Robustness and Stochastic Programming

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Outline

- Introduction
- Robust Optimization
- Robust Linear Programming
- Stochastic Programming
- Stochastic Network Design
- Computational Results

Uncertainty : Robustness or Stochastic

- Two approaches to handle uncertainty in combinatorial optmization:
  1. Robust Optimization
     - No random variables
     - Uncertainty expressed by scenarii under the form of :
       (a) discrete sets,
       (b) intervals,
       (c) spheres.
  2. Stochastic Optimization
     - Random variables,
     - Simulations,
     - Confidence intervals.

Robust Optimization

- Let
$$\min_x f(x,\omega)$$
be a family of optimization problems.

- if ω is an uncertain parameter and if the information is available under the form $\omega \in \Omega$, the robust problem can be written as :

$$(P_\Omega)\ \min_x \sup_{\omega\in\Omega} f(x,\omega)$$

Robust Optimization -discrete case-

- Let S be a set of senarios,
- Let X be a set of decisions variables,
- Let D be a set of input data,
- Let D^s be data instance related to scenario s,
- Let F^s the set of decision variables when s is realised.

The optimization "deterministic" problem for one scenario s is :

$$z^s = f(X_s^*, D^s) = \min_{X \in F^s} f(X, D^s)$$

Optimisation Robuste -cas discret-

Kouvelis et Yu proposed three robustness formulations :

- Absolute robust decision :

$$z_A = \max_{s \in S} f(X_A, D^s) = \min_{X \in \bigcap_{s \in S} F^s} \max_{s \in S} f(X, D^s)$$

- Relative deviation decision :

$$z_D = \max_{s \in S}(f(X_D, D^s) - f(X_s^*, D^s)) = \min_{X \in \bigcap_{s \in S} F^s} \max_{s \in S}(f(X, D^s) - f(X_s^*, D^s))$$

- Robust Relative Deviation :

$$z_R = \max_{s \in S} \frac{(f(X_R, D^s) - f(X_s^*, D^s))}{f(X_s^*, D^s)} = \min_{X \in \bigcap_{s \in S} F^s} \max_{s \in S} \frac{(f(X_R, D^s) - f(X_s^*, D^s))}{f(X_s^*, D^s)}$$

Uncertainty and Linear Programming

- Let's consider the following LP : $\min_{x \geq 0} \{c^t x : \ \bar{A}x + b \leq 0\}$,

- It can also be written as :

$$\min_{x \geq 0, \ x_n = 1} \{c^t x : Ax \leq 0\}, \text{ où } A := (\bar{A}, b) \in \mathcal{R}^{m \times n}, b \in \mathcal{R}^m \text{ et } c, x \in \mathcal{R}^n.$$

- Let $A = A(\omega)$, with an uncertain parameter ω, the robust LP can be written as follows :

$$(LP_\Omega) \min_{x \geq 0, \ x_n = 1} \{c^t x : \ A(\omega)x \leq 0, \forall \omega \in \Omega\}.$$

- x can be a discrete variable.

Uncertainty and Linear Programming -contd-

- (LP_Ω) can be written as :
  $\min_{x\geq 0,\ x_n=1}\{c^t x : \ \sup_{\omega\in\Omega} A_i(\omega)x \leq 0, i=1,\ldots,m\}$.

- If A is linearly dependant of the uncertain parameter, then: $A(\omega) = \overset{\circ}{A} + \sum_{i=1}^{p}\omega_j A^j$.

- Let $\Omega = \rho B$ where B is a closed unit ball for the euclidian norm and $\rho \geq 0$, we consider the following family of problems :

  $(LP_\rho)\min_{x\geq 0,\ x_n=1}\{c^t x : \ A(\omega)x \leq 0,$ si $\|w\| \ \leq \rho\}$.

- Then $A_i(\omega)x = \overset{\circ}{A_i}x + \sum_{j=1}^{p} w_j A^i_j x = \overset{\circ}{A_i}x + R^i(x)^t w$,
  where $R^i_j(x) = A^i_j x$ and $\sup_{\omega\in\rho B} A_i(\omega)x = \overset{\circ}{A_i}x + \rho\|R^i(x)\|$.

Uncertainty and Linear Programming -contd-

- From analytical point of view, robust LP can be written as :

$$(LP_\rho) \min_{x\geq 0,\ x_n=1} \{c^t x :\ \overset{\circ}{A_i} x + \rho \|R^i(x)\| \leq 0, i = 1, \ldots, m\}.$$

- Since $R(.)$ is affine, (LP_ρ) is a Second Order Cone Programming problem (See Bonnans).

