INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Optimal concave costs in the SDH context

Sébastien Choplin — Jérôme Galtier — Stéphane Pérennes

N° 5201

Mai 2004

Thème COM

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--5201--FR+ENG

Optimal concave costs in the SDH context

Sébastien Choplin[*] , Jérôme Galtier[†] , Stéphane Pérennes

Thème COM — Systèmes communicants
Projet Mascotte

Rapport de recherche n° 5201 — Mai 2004 — 11 pages

Abstract: We address a problem of network design with minimum cost, and uniform all-to-all demands between the vertices. We deal with the case of concave increasing link cost function f depending of the capacity over directed arcs. We obtain lower bounds for this problem. In the generic case $f : x \mapsto x^{\alpha}$, where $\alpha \in [0; 1]$, we exhibit some families that constitute an 1.12 asymptotical approximation of the optimal network.

Key-words: Network design, SDH, buy-at-bulk, optimisation.

[*] LaRIA - Univ. de Picardie J. Verne, travail effectué quand l'auteur était membre de Mascotte.
[†] France Télécom R&D

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

Coûts concaves optimaux dans le contexte SDH

Résumé : Nous traitons d'un problème de conception de réseau à coût minimal, avec une demande tous-vers-tous uniforme entre les sommets. Nous traitons du cas des fonctions de coût concaves et croissantes sur les arcs dépendant de la capacité. Nous obtenons des bornes inférieures pour ce problème. Dans le cas générique $f : x \mapsto x^{\alpha}$, où $\alpha \in [0; 1]$, nous mettons en évidence des familles qui atteignent une 1.12 approximation assymptotique du réseau optimal.

Mots-clés : Conception de réseaux, SDH ,optimisation.

1 Introduction

The following study was motivated by a question asked by France Telecom operational services concerning the cost in SDH loop networks. In such networks, the traffic requests (or demands) are routed using a set of containers of fixed capacity, where each container connects two nodes of the network. A demand can be fulfilled by several containers forming a path between its endpoint nodes. The establishment cost of the containers is a function of its capacity, and we aim at minimizing the total cost of the containers.

The set of these containers can be considered as a virtual (or logical) network in which the communications are achieved. In this paper, we study the design of optimal SDH loop networks given a concave increasing link cost function and a All-to-All unitary set of requests. When a container of a given capacity is established in a SDH loop, the same capacity is reserved on the remaining part of the loop, so we do not consider the problem of cumulating the containers' capacities passing through a given link. We model the set of containers by a directed graph where the capacity of each arc corresponds to the capacity of the associate container. Then we only have to find which graph (associated with capacities) can achieve a multi-flow of one unit between each pair of nodes.

Our optimization model associates to each capacity a concave cost. In the particular case where the cost is given by $f : x \mapsto x^\alpha$, where $\alpha \in [0;1]$, we identify assymptotically the cost of the optimal network when the number n of nodes grows - that is, up to a multiplicative constant less than 1.12 - by some simple families of graphs. Those families are simply summarized as follows:

$\alpha = 0$:
: The family of *rings of* n *nodes* achieves the optimal cost.

$\eta(k) < \alpha < \eta(k-1)$, for $k \in \mathbb{N},\ k \geq 2$:
: The family of *stars of* $(k+1)$*-rings* achieves a fair assymptotical approximation ratio (at worst, 1.12). The value of η is given by:

$$\eta(k) = \frac{\ln((k+1)^2/(k^2+2k))}{\ln((k+1)/k)}.$$

$\eta(1) < \alpha < 1$:
: The family of *stars* gives a solution with an assymptotical cost *equivalent* to the optimal solution for $\alpha > 0.585$ and gives a fair approximation otherwise.

$\alpha = 1$:
: The family of *cliques of* n *nodes* achieves the optimal cost.

Two interesting phenomenas occur. On the one hand, when α tends to 0, the assymptotically best family tends to have larger and larger rings and finally reaches the shape of the ring, that is optimal when $\alpha = 0$. On the other hand, when α tends to 1, the shape of the assymptotically optimal solution suddenly changes from the star to the clique. Along with this change, the cost falls abruptly by a factor of 2 at the point $\alpha = 1$ (see Figure 1).

2 Related problems

The problem of minimizing the cost of a multicommodity network flow problem with link cost functions is a classical one. If the cost functions are convex, LP relaxation can be applied and many problems have been resolved, see [4] for a survey. In [3], authors study the case where the cost functions are discontinuous step increasing and, using large scale LP relaxations, describe an algorithm able to give a lower bound of the optimal cost. Often, the network supporting the flow is given and the goal is to find the routing and the capacities of the links to achieve the flow, see for example [1, 5]. We study a problem where the communications have to be routed on a virtual network supported by the physical one. Then each virtual link between two nodes can be

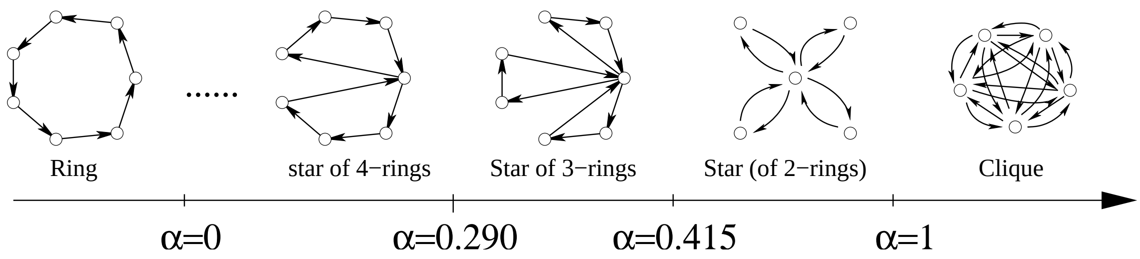

Figure 1: Quasi-optimal families.

interesting for us and so the support for this virtual network is dense and classical approaches used when the support network is the physical one (which is often sparse) are not efficient. This problem can also be viewed as finding the physical network which can achieve all requests minimizing its global cost, see [2] for a polyhedral approach.

An alternative problem to our that also need to be mentioned is the Ring Loading Problem [6]. The essential difference lies on the close (but not equivalent) norms SDH/Sonet. Both of them rely on optical rings where each link is made by a pair of fibers connecting the two endpoints both ways. The Sonet that turns to be in force essentially in North America, uses the pairs in the nominal mode both ways, dealing with the faults in a best effort fashion. On the other side, the SDH norm uses the pair just one way in the nominal mode, the other being used only to prevent failures. In case one of the links falls, all the reverse fibers of the remaining links are used to create an artificial replacement link.

3 The SDH-loop model

In this paper, we want to determine a logical SDH network of minimum cost given a set of requests to achieve. As an important part of the cost of a link is its deployment, we consider that the cost of a logical link follows a concave increasing function of its capacity which depends on the material. In SDH loops, when a logical link with a given rate is established from a node u to a node v, the logical space assigned to it is reserved throughout all the loop. Then, for each established connection, the whole physical network is charged with the given rate. According to this consideration, we can forget the loop topology and only consider the logical one.

The logical network is represented by the complete directed graph with n vertices K_n and a capacity function $c : A \rightarrow I\!R^+$ on the arcs of K_n. $V(K_n)$ will denote the set of vertices of K_n and $A(K_n)$ its set of arcs.

The cost function $\mathcal{C} : I\!R^+ \rightarrow I\!R^+$ of the capacity of an arc is assumed to have the following properties :

- $\mathcal{C}$ is concave, increasing,
- $\mathcal{C}(0) = 0$ and $\mathcal{C}(1) = 1$
- $\mathcal{C}(n) = o(n)$.

The cost of a network is then simply $\sum_{a \in A(K_n)} \mathcal{C}(c(a))$.

Definition 1 *A network (K_n, c) is said to be **feasible** for a set of rated requests $\mathcal{R}$ if there exists a multi-commodity flow which can achieve the requests according to the capacity function c. A network (K_n, c) is said to be **optimal** for $\mathcal{C}$ and $\mathcal{R}$ if its cost $\sum_{a \in A(K_n)} \mathcal{C}(c(a))$ is minimal over all capacity functions which make it feasible for $\mathcal{R}$.*

In this context, our concern is to determine the cheapest topology that can carry a given traffic. We can focus on two different cases:

- the capacities can take any positive value,
- the capacities are constrained to be in a determined fixed set, such as $\{1, 4, 16, 64\}$.

We concentrate on the first case, and we identify the asymptotical optimal costs for an All-to-All set of requests where all rates are equal to 1. Under this assumptions, a feasible network will be called a ***n*-Gossip network**. We will almost always identify any Gossip network with the capacity allocation defined on its arcs.
Notation : Given $\mathcal{C}$, the optimal cost of a n-Gossip network will be denoted by $\mathcal{K}_{\mathcal{C}}(n)$.

4 Lower bounds and cheap topologies in the continuous case

In the continuous case, it is beneficial to set the capacities of the arcs at their load (the amount of flow passing through an arc) in an optimal network. Therefore the capacity $c(a)$ of an arc a will also represent its load in the current configuration. The case where $\mathcal{C}(c) = \alpha c + \beta$ (with α, β fixed reals) is the classical one where the cost is a linear function with the capacity. In the case $\beta = 0$, the optimal solution is the request graph (see $G4$ in figure 4).

Definition 2 (Degree) *We call* ***degree*** *of a vertex the number of arcs with non zero capacity leaving it.*

Notation : We denote for a vertex $x \in V(K_n), c^+(x) = \sum_{a=(x,y)\in A(K_n)} c(a)$.

Lemma 3 (Obvious bounds) $n\mathcal{C}(n-1) \leq \mathcal{K}_{\mathcal{C}}(n) \leq 2n\mathcal{C}(n-1)$

PROOF. Lower bound: each vertex has to send a total flow of $n-1$, which requires at least a set of arcs starting from that vertex with cumulated capacity of $n-1$ i.e. $c^+(x) \geq n-1$ for all vertex x. As the cost function is concave, the cost of these arcs is at least $\mathcal{C}(n-1)$. Upper bound: it comes from the example of the star network, where a central vertex receives and sends the requests to all the others (see Figure 2 for the example with $n = 5$). $\square$

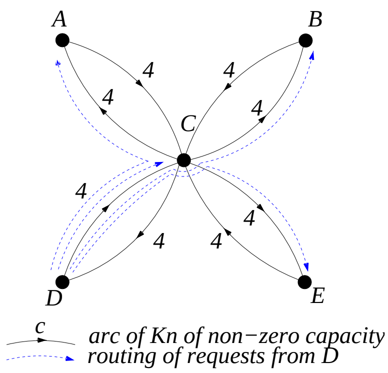

Figure 2: The star network.

In the lower bound of Lemma 3, we just take into account the first arc of each path starting from a vertex. But if each vertex has a high degree, it will be not efficient as the cost function is concave, then the number of vertices with high degree has to be small and then some additional capacity will be required. First we remark that, in an optimal n-Gossip network, there exist $\mathcal{O}(n)$ vertices that inform $\mathcal{O}(n)$ vertices along dipaths of length at least 2.

Lemma 4 *There exist a network of optimal cost with integral flow over all the arcs.*

PROOF. Suppose, for instance, in an optimal network of capacity c, two distinct dipaths P_1 and P_2 are both used to carry a flow of at least $\varepsilon > 0$ between a source and a destination. Let $\chi(P_i)$ (for $i = 1, 2$), be the function that equals 1 on the arcs of P_i and 0 otherwise, and consider $c' = c + \varepsilon\chi(P_1) - \varepsilon\chi(P_2)$ and $c'' = c + \varepsilon\chi(P_2) - \varepsilon\chi(P_1)$. Both c' and c'' are n-Gossip networks and optimal since $\mathcal{C}$ is concave. □

Notation : N_Δ denotes the number of vertices with degree at most Δ.

Lemma 5 *Let $\Delta(n) = \sqrt{2n\mathcal{C}(n)}$, then in any optimal n-Gossip network, if $N_{\Delta(n)}$ is the number of vertices of degree at most $\Delta(n)$, we have $N_{\Delta(n)} \geq n - \Delta(n)$.*

PROOF. $n - N_\Delta$ vertices of degree at least $\Delta + 1$ induce a cost of at least $(n - N_\Delta)(\Delta + 1)\mathcal{C}(1) = (n - N_\Delta)(\Delta + 1)$ (as Lemma 4 show that the flow is integral). As Lemma 3 gives $\mathcal{K}_\mathcal{C}(n) \leq 2n\mathcal{C}(n)$, we have $(n - N_\Delta)\Delta \leq 2n\mathcal{C}(n)$, so if $\Delta(n) = \sqrt{2n\mathcal{C}(n)}$ then $N_{\Delta(n)} \geq n - \Delta(n)$. □

Remark : As $\mathcal{C}(n) = o(n)$ (by definition), in an optimal Gossip, $\Delta(n) = o(n)$ and so $N_{\Delta(n)} = n + o(n)$.

To improve the lower bound, the argument will be to split the paths into two parts, an initial part which consists in the first arc of each path and induces a capacity function c_0, and the remaining part which consists in the other arcs of the paths and induces a capacity function c_1. The initial part (c_0) is such that for any x, $c_0^+(x) \geq n - 1$, while the remaining part allows almost to perform a Gossip, indeed c_1 informally allows each vertex to inform all the others starting from some subset of V whose size is at most its degree. In order to capture those two properties we define Quasi-Gossip networks.

4.1 Quasi-Gossip

In order to derive a lower bound we study a relaxed network, that we call *Quasi-Gossip*. Informally a Quasi-Gossip network is the superposition of two capacity functions c_0 and c_1, c_0 is such that every vertex have an outgoing capacity of $n - 1$ and c_1 allows to perform some (large) set of broadcast