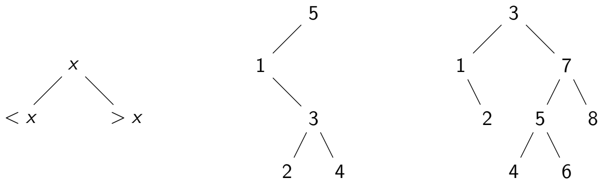






Lien avec l'ordre faible

Étiquetage canonique



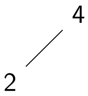
Insertion dans un arbre binaire de recherche

4

15324 $\rightarrow$

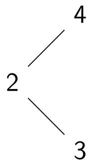
Insertion dans un arbre binaire de recherche

$15324 \rightarrow$



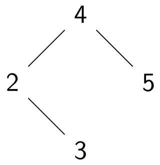
Insertion dans un arbre binaire de recherche

15324 →



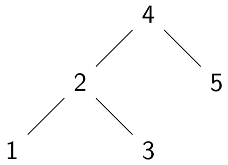
Insertion dans un arbre binaire de recherche

15324 $\rightarrow$



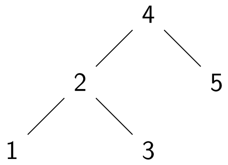
Insertion dans un arbre binaire de recherche

15324 $\rightarrow$

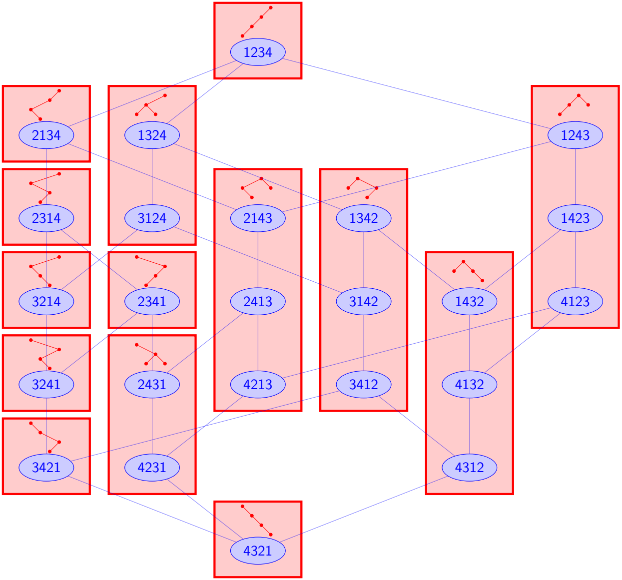


Insertion dans un arbre binaire de recherche

15324 $\rightarrow$



Caractérisation : les permutations qui correspondent à un arbre donné sont ses extensions linéaires
$15324, 31254, 35124, 51324, \ldots$


1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

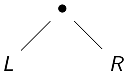
Polynômes de Tamari

On définit récursivement $\mathcal{B}_T$ par

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$

avec T =



Théorème (Châtel, P.)

$\mathcal{B}_T$ compte le nombre d'arbres inférieurs ou égaux à T dans le treillis de Tamari en fonction du nombre de nœuds sur leur branche gauche.

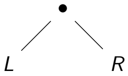
Polynômes de Tamari

On définit récursivement $\mathcal{B}_T$ par

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

avec T =



Théorème (Châtel, P.)

$\mathcal{B}_T$ compte le nombre d'arbres inférieurs ou égaux à T dans le treillis de Tamari en fonction du nombre de nœuds sur leur branche gauche.

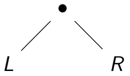
Polynômes de Tamari

On définit récursivement $\mathcal{B}_T$ par

$$\mathcal{B}_\emptyset := 1$$

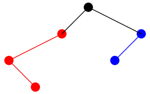
$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

avec T =



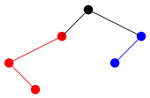
Théorème (Châtel, P.)

$\mathcal{B}_T$ compte le nombre d'arbres inférieurs ou égaux à T dans le treillis de Tamari en fonction du nombre de nœuds sur leur branche gauche.



$$\mathcal{B}_{\emptyset} := 1$$

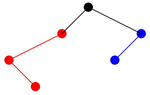
$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$



$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

$$\mathcal{B}_L(x) = x^3 + x^2$$

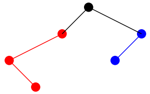


$$\mathcal{B}_{\emptyset} := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1}$$

$$\mathcal{B}_L(x) = x^3 + x^2$$

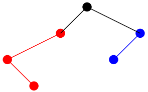
$$\mathcal{B}_R(x) = x^2$$



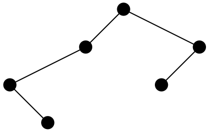
$$\mathcal{B}_\emptyset := 1$$

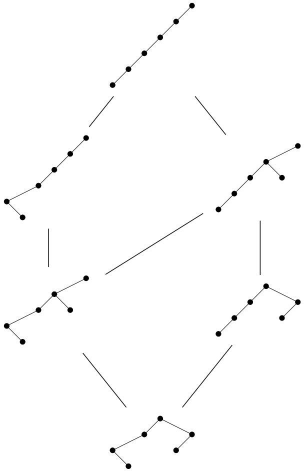
$$\mathcal{B}_T(x) := x(x^3 + x^2)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$

$$\mathcal{B}_R(x) = x^2$$

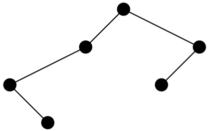


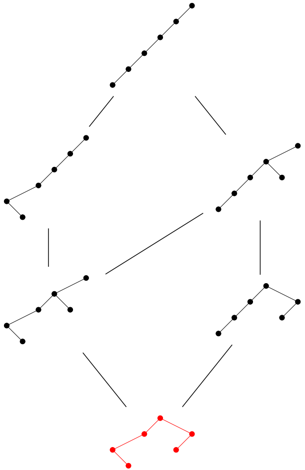
$$\mathcal{B}_\emptyset := 1$$
$$\mathcal{B}_T(x) := x(x^3 + x^2)(1 + x + x^2)$$



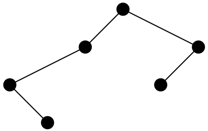


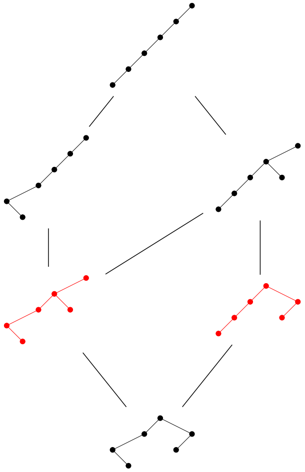
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



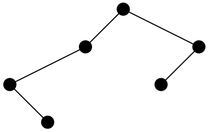


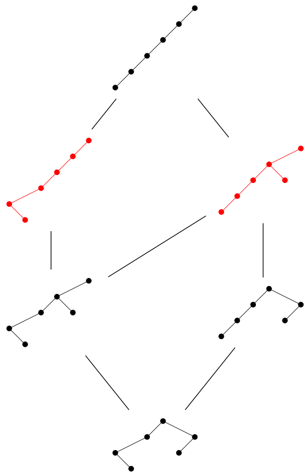
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



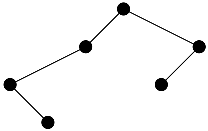


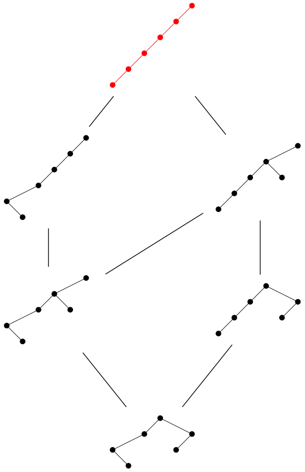
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



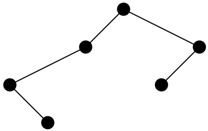


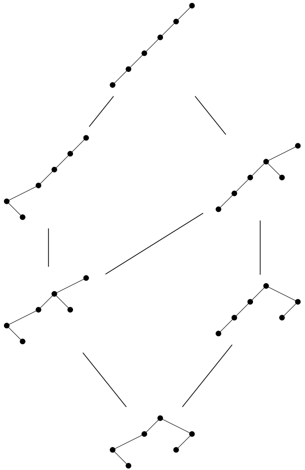
$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$



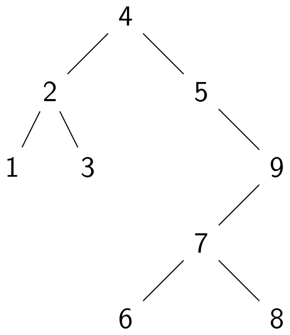


$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$

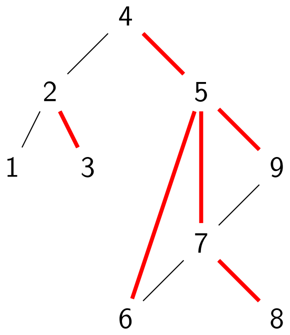


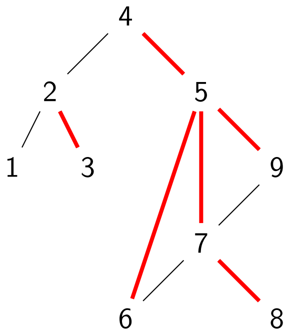


$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$
$$\mathcal{B}_T(1) = 6$$

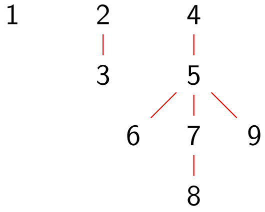

4
2
5
1
3
9
7
6
8

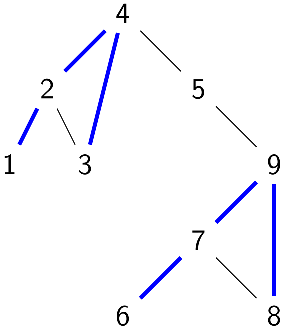
Forêt finale $F_{\geq}(T)$



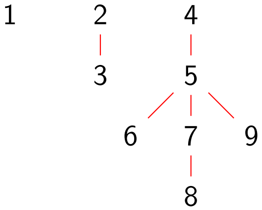


Forêt finale $F_{\geq}(T)$

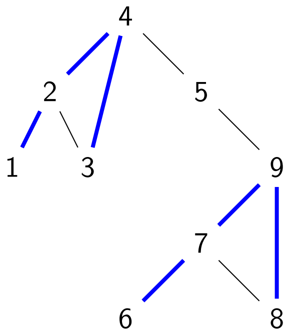




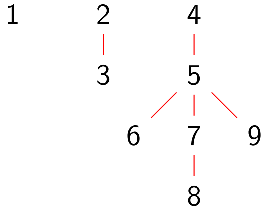
Forêt finale $F_{\geq}(T)$



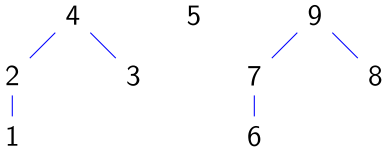
Forêt initiale $F_{\leq}(T)$

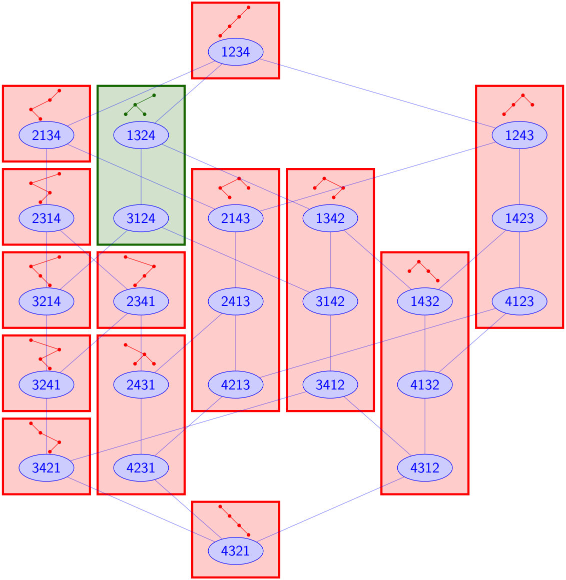


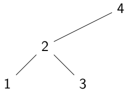
Forêt finale $F_{\geq}(T)$

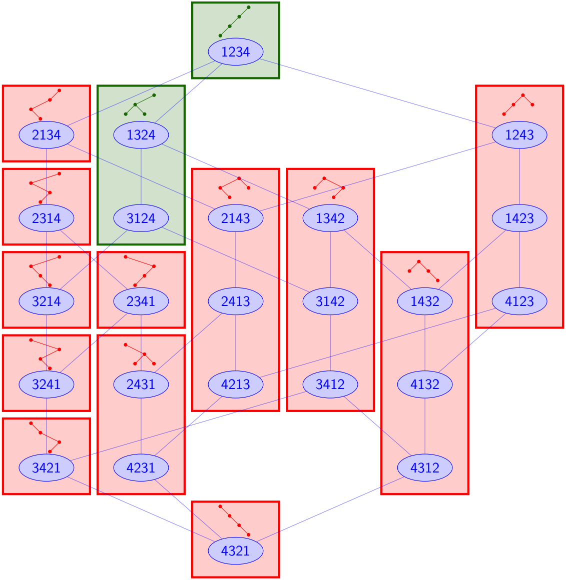


Forêt initiale $F_{\leq}(T)$

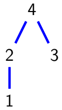


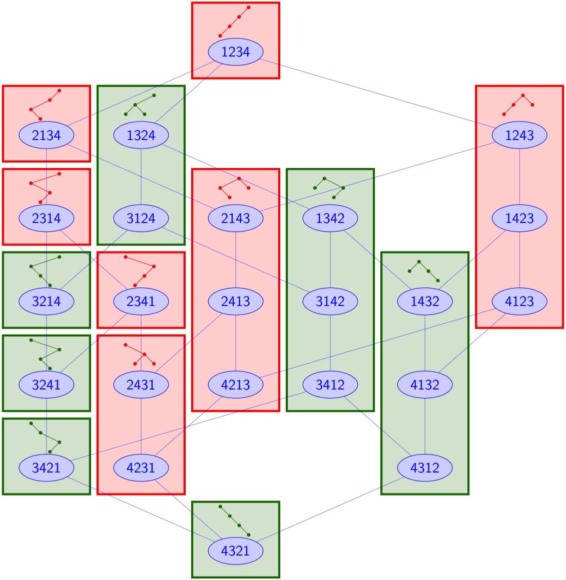

1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321


4
2
1
3


1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

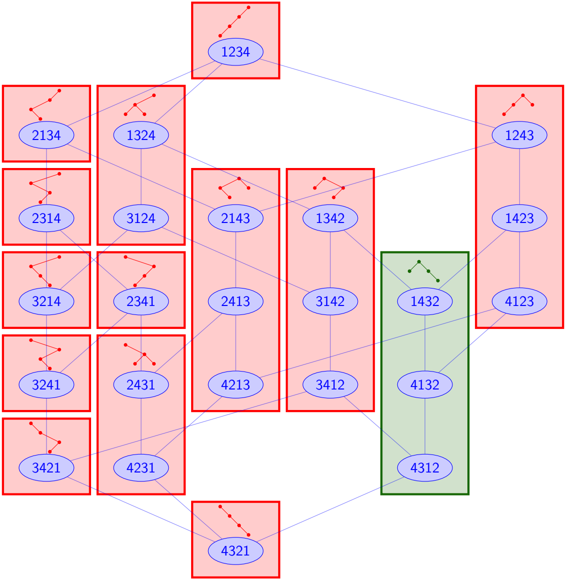
$F_{\leq}(T)$

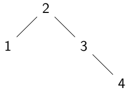

4
2
3
1

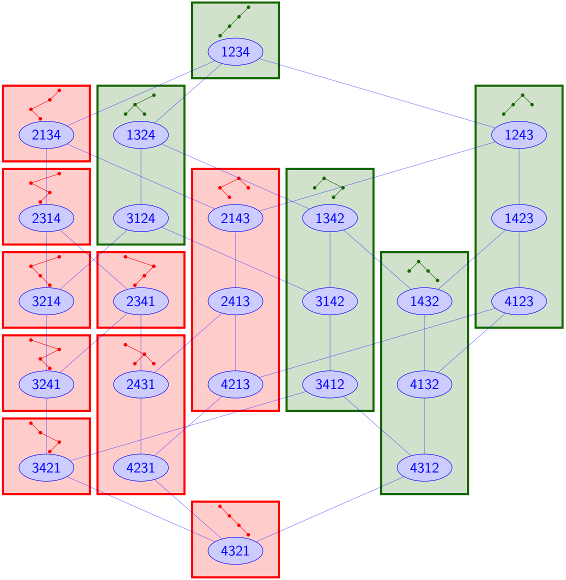

1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

$$F_{\geq}(T)$$

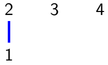

1
2
4
3

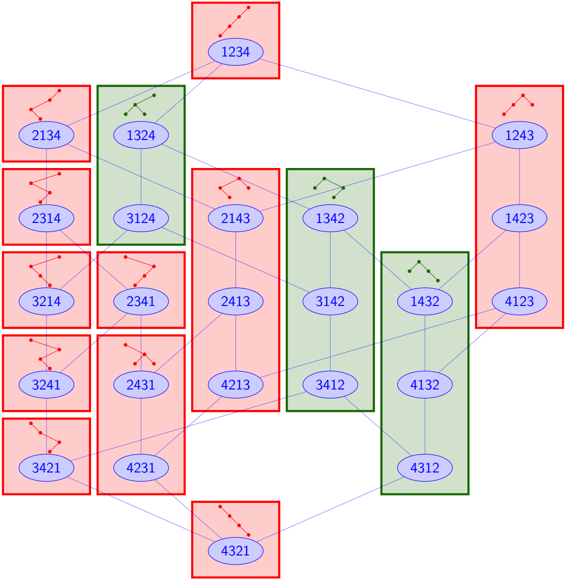

1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
3142
2413
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321


2
1
3
4


1234
2134
1324
1243
2314
3124
2143
1342
1423
3214
2341
2413
3142
1432
4123
3241
2431
4213
3412
4132
3421
4231
4312
4321

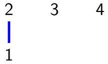
$F_{\leq}(T')$


2
3
4
1



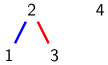
$F_{\geq}(T)$

$F_{\leq}(T')$



Intervalle-poset

$[T, T']$



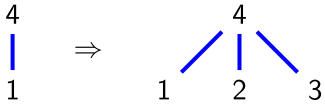
Théorème (Châtel, P.)

Les intervalles de Tamari sont en bijection avec les posets étiquetés sur $1, \dots, n$ *de taille* n *vérifiant :*

Théorème (Châtel, P.)

Les intervalles de Tamari sont en bijection avec les posets étiquetés sur $1, \dots, n$ *de taille* n *vérifiant :*

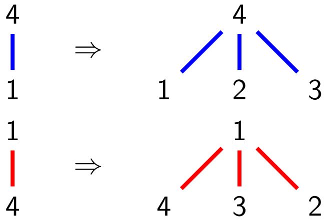
- *Si* $a < c$ *et* $a \lhd c$ *alors* $b \lhd c$ *pour tout* $a < b < c$.

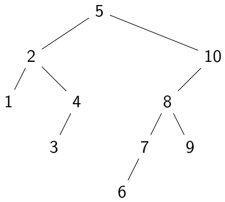


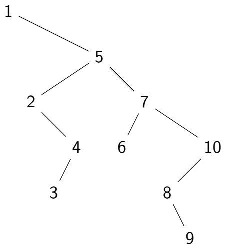
Théorème (Châtel, P.)

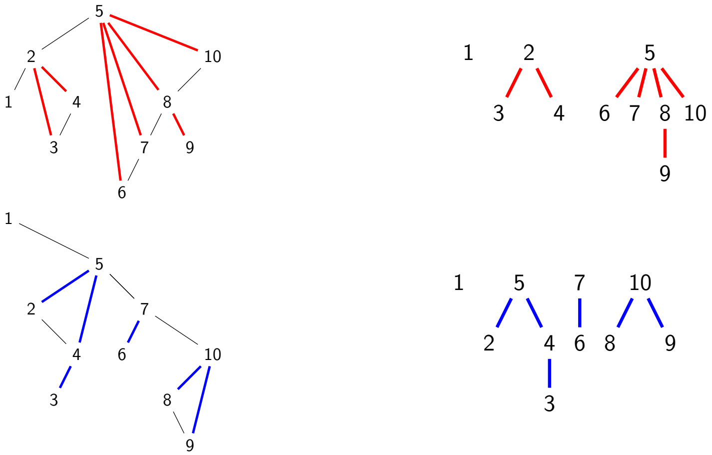
Les intervalles de Tamari sont en bijection avec les posets étiquetés sur $1, \ldots, n$ *de taille* n *vérifiant :*

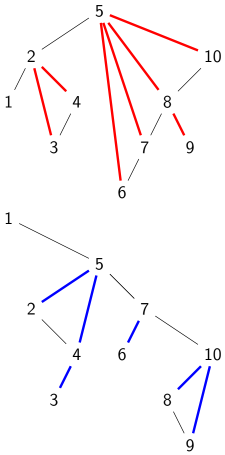
- *Si* $a < c$ *et* $a \lhd c$ *alors* $b \lhd c$ *pour tout* $a < b < c$.
- *Si* $a < c$ *et* $c \lhd a$ *alors* $b \lhd a$ *pour tout* $a < b < c$.

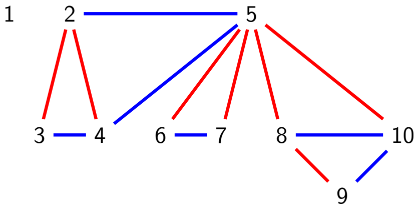


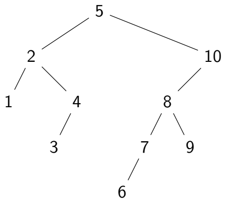

5
2
10
1
4
8
3
7
9
6

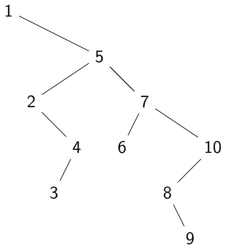

1
5
2
7
4
6
10
3
8
9

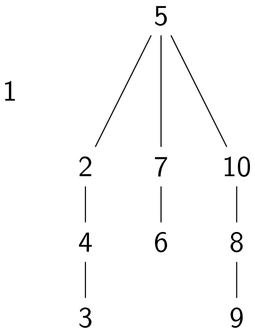

5
2 10
1 4 8
3 7 9
6
1 2 5
3 4 6 7 8 10
9
1
5
2 7
4 6 10
3 8
9
1 5 7 10
2 4 6 8 9
3

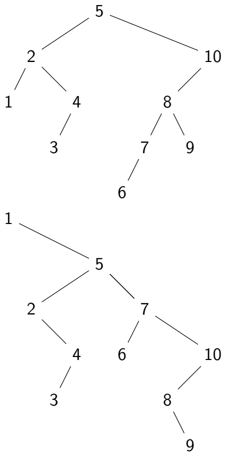

5
2
10
1
4
8
3
7
9
6
1
5
2
7
4
6
10
3
8
9

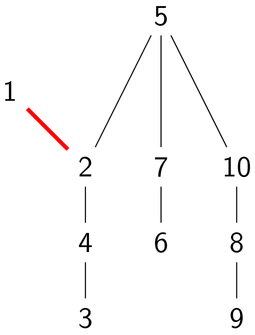

1
2
5
3
4
6
7
8
10
9

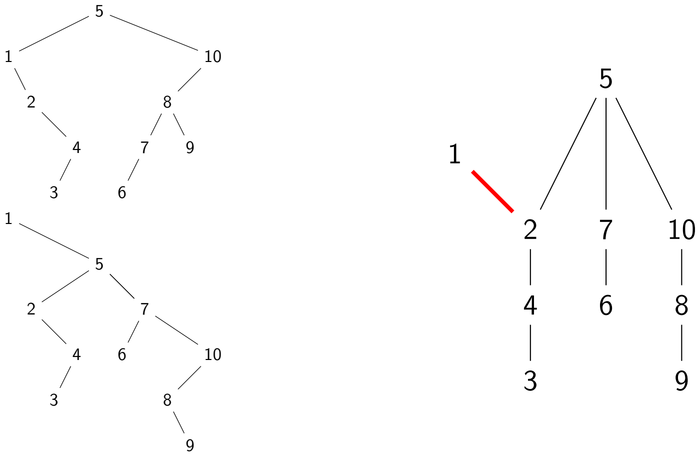

5
2
10
1
4
8
3
7
9
6

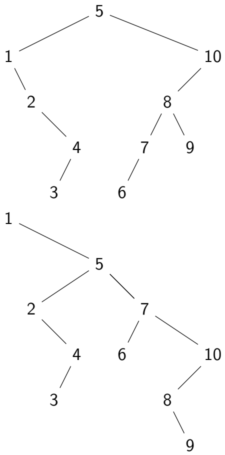

1
5
2
7
4
6
10
3
8
9

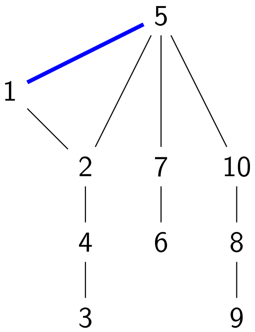

5
1
2
7
10
4
6
8
3
9

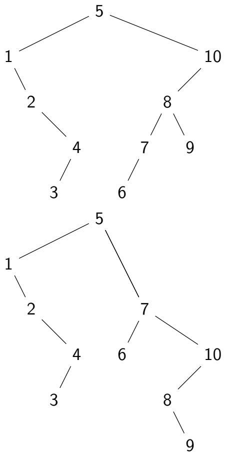

5
2
10
1
4
8
3
7
9
6
1
5
2
7
4
6
10
3
8
9

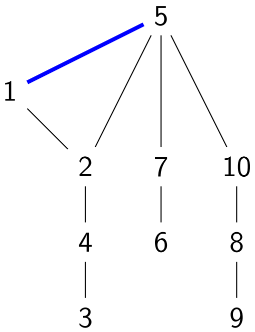

5
1
2
7
10
4
6
8
3
9

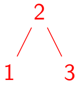

5
1
10
2
8
4
7
9
3
6
1
5
2
7
4
6
10
3
8
9
5
1
2
7
10
4
6
8
3
9

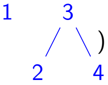

5
1
10
2
8
4
7
9
3
6
1
5
2
7
4
6
10
3
8
9


5
1
2
7
10
4
6
8
3
9

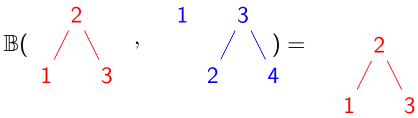

5
1
10
2
8
4
7
9
3
6
5
1
2
7
4
6
10
3
8
9

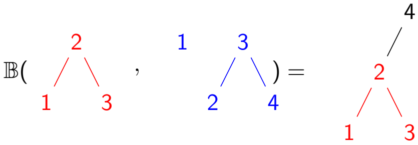

5
1
2
7
10
4
6
8
3
9

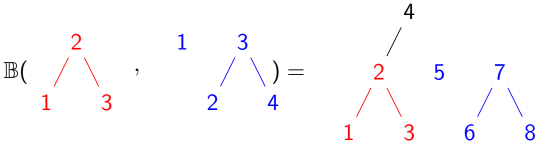
$\mathbb{B}($ 

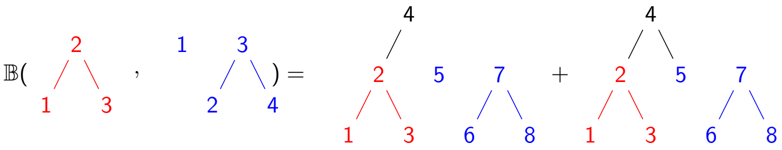
, 

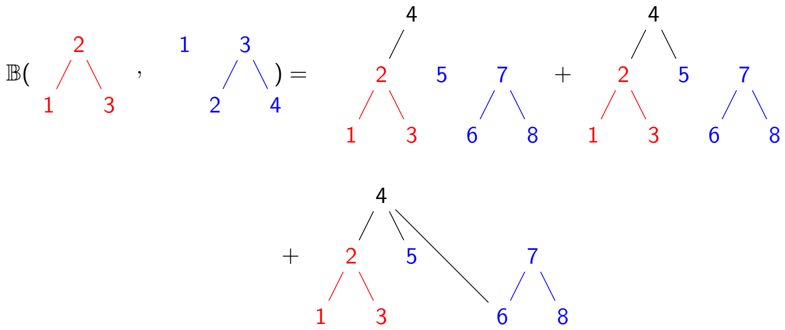
$) =$

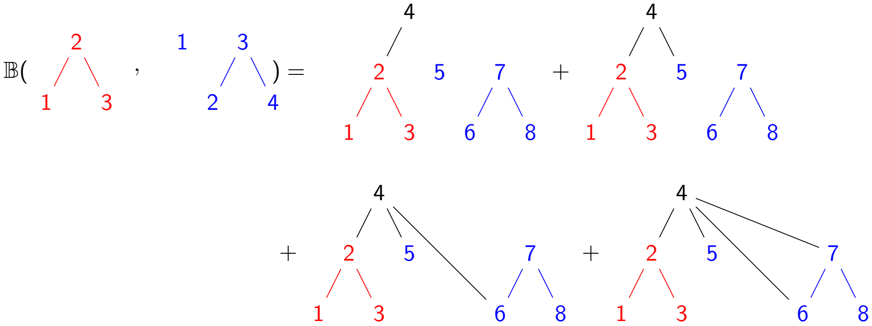

𝔹(
2
1
3
,
1
3
2
4
) =
2
1
3

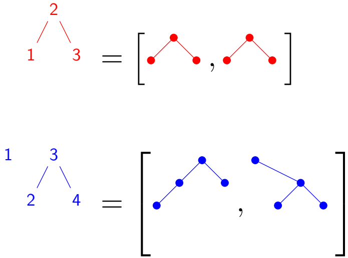

𝔹(
2
1
3
,
1
3
2
4
) =
4
2
1
3

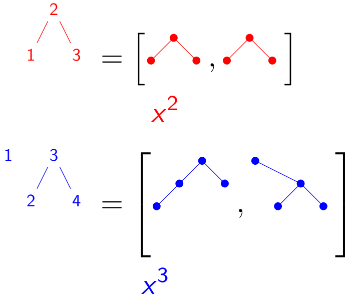

𝔹(
2
1 3
,
1 3
2 4
) =
4
2 5 7
1 3 6 8

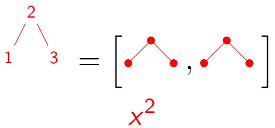

$\mathbb{B}($
,
$) =$
+
2
1
3
1
3
2
4
4
2
5
7
1
3
6
8
4
2
5
7
1
3
6
8

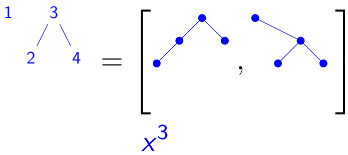

$\mathbb{B}($
$) =$
+
+

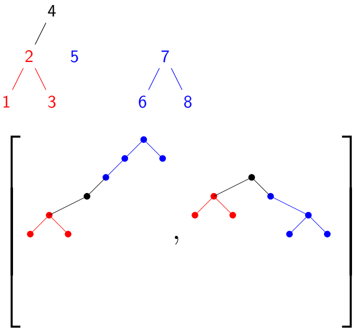

$\mathbb{B}($, $) =$
+
+
+

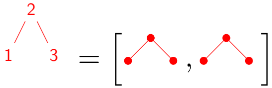

2
1
3
1
3
2
4


2
1 3
=
x2
1 3
2 4
=
x3

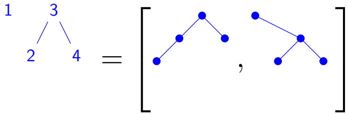

2
1
3
=
x^2


1
3
2
4
=
x^3

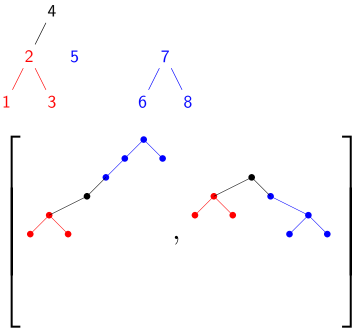

4
2
5
7
1
3
6
8



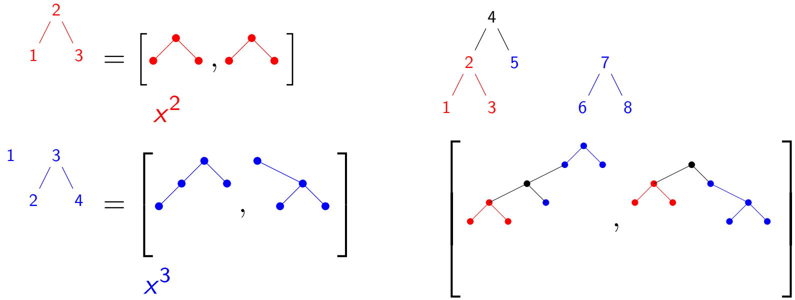
x^2



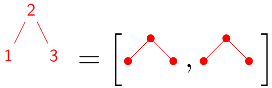
x^3



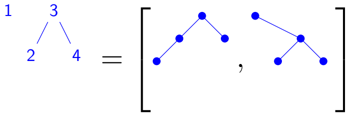
$x^2.x.x^3$



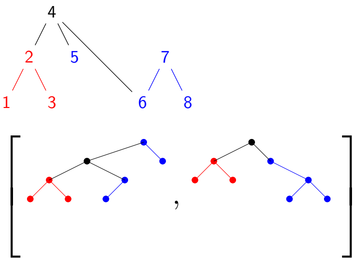
$$x^2.x.x^3 + x^2.x.x^2$$



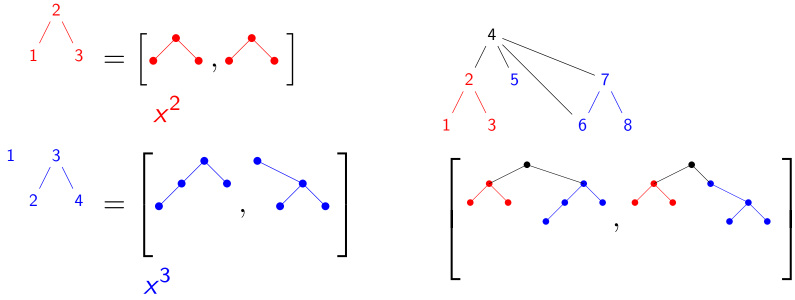
x^2



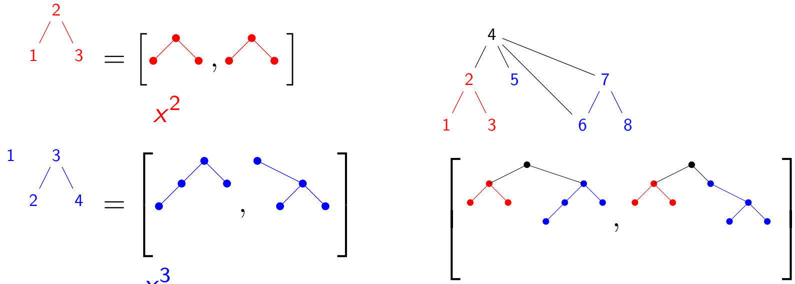
x^3



$$x^2.x.x^3 + x^2.x.x^2 + x^2.x.x$$



$$x^2.x.x^3 + x^2.x.x^2 + x^2.x.x + x^2.x$$



x^2

x^3

$$x^2.x.(x^3 + x^2 + x + 1)$$

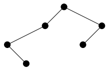
Théorème (Chapoton)

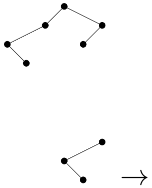
La série génératrice des intervalles de Tamari vérifie l'équation fonctionnelle

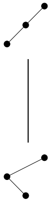
$$\Phi(x, y) = \mathrm{B}(\Phi, \Phi) + 1$$

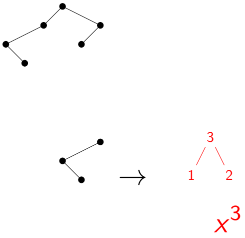
où

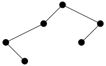
$$\mathrm{B}(f, g) = xyf(x, y)\frac{xg(x, y) - g(1, y)}{x - 1}$$

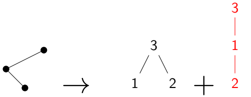




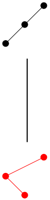


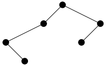

3
1
2
x^3

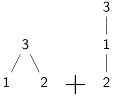




$$x^3 + x^2$$



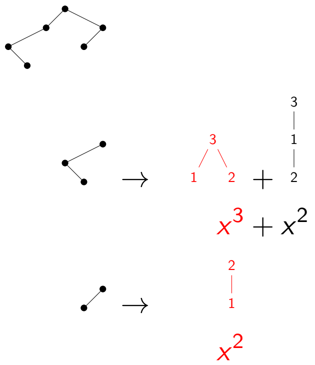


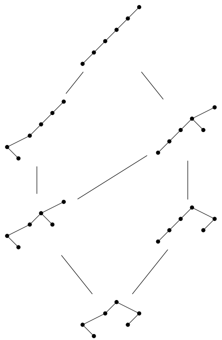
 $\rightarrow$ 

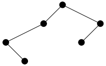
$$x^3 + x^2$$

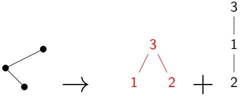
 $\rightarrow$ 

$$x^2$$

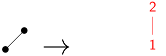

3
1
2
3
1
2
$x^3 + x^2$
2
1
x^2



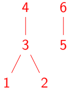




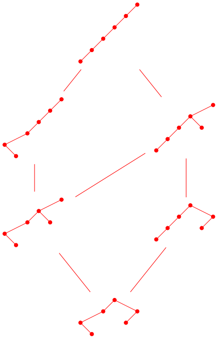
$$x^3 + x^2$$

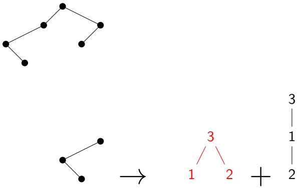


$$x^2$$

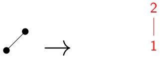


$$x^3.x.x^2$$

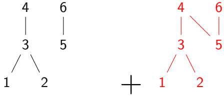




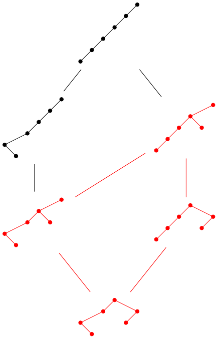
$$x^3 + x^2$$

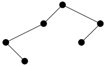


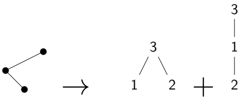
$$x^2$$



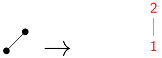
$$x^3.x.x^2 \quad + x^3.x.x$$



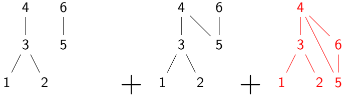




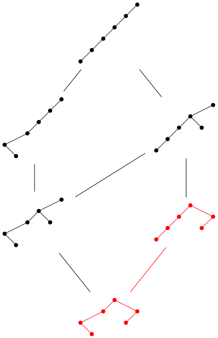
$$x^3 + x^2$$

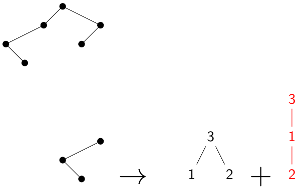


$$x^2$$

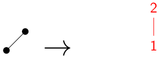


$$x^3.x.x^2 \quad + x^3.x.x + x^3.x$$

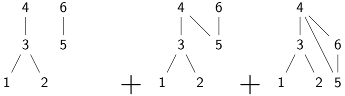




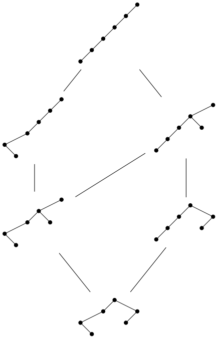
$$x^3 + x^2$$

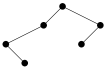


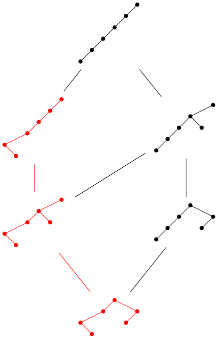
$$x^2$$

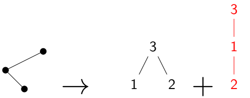


$$x^3.x.x^2 \quad + x^3.x.x + x^3.x$$

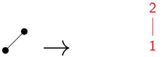




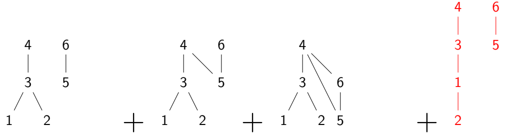




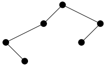
$$x^3 + x^2$$

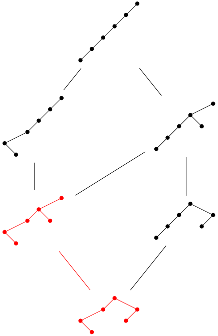


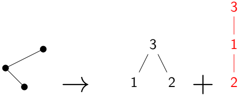
$$x^2$$



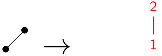
$$x^3.x.x^2 \quad + x^3.x.x + x^3.x \quad + x^2.x.x^2$$



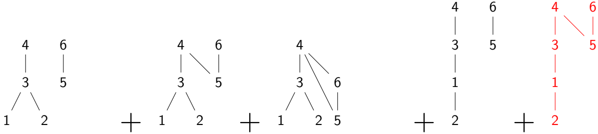




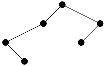
$$x^3 + x^2$$

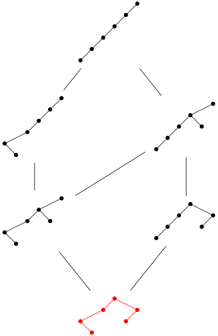


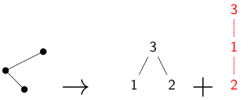
$$x^2$$



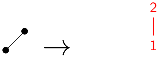
$$x^3.x.x^2 \quad + x^3.x.x + x^3.x \quad + x^2.x.x^2 + x^2.x.x$$



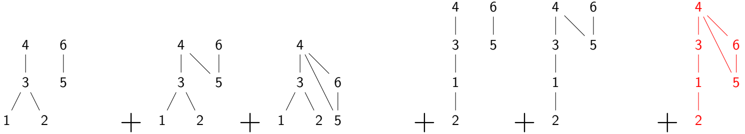




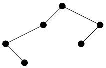
$$x^3 + x^2$$

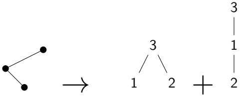


$$x^2$$

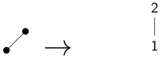


$$x^3.x.x^2 + x^3.x.x + x^3.x + x^2.x.x^2 + x^2.x.x + x^2.x$$

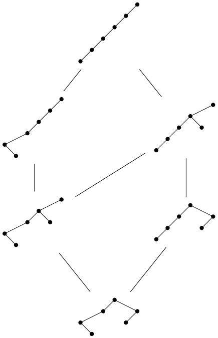




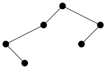
$$x^3 + x^2$$

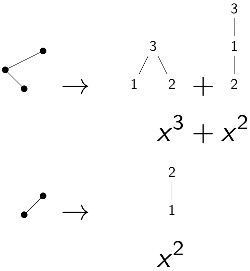


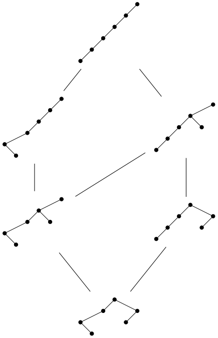
$$x^2$$



$$(x^3 + x^2).x.(x^2 + x + 1)$$







$$x\,\mathcal{B}_L(x)\frac{\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$

Posets de m-Tamari

(2011-2012) Bergeron, Préville-Ratelle, *Higher trivariate diagonal harmonics via generalized Tamari posets.*

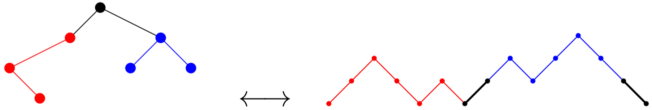
Posets de m-Tamari

(2011-2012) Bergeron, Préville-Ratelle, *Higher trivariate diagonal harmonics via generalized Tamari posets.*

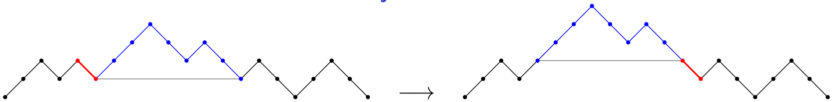
Structure de treillis, intervalles

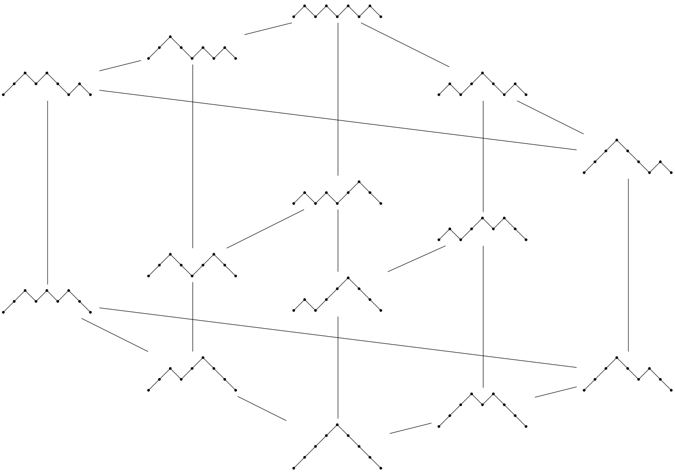
(2011) Bousquet-Mélou, Fusy, Préville-Ratelle, *The number of intervals in the m-Tamari lattices.*

Bijection arbres - chemins



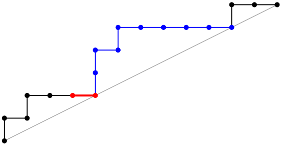
Rotation sur les chemins de Dyck



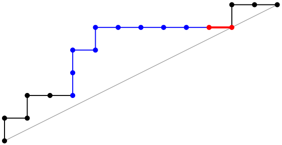


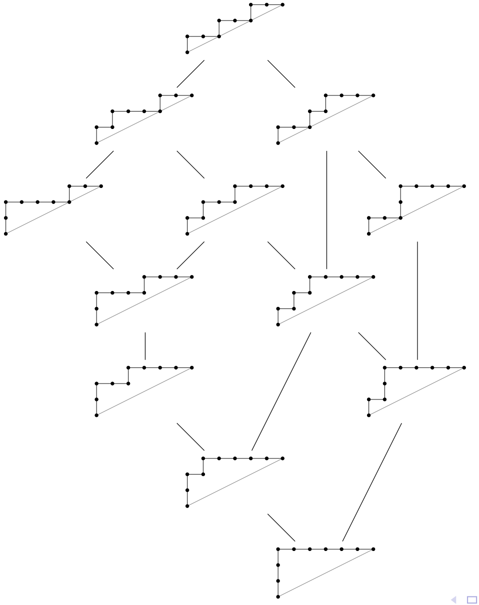
Chemins m-Ballots

Exemple $m = 2$.



$\longrightarrow$

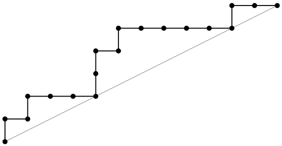




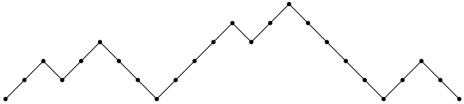
Les posets de *m*-Tamari sont des treillis

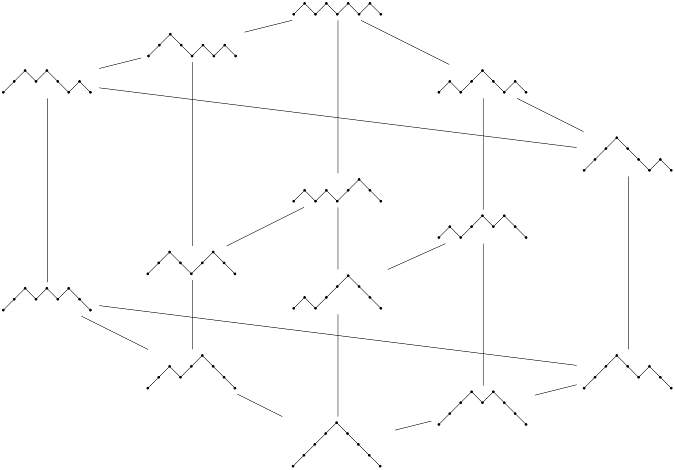
Bousquet-Mélou, Fusy, Préville-Ratelle : plongement des posets *m*-Tamari dans le treillis de Tamari $n \times m$.

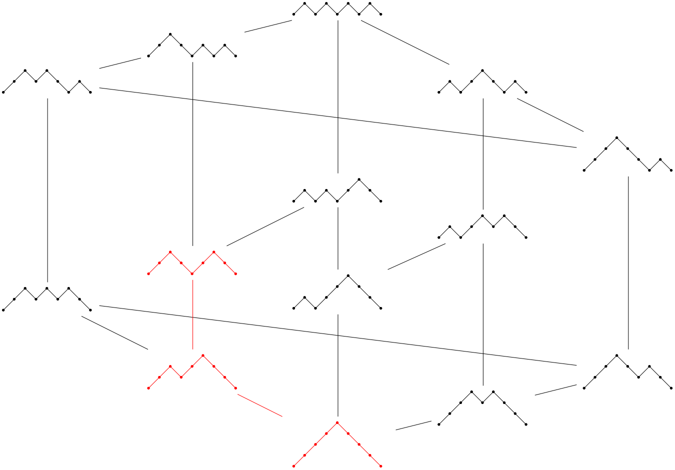
Chemin *m*-Ballot

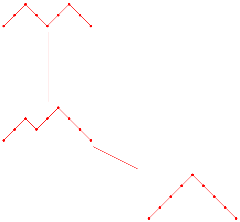


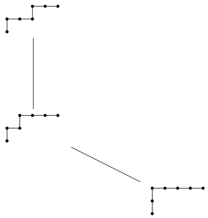
chemin de Dyck



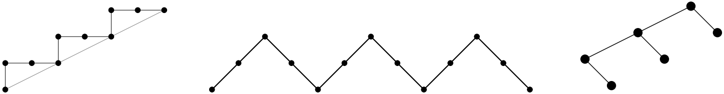


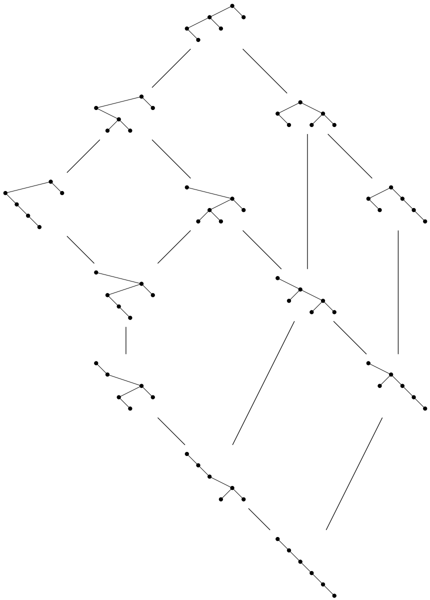




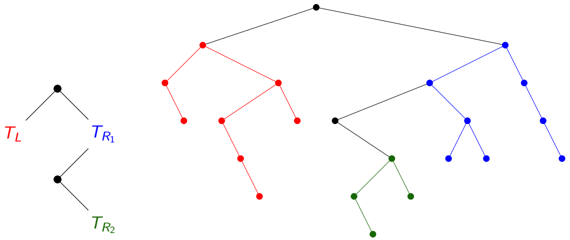


Arbres m-binaires





Structure ternaire



Cas du treillis de Tamari

(Chapoton)

$$\Phi = 1 + \mathrm{B}(\Phi, \Phi)$$
$$\mathrm{B}(f, g) = xf(x)\Delta(g)$$
$$\Delta(g) = \frac{xg(x) - g(1)}{x - 1}$$

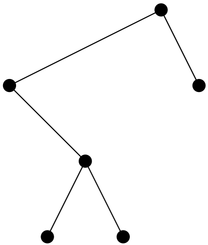
Cas des treillis *m*-Tamari

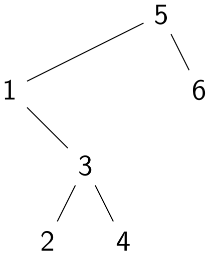
(Bousquet-Mélou, Fusy, Préville-Ratelle)

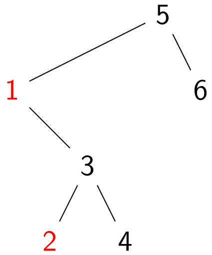
$$\Phi^{(2)} = 1 + \mathrm{B}^{(2)}(\Phi^{(2)}, \Phi^{(2)}, \Phi^{(2)})$$

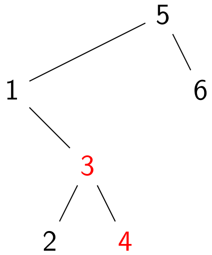
$$\mathrm{B}^{(2)}(f, g_1, g_2) = xf(x)\Delta(g_1\Delta(g_2))$$

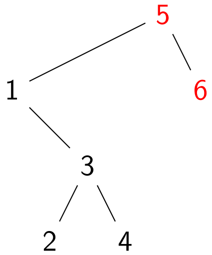
$$\Delta(g) = \frac{xg(x) - g(1)}{x - 1}$$

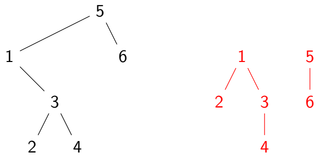


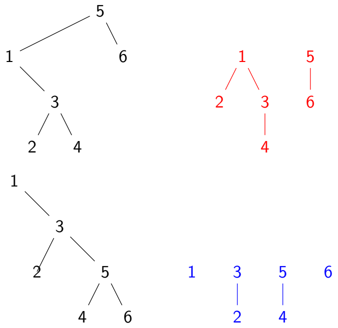

5
1
6
3
2
4

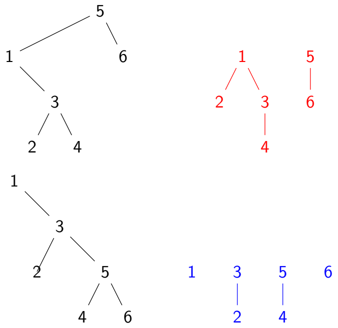

5
1
6
3
2
4

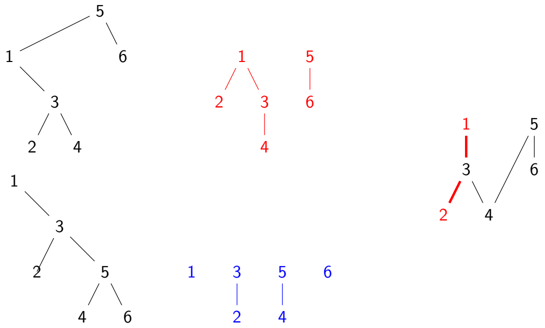

5
1
6
3
2
4

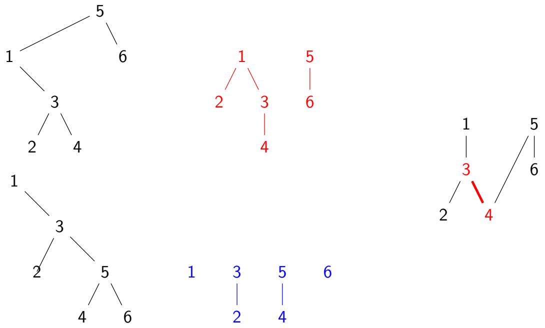

5
1
6
3
2
4

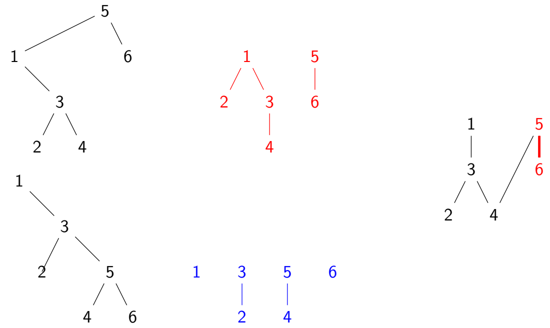

5
1
6
3
2
4
1
5
2
3
6
4

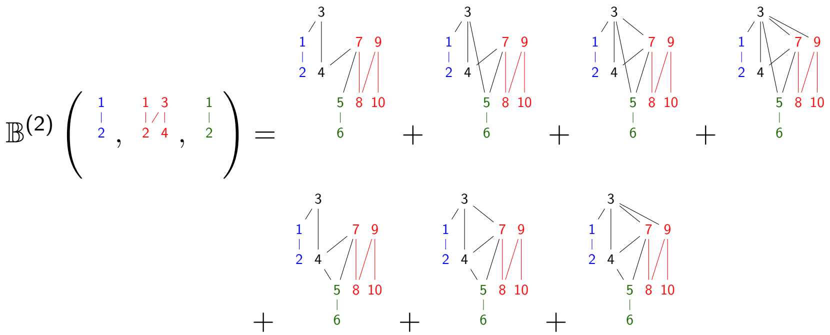

5
1
6
3
2
4
1
2
3
5
6
4
1
3
2
5
4
6
1
3
5
6
2
4


5
1
6
3
2
4
1
5
2
3
6
4
1
5
3
6
2
4
1
3
2
5
4
6
1
3
5
6
2
4


5
1
6
3
2
4
1
5
2
3
6
4
1
5
3
6
2
4
1
3
2
5
4
6
1
3
5
6
2
4


5
1
6
3
2
4
1
3
2
5
4
6
1
5
2
3
6
4
1
3
5
6
2
4
1
5
3
6
2
4



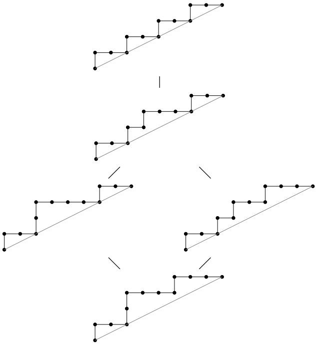


Théorème (Châtel, P.)

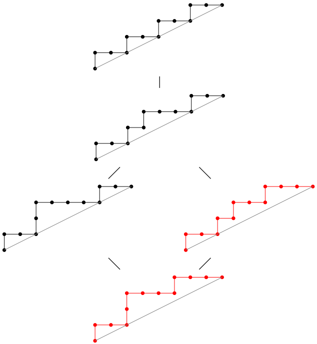
Soit T un élément de m-Tamari composé des éléments $L, R_1, \ldots, R_m$.
On définit récursivement $\mathcal{B}_T^{(m)}$ par

$$\mathcal{B}_{\emptyset}^{(m)} := 1$$
$$\mathcal{B}_T^{(m)}(x) := \mathrm{B}^{(\mathrm{m})}(L, R_1, \ldots, R_m)$$

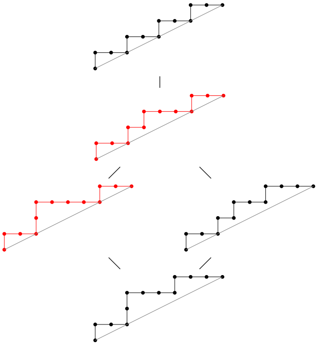
Alors $\mathcal{B}_T^{(m)}$ compte le nombre d'éléments inférieurs ou égaux à T dans le treillis de m-Tamari.



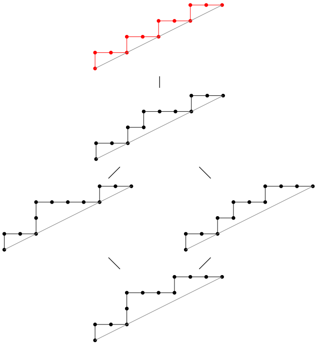
$$\mathcal{B}_T^{(2)} = \mathrm{B}^{(2)}(x,x,x)$$
$$= x^2\Delta(x\Delta(x))$$
$$= x^2\Delta(x(1+x))$$
$$= x^2(2+2x+x^2)$$
$$= 2x^2+2x^3+x^4$$



$$\mathcal{B}_T^{(2)} = \mathrm{B}^{(2)}(x,x,x)$$
$$= x^2\Delta(x\Delta(x))$$
$$= x^2\Delta(x(1+x))$$
$$= x^2(2+2x+x^2)$$
$$= 2x^2 + 2x^3 + x^4$$



$$\begin{aligned}\mathcal{B}_T^{(2)} &= \mathrm{B}^{(2)}(x,x,x)\\ &= x^2\Delta(x\Delta(x))\\ &= x^2\Delta(x(1+x))\\ &= x^2(2+2x+x^2)\\ &= 2x^2 + {\color{red}2x^3} + x^4\end{aligned}$$



$$
\begin{aligned}
\mathcal{B}_T^{(2)} &= \mathrm{B}^{(2)}(x,x,x) \\
&= x^2\Delta(x\Delta(x)) \\
&= x^2\Delta(x(1+x)) \\
&= x^2(2+2x+x^2) \\
&= 2x^2+2x^3+x^4
\end{aligned}
$$

Intervalles-posets

Intervalles-posets

- bijection avec les triangulations planes, statistiques symétriques, etc.

Intervalles-posets

- bijection avec les triangulations planes, statistiques symétriques, etc.
- polynômes de Tamari multivariés et flots

Intervalles-posets

- bijection avec les triangulations planes, statistiques symétriques, etc.
- polynômes de Tamari multivariés et flots
- treillis cambriens et généralisations

Intervalles-posets

- bijection avec les triangulations planes, statistiques symétriques, etc.
- polynômes de Tamari multivariés et flots
- treillis cambriens et généralisations

Structures "m"

Intervalles-posets

- bijection avec les triangulations planes, statistiques symétriques, etc.
- polynômes de Tamari multivariés et flots
- treillis cambriens et généralisations

Structures "m"

- structures algébriques

Intervalles-posets

- bijection avec les triangulations planes, statistiques symétriques, etc.
- polynômes de Tamari multivariés et flots
- treillis cambriens et généralisations

Structures "m"

- structures algébriques
- treillis des chaînes de permutations