- We can also introduce problems with recourse i.e. we consider two variables x and y but only x is chosen before event of random parameter. Resulting problem is : $\min_{x,y} f(x, y, \omega)$ and the robust counterpart is :

$$(P_\Omega) \min_x \sup_{\omega\in\Omega} \inf_y f(x, y, \omega).$$

Uncertainty and Linear Programming -contd-

- Robustness on progress work at GrafComm at LRI : Lynda Gastal is studying robust shortest paths and applications in telecommunications routing problems.

- Approximated algorithm for Robust Spanning Tree problem (Joint work with C. Kenyon).

- Our robust problems are inspired from meshed and cellular telecommunication network design area.

Stochastic Programming

- Linear Program with random parameters :

$$\min_{x \in R^n_+} \{cx : Ax = b, Tx \geq h\}$$

- Assumptions:
  1. The value of (T,h) is unknown,
  2. Uncertainty expressed by probability distribution, e.g. "scenarios" $Pr\{(T,h) = (T^s, h^s)\} = p_s$, $s = 1, \ldots, S$
  3. Distributions are known (statistics, experts, . . .)

- Stochastic LP : Decide variables x "here and now" without knowing real value of (T, h) but only its probability distribution.

Stochastic Programming

- SPs can be express analytically (at least) twofold:
- By deterministic linear or nonlinear programs,
- By replacing random variables by representative parameters (for instance moments).

Solution Methods

- **Fat Solution** ⇒ satisfy SP constraints for all scenarios
  - Advantages → Deterministic equivalent LP,
  - Drawbacks → No feasible solution in almost all cases.

- **Expected Value** ⇒ Replace random variable by its mean i.e $\bar{d} = \sum_s p_s d_s$
  - Advantages → Deterministic equivalent LP,
  - Drawbacks → Risk not taken into account, feasibility for only few scenarios!

- **Two-stage recourse model** ⇒ introduce explicitly corrective actions in the LP. Replace decision variable vector of the second stage by a vector related to the event of random variables.
  - Advantages → risk explicitly considered, costs depends on recourse actions. Hude size LP !
  - Drawbacks → Difficult to solve directly, large size LP.

- **Chance constraint**

$$\min_{x \in R^n_+} \{cx : Ax = b, Pr\{Tx \leq h\} \geq p\}$$

Very difficult problem to solve !

Multicommodity flows

- Let $G = (V, E)$ a non-oriented graph with $|V| = n$ nodes and $|E| = m$ edges.

- let x_{ij}^k decision variable corresponding to traffic demand flow fraction of commodity k routed on edge (i, j). Let x^k flow vector for commodity k.

- Let y_{ij} capacity integer decision variable on edge (i, j).

The flow formulation can be expressed as follows :

Multicommodity flows -contd-

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in E} r_{ij} y_{ij} && && \text{(1a)}\\
\text{s.t.} \quad & \mathcal{N}\left|x^k\right| = r^k, && \forall k \in \mathcal{K}, && \text{(1b)}\\
& \sum_{k\in\mathcal{K}} \left|x^k_{ij}\right| \le y_{ij}, && \forall (i,j) \in E. && \text{(1c)}
\end{aligned}
$$

where:

- $\mathcal{N}$ is the node-edge incidence matrix. Its columns are composed of two nonzero components $\{+1,-1\}$ according to (i,j)-edge orientation.

- $\mathcal{K}$ is set of commodities d^k to route between s^k and t^k.

Multicommodity flows -contd-

An equivalent linear programming formulation is obtained by introducing non-negative variables, x_{ij}^{k+} and x_{ij}^{k-}, such that $x_{ij}^{k} = x_{ij}^{k+} - x_{ij}^{k-}$ and $\left|x_{ij}^{k}\right| = x_{ij}^{k+} + x_{ij}^{k-}$,

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in E} r_{ij} y_{ij} && \text{(2a)}\\
\text{s.t.} \quad & \mathcal{N}\left(x^{k+} - x^{k-}\right) = r^{k}, \quad \forall k \in \mathcal{K}, && \text{(2b)}\\
& \sum_{k\in\mathcal{K}} \left(x_{ij}^{k+} + x_{ij}^{k-}\right) \le y_{ij}, \quad \forall (i,j) \in E, && \text{(2c)}\\
& x_{ij}^{k+}, x_{ij}^{k-} \ge 0, \quad \forall k \in \mathcal{K}, \quad \forall (i,j) \in E && \text{(2d)}\\
& y_{ij} \ge 0, \text{integer } \forall (i,j) \in E. && \text{(2e)}
\end{aligned}
$$

Stochastic Optimization in telecommunications networks

Uncertainty Sources in telecommunications :

- Traffic demand ⇒ RHS of multicommodity LP,
- Cost function ⇒ LP objective function,
- Network Topology i.e. underlying graph ⇒ constraint matrix,
- Flow variables,...

Stochastic Optimization in telecommunications networks -contd-

$$\begin{aligned}
\min \quad & \mathbb{E}_r \sum_{(i,j)\in E} r_{ij} y_{ij} && \text{(3a)}\\
\text{s.t.} \quad & \mathcal{N}\left(x^{k+} - x^{k-}\right) = \mathbb{E}_d r^k, \quad \forall k \in \mathcal{K}, && \text{(3b)}\\
& \sum_{k\in\mathcal{K}} \left(x_{ij}^{k+} + x_{ij}^{k-}\right) \leq y_{ij}, \quad \forall (i,j) \in E, && \text{(3c)}\\
& x_{ij}^{k+}, x_{ij}^{k-} \geq 0, \quad \forall k \in \mathcal{K}, \quad \forall (i,j) \in E. && \text{(3d)}
\end{aligned}$$

Question : What are relevant random variables for telecommucations companies among $\mathcal{N}?, r, d$?

Answer : Traffic Demand (competition, deregulated markets, . . .).

Telecommunication Network Capacity Design for Uncertain Demand -Assumption-

- If traffic demand is not satisfied (lack of capacity), telecommunication operator has to pay a penalty π.

- The problem consists in investing in network links interms of capacities ($y \in Y \cap \mathbb{Z}^m$) in order to minimize network total costs.

- Cost function is composed of investment costs ($r^\top y$) and expectation function costs generated by the penalties ($E_\xi[Q(y,\xi)]$).

- Two stage stochastic programming can be written as :
  - $r \in \mathbb{R}^m$ is cost vector associated with first stage integer variables $y \in Y \cap \mathbb{Z}^m$.
  - E_ξ is expectation function associated with random variable ξ.

Now SP can be written as :

Telecommunication Network Capacity Design for Uncertain Demand -contd-

$$(\mathrm{P1}) \quad \min_{y \in Y \cap \mathbb{Z}^m} \{r^\top y + E_\xi[Q(y,\xi)]\} \tag{4}$$

For given y and ξ, penalty cost $Q(y,\xi)$ is given by solving the second stage problem :

$$\begin{aligned}
(\mathrm{P2})\ Q(y,\xi) &= \min \sum_{k=1}^{K_\xi} \pi_k s_k(\xi) && (5)\\
s.t. \quad & \sum_{k=1}^{K_\xi} (x^{k+}(\xi) + x^{k-}(\xi)) \leq u + y, && (6)\\
& \mathcal{N}(x^{k+}(\xi) - x^{k-}(\xi)) + \Gamma^k(\xi) = d^k(\xi),\ \forall\, k, && (7)\\
& x^{k+}(\xi),\ x^{k-}(\xi),\ s_k(\xi) \geq 0,\ \ \forall\, k, && (8)
\end{aligned}$$

Monte Carlo Scenario Modelling

- Consider discrete and finite set T of random variable realizations ξ, $\{\xi^1, \xi^2, \cdots, \xi^T\}$ and $prob(\xi^t) = 1/T$, $t = 1, ..., T$.
- (P1) can be written as follows :
  (P3) $\min_{y \in Y \cap \mathbb{Z}^m} \{r^\top y + T^{-1} \sum_{t=1}^{T} Q(y, \xi^t)\}$
- T realizations of ξ are input data for our model. Traffic distributions are also given.
- Each scenario t, $t = 1, ..., T$, has $K_t \in \{1, 2, 3, 4, \cdots\}$ demands given by Poisson distribution function with mean λ.
- For each demand k, $k = 1, ..., K_t$, we associate a bandwidth b_k given by normal or log-normal random variable.

Solution Methods

Three approaches were studied and compared :

- L-Shaped Decomposition.
- Stochastic Subgradient.
- Volume Algorithm.

Numerical Results

- T is number of scenarios.
- Ng is node groups number.
- λ is demand average number and β is bandwidth of each demand.
- σ^2 is bandwidth variance.
- M_β is maximum bandwidth of each demand.

Table 1: Test problem features.

n	m	T	Ng	λ	M_λ	β	σ^2	M_β	Inv	r	π_k	u
7	12	250	2	15	20	5	1	1024	1000	3	21	3
8	28	250	2	20	25	400	1	1024	2000	3	24	300
10	45	250	2	25	30	450	1	1024	1000	3	30	350
15	105	250	3	50	55	550	1	1024	5000	3	45	400
21	210	250	3	45	50	2500	1	10000	15000	3	63	2000
41	70	5	3	85	90	50	1	1024	1000	3	123	36

Table 2: Summary of the results.

n	Alg	$Distr$	UB	$CstInv$	$Iter$	CPU	$Gap(\%)$
7	Ben	nor	343.122	291	7	0:06	0.0000
7	Vol	nor	343.122	291	61	0:25	0.0000
7	Sub	nor	343.122	291	39	0:25	0.0000
7	Ben	log	1326.630	759	7	0:05	0.0000
7	Vol	log	1326.630	759	32	0:23	0.0000
7	Sub	log	1326.670	759	82	0:23	0.0030
8	Ben	nor	18503.000	15315	7	0:11	0.0000
8	Vol	nor	19335.900	15033	100	1:49	4.5014
8	Sub	nor	19333.200	15024	100	1:50	4.4868
8	Ben	log	34935.000	27738	9	0:14	0.0000
8	Vol	log	36168.700	28536	100	1:49	3.5314
8	Sub	log	36166.100	28524	100	1:49	3.5240
10	Ben	nor	17558.300	13116	5	0:18	0.0000
10	Vol	nor	18105.700	13242	100	05:04	3.1176
10	Sub	nor	18106.600	13230	100	09:58	3.1227
10	Ben	log	36958.500	29091	7	0:26	0.0000
10	Vol	log	38379.200	29670	100	10:02	3.8446
10	Sub	log	38374.900	29655	100	05:07	3.8324

Table 3: Summary of the results.

n	Alg	$Distr$	UB	$CstInv$	$Iter$	CPU	$Gap(\%)$
15	Ben	nor	34619.100	29157	5	2:42	0.0000
15	Vol	nor	39234.200	35109	100	40:24	13.3312
15	Sub	nor	28534.200	20787	100	53:45	9.3545
15	Ben	log	45602.000	34737	8	9:53	0.0000
15	Vol	log	49765.500	37785	100	111:34	9.1301
15	Sub	log	49749.200	37776	100	56:38	9.0943
21	Ben	nor	2516.980	0	1	2:18	0.0000
21	Vol	nor	2516.980	0	2	79:22	0.0000
21	Sub	nor	2516.980	0	2	78:31	0.0000
21	Ben	log	260787.000	192618	5	13:52	0.0000
21	Vol	log	335897.000	184698	100	290:31	28.8013
21	Sub	log	335971.000	184308	100	160:51	28.8297
41	Ben	nor	314057.000	*	6	*	*
41	Vol	nor	44511.000	44511	98	2:58	*
41	Sub	nor	44169.600	44145	100	2:24	*
41	Ben	log	305139.000	102315	100	3:51	0.30529
41	Vol	log	307044.000	104697	97	4:19	*
41	Sub	log	307202.000	104916	100	3:42	*

Additional Results

$$\begin{aligned} \text{(P)} \quad \min \quad & r^\top y + E_\xi[Q(y,\xi)] \\ s.t. \quad & y \in Y \cap I^m. \end{aligned} \tag{9}$$

- Notice that for the model with integer (respectively 0-1) first-stage variables, the set I denotes $\mathbb{Z}$ (respectively $\{0,1\}$).

- For given y and ξ, the penalty cost $Q(y,\xi)$ is obtained by solving the second-stage problem:

$$
\begin{aligned}
\text{(P2)}\ \ Q(y,\xi) \quad &= \quad \min \sum_{k=1}^{\mathcal{K}_\xi} \pi_k s_k(\xi) && (10)\\
s.t. \quad & \quad \sum_{k=1}^{\mathcal{K}_\xi} |x^k(\xi)| \le u + Ky, && (11)\\
& \quad \mathcal{N}x^k + \Gamma^k(\xi) = d^k(\xi),\ k = 1, ..., \mathcal{K}_\xi, && (12)\\
& \quad s_k(\xi) \ge 0,\ k = 1, ..., \mathcal{K}_\xi, && (13)
\end{aligned}
$$

Additional Results -contd-

- We study two versions :
  - $y \in Y \cap \{0,1\}^m$.
  - $y \in Y \cap \mathbb{Z}^m$.

- We propose for both problems (see the references below):
  - Branch and Bound algorithms.
  - Branch and Cuts algorithm with adaptive cuts.

- References:
  - Telecommunication Network Capacity Design for Uncertain Demand R. Andrade, A. Lisser, N. Maculan, G. Plateau. Computational Optimization and Applications. Nov 2004. Vol. 29, Iss. 2; p. 127
  - B&B frameworks for the capacity expansion of high speed telecommunication networks under uncertainty, R. Andrade, A. Lisser, N. Maculan, G. Plateau, AOR, Vol. 140, pp. 49-65, 2005.

- Enhancing a branch and bound algorithm for two stage stochastic integer network design based models, R. Andrade, A. Lisser, N. Maculan, G. Plateau, to appear in Managment Science, september 2006.

Deterministic FAP

FAP problem : Assign n frequencies to m sites in order to satisfy given demands for frequencies and minimize the interferences between different frequencies.

The frequencies are represented as a set of positive integers $i = 1, .., n$.

For every pair (i, j) of frequencies the distance ρ_{ij} is defined:

$$\rho_{ij} = |i - j|$$

Deterministic FAP. contd

Let d_i be the demand for frequencies for the site i,

w_{ij}^{kl} is the interference attained if frequency k is assigned to site i and frequency l is assigned to site j,

and x_i^k is a decision binary variable which equals 1 if frequency k is assigned to site i and zero otherwise;

Let N be the set of sites and M the set of frequencies.

The FAP can be written as :

FAP

$$\min_{x_i^k \in \{0,1\}} \sum_{i,j,k,l} w_{ij}^{kl} x_i^k x_j^l \tag{14}$$

subject to :

$$\sum_k x_i^k = d_i, \ \forall i \tag{15}$$

$$x_i^k + x_j^l \leq 1, \ \forall i,j,k,l : \ i,j \text{ adjacent}, \ |k-l| \leq c_1 \tag{16}$$

$$x_i^k + x_i^l \leq 1, \ \forall i,k,l : \ |k-l| \leq {c_2}^1 \tag{17}$$

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[1] It is common to set $c_1 = 2$ and $c_2 = 3$ in practice. We will use such constants both for our modelling or numerical experiments.

FAP Semidefinite relaxation

Let $y_{i,j}^{l,k} = x_i^l x_j^k$. The FAP can be written as :

$$\sum_{l=1}^{m} y_{i,i}^{l,l} = d_i, \forall i \in N$$

$$y_{i,i}^{l,k} = 0, \forall i \in N, \text{ and } |l-k| \leq 3.$$

or

$$\left\{ \begin{array}{l} y_{i,i}^{l,l+1} = 0 \\ y_{i,i}^{l,l+2} = 0 \quad \forall i \in N, \forall l \in M \\ y_{i,i}^{l,l+3} = 0 \\ \\ \end{array} \right.$$

FAP Semidefinite relaxation

and

$$y_{i,j}^{l,k} = 0, \forall i \in N, \text{ and } |l-k| \leq 2.$$

or

$$\begin{cases} y_{i,j}^{l,l} = 0 \\ y_{i,j}^{l,l+1} = 0 \quad 1 \leq i,j \leq n, \ j \text{ co-site of } i \text{ and } l \in M \\ y_{i,j}^{l,l+2} = 0 \\ \end{cases}$$

FAP Semidefinite relaxation

Let the matrices Y and W defined as :

$$Y = \begin{bmatrix} Y_{1,1} & \cdots & Y_{1,n} \\ \vdots & \ddots & \vdots \\ Y_{n,1} & \cdots & Y_{n,n} \end{bmatrix}, \text{ where } Y_{i,j} = \begin{bmatrix} y_{i,j}^{1,1} & \cdots & y_{i,j}^{1,m} \\ \vdots & \ddots & \vdots \\ y_{i,j}^{m,1} & \cdots & y_{i,j}^{m,m} \end{bmatrix},$$

$$\text{and } W = \begin{bmatrix} W_{1,1} & \cdots & W_{1,n} \\ \vdots & \ddots & \vdots \\ W_{n,1} & \cdots & W_{n,n} \end{bmatrix}, \text{ where } W_{i,j} = \begin{bmatrix} w_{i,j}^{1,1} & \cdots & w_{i,j}^{1,m} \\ \vdots & \ddots & \vdots \\ w_{i,j}^{m,1} & \cdots & w_{i,j}^{m,m} \end{bmatrix}$$

The SDP relaxed FAP can be written as :

$$(SDPFAP)\left\{\begin{array}{lll}
Min & Trace(W*Y) & \\
s.c & Trace(Y_{ii}) = d_i & i \in N \\
 & \left\{\begin{array}{l} y_{i,i}^{l,l+1} = 0 \\ y_{i,i}^{l,l+2} = 0 \\ y_{i,i}^{l,l+3} = 0 \end{array}\right. & i \in N,\ l \in M \\
 & & \\
 & \left\{\begin{array}{c} y_{i,j}^{l,l} = 0 \\ y_{i,j}^{l,l+1} = 0 \\ y_{i,j}^{l,l+2} = 0 \end{array}\right. & 1 \leq i,j \leq n, \ \ \text{j co-site of i and } l \in M \\
 & diag(Y) = y & \\
 & Y - yy^t \succeq 0. &
\end{array}\right. \tag{18}$$

where $y = (x_i^l)_{1 \leq l \leq m, 1 \leq i \leq n}$ is the decision variable vector.

FAP SDP relaxation

Let Z_{il} and Z_{ijl} be a determined matrices for each co-site and co-station constraint respectively. The (18) can be also written as :

$$(SDPFAP1)\left\{\begin{array}{lll} Min & Trace(W*Y) & \\ s.c & & \\ & Trace(Y_{ii}) = d_i & i \in N \\ & Trace(Z_{il}Y_{il}) = 0 & \forall i, l \\ & Trace(Z_{ijl}Y_{ijl}) = 0 & \forall i, j, l, \text{ i and j are cosite} \\ & diag(Y) = y & \\ & Y - yy^t \succeq 0 & \end{array}\right. \tag{19}$$

Stochastic problem

The main uncertainty sources in the FAP are the following :

- Interference may change due to changing atmospheric conditions, time of the day and of the year, etc.

  It is more realistic to assume that interferences are described by joint probabilistic distribution $H(w)$.

- Assume that demand at sites $i = 1,,,m$ is a random vector with joint distribution $P(d)$.

We consider the case where the frequency assignment is based on the probabilistic description of demand and interference patterns $i.e.$ non adaptive case.

Non adaptive FAP

Frequency assignment decision is made before any specific demand realization becomes known and is not changed afterwards.

The objective function can be written as:

$$\min_{x_i^k} \mathbb{E}_w \sum_{i,j,k,l} w_{ij}^{kl} x_i^k x_j^l + c\mathbb{E}_d \sum_i \max\left\{0, d_i - \sum_k x_i^k\right\} \tag{20}$$

As

$$\mathbb{E}_w \sum_{i,j,k,l} w_{ij}^{kl} x_i^k x_j^l = \sum_{i,j,k,l} \left(\mathbb{E}_w w_{ij}^{kl}\right) x_i^k x_j^l = \sum_{i,j,k,l} \bar{w}_{ij}^{kl} x_i^k x_j^l$$

Non adaptive Case

- The demand distribution $P(d)$ is concentrated in a finite number of points $d^r = (d_1^r, ..., d_m^r)$ with weights p_r, $r = 1, ..., R$.
- These points will be called *demand scenarios*.

The problem (20) is equivalent to the following:

$$\min_{x_i^k} \sum_{i,j,k,l} \bar{w}_{ij}^{kl} x_i^k x_j^l + c \sum_r \sum_i p_r \max \left\{ 0, d_i - \sum_k x_i^k \right\} \qquad (21)$$

Non adaptive Case

After introducing auxiliary variables v_i^r for each site $i = 1, ..., m$ and for each scenario $r = 1, ..., R$, we obtain the following familiar quadratic problem:

$$\min_{x_i^k, v_i^r} \sum_{i,j,k,l} \bar{w}_{ij}^{kl} x_i^k x_j^l + c \sum_r \sum_i p_r v_i^r \tag{22}$$

$$\sum_k x_i^k + v_i^r = d_i^r, \quad \forall i, r \tag{23}$$

where v_i^r is integer and nonnegative. Constraints (16)-(17) can be added to this formulation.

Semidefinite relaxations for stochastic FAP

Assume that $v_i^r \in \{0,1\}$. As for the deterministic FAP, we introduce $y_{i,j}^{l,k} = x_i^l x_j^k$ together with the following notations:

$$Y = \begin{bmatrix} Y_{1,1} & \cdots & Y_{1,n} \\ \vdots & \ddots & \vdots \\ Y_{n,1} & \cdots & Y_{n,n} \end{bmatrix}, \text{ with } Y_{i,j} = \begin{bmatrix} y_{i,j}^{1,1} & \cdots & y_{i,j}^{1,m} \\ \vdots & \ddots & \vdots \\ y_{i,j}^{m,1} & \cdots & y_{i,j}^{m,m} \end{bmatrix},$$

$$\tilde{W} = \begin{bmatrix} \tilde{W}_{1,1} & \cdots & \tilde{W}_{1,n} \\ \vdots & \ddots & \vdots \\ \tilde{W}_{n,1} & \cdots & \tilde{W}_{n,n} \end{bmatrix}, \text{ with } \tilde{W}_{i,j} = \begin{bmatrix} \tilde{w}_{i,j}^{1,1} & \cdots & \tilde{w}_{i,j}^{1,m} \\ \vdots & \ddots & \vdots \\ \tilde{w}_{i,j}^{m,1} & \cdots & \tilde{w}_{i,j}^{m,m} \end{bmatrix}$$

$$V_r = (v_1^r, ..., v_N^r),\ U = \begin{bmatrix} Y & 0 & \cdots & 0 \\ 0 & V_1V_1^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V_RV_R^T \end{bmatrix}$$

$$D = \begin{bmatrix} \tilde{W} & 0 & \cdots & 0 \\ 0 & cp_1I_N & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & cp_NI_N \end{bmatrix}$$ where I_N is NxN unit matrix.

To represent demand constraints, we consider a submatrix U_{ir} of matrix U defined by:

$$U_{ir} = \begin{bmatrix} Y_{ii} & 0 \\ 0 & (v_i^r)^2 \end{bmatrix}$$

Then the corresponding constraint is

$$trace(U_{ir}) \geq d_i^r, \forall i, r \tag{24}$$

Then SDP lifting of (22)-(23) is the following:

$$\min_{U} trace(D * U) \tag{25}$$

subject to

$$trace(U_{ir}) \geq d_i^r, \forall i, r \tag{26}$$

$$U - diag(U)diag(U)^T \succeq 0 \tag{27}$$

Thus, positive semidefinite relaxation of the stochastic FAP is the problem (25)-(27) where co-cite and co-station constraints should be added similarly to the deterministic FAP.

SDP relaxation within decomposition algorithm

Generic scheme for the Benders decomposition algorithm is:

- ***Initialization***.
  Select a feasible frequency assignment x_i^{k0} and let $u^{+0} = +\infty$, $u^{-0} = -\infty$ be the current upper and lower bound respectively.

- ***Generic step***.
  At step s, let x_i^{ks} be the current frequency assignment and u^{+s}, u^{-s} the upper and lower bounds resp. Then at step s we do the following:
  - Solve subproblem,
  - Add new cut to the master program.

Benders decomposition steps

Solve the subproblem

$$\min_{v_i^r} cp_r v_i^r \tag{28}$$

$$\sum_k x_i^{ks} + v_i^r \geq d_i^r, \quad \forall i, r \tag{29}$$

whose dual

$$\max_{\mu_i^r \geq 0} \sum_r \sum_i \mu_i^r \left(d_i^r - \sum_k x_i^{ks} \right)$$

$$0 \leq \mu_i^r \leq cp_r, \quad \forall i, r$$

has an explicit solution

$$\mu_i^{rs} = \begin{cases} cp_r & \text{if} \quad d_i^r \geq \sum_k x_i^{ks} \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

Then, **Add the cut**

$$z_0 \geq \sum_{i,j,k,l} \bar{w}_{ij}^{kl} x_i^k x_j^l + \sum_i \sum_r \left(d_i^r - \sum_k x_i^k \right) \mu_i^{rs}$$

to the master problem and compute the upper bound by

$$u^{+0} = \min \left\{ u^{+0}, \sum_{i,j,k,l} \bar{w}_{ij}^{kl} x_i^{ks} x_j^{ls} + \sum_i \sum_r \left(d_i^r - \sum_k x_i^{ks} \right) \mu_i^{rs} \right\}$$

Solve the master problem using SDP relaxation :

$$\bar{z}_0 = \min_{z_0, x_i^k} z_0 \tag{31}$$

$$z_0 - \sum_{i,j,k,l} \bar{w}_{ij}^{kl} x_i^k x_j^l + \sum_i \sum_k x_i^k \sum_r \mu_i^{rq} \geq \sum_i \sum_r d_i^r \mu_i^{rq}, \ q = 1, ..., s \tag{32}$$

and

- let $u^{-0} = \bar{z}_0$.
- Stop if $u^{-0} - u^{-0} < \epsilon$ where ϵ is some prespecified tolerance.
- Otherwise, let $x_i^{k,s+1}$ be the solution of (31)-(32) and go to the step $s + 1$.

We have to lift inequality (32) into the cone of positive semidefinite matrices.

Observe that this inequality can be expressed as follows:

$$trace\left(\left[\begin{array}{cc} 1 & b_q^T \\ b_q & -\tilde{W} \end{array}\right] * \left[\begin{array}{cc} z_0 & x^T \\ x & Y \end{array}\right]\right) \geq \sum_i \sum_r d_i^r \mu_i^{rq}$$

with

$$trace\left(\left[\begin{array}{cc} 0 & -\frac{1}{2}1_{nm}^T \\ -\frac{1}{2}1_{nm} & I_{nm} \end{array}\right] * \left[\begin{array}{cc} z_0 & x^T \\ x & Y \end{array}\right]\right) = 0$$

where

$$x = (x_1^1, ..., x_1^m, ..., x_n^1, ..., x_n^m),\ b_q = \frac{1}{2}\left(\overbrace{b_{1q}, ..., b_{1q}}^{n\ \text{times}}, ..., \overbrace{b_{nq}, ..., b_{nq}}^{n\ \text{times}}\right),\ b_{iq} = \sum_r \mu_i^{rq}$$

Denoting now

$$U = \begin{bmatrix} z_0 & x^T \\ x & Y \end{bmatrix},\ A = \begin{bmatrix} 1 & 0_{nm}^T \\ 0_{nm} & 0_{nmxnm} \end{bmatrix},$$

$$B_q = \begin{bmatrix} 1 & b_q^T \\ b_q & -\tilde{W} \end{bmatrix},\ C = \begin{bmatrix} 0 & -\frac{1}{2}1_{nm}^T \\ -\frac{1}{2}1_{nm} & I_{nm} \end{bmatrix},\ D = \begin{bmatrix} 0 & 0_{nm} \\ 0_{nm} & I_{nm} \end{bmatrix}$$

we obtain the following relaxation of the master problem:

$$\min_U trace(AU) \tag{33}$$

$$trace(B_qU) \geq \sum_i \sum_r d_i^r \mu_i^{rq},\ q = 1, ..., s \tag{34}$$

$$trace(CU) = 0 \tag{35}$$

$$DU - diag(DU)diag(DU)^T \succeq 0 \tag{36}$$

Numerical Results

Our first numerical experiments concerned small instances.

Table 4: Instances testés

Instances	$\#sites$	$\#frequencies$
Fap1	3	12
Fap2	5	14
Fap3	6	16
Fap4	8	20

We solved 4 variants of each instance with $5, 10, 15$ and 30 scenarios respectively.

LP equivalent problems

Table 5: LP equivalent sizes

Instances	S=5		S=10		S=15		S=30	
	$\#var$	$\#const$	$\#var$	$\#const$	$\#var$	$\#const$	$\#var$	$\#const$
Fap1	717	2076	732	2091	747	2106	792	2151
Fap2	2580	7765	2605	7790	2630	7815	2705	7890
Fap3	4782	14172	4812	14202	4842	14232	4932	14322
Fap4	13080	38976	13120	39016	13160	39056	13280	39176

- Linearization leads to a large number of binary variables and constraints,
- CPLEX solved to optimality only the first two instances,
- Weak lower bounds of the LP relaxation.

Benders results

Table 6: Test SDP Sizes

Instances	Master SDP Program					
	#var	#const	MP iterations number			
			S=5	S=10	S=15	S=30
Fap1	666	100	4	3	7	4
Fap2	2484	356	7	7	10	9
Fap3	4656	271	7	9	6	7
Fap4	12880	457	15	19	16	17

- The number of constraints is related to the last master program solved,
- SDP master program iterations are less than 20 for all instances.

Numerical Results

Table 7: Lower and Upper Bounds

Data	S=5				S=10			
	LB	Opt	UB	Gap	LB	Opt	UB	Gap
Fap1	312	320	324	3.7	292	301	310	5.8
Fap2	766	776	803	4.6	741	759	807	8.1
Fap3	915	940†	955	4.1	820	863†	907	9.5
Fap4	9772	11157†	10186	4.0	9850	‡	10250	3.9

Table 8: Lower and Upper Bounds

Data	S=15				S=30			
	LB	Opt	UB	Gap	LB	Opt	UB	Gap
Fap1	296	302	304	2.6	299	299	299	0
Fap2	932	965	1002	6.9	970	983	1004	3.3
Fap3	875	903†	910	3.6	996	1017†	1029	3.2
Fap4	10336	‡	11036	6.3	9428	‡	9837	4.1

†: Best solution given by CPLEX ‡: No solution given by CPLEX

Numerical Results

Table 9: Costs vs penalties

Data	S=5			S=10			S=15			S=30		
	UB	cost	Penalty	UB	cost	Penalty	UB	cost	Penalty	UB	cost	Penalty
Fap1	324	164	160	310	145	165	304	122	182	299	80	219
Fap2	803	223	620	807	267	540	1002	214	978	832	200	632
Fap3	955	279	676	907	237	570	910	250	660	1029	199	830
Fap4	10186	2286	7900	10250	3050	7200	11036	1122	9914	9837	1016	8821

Conclusion

- SDP helps better solving the stochastic FAP than LP approaches,
- Size limit depends of the ability of SDP packages,
- New stochastic FAP variants are under tests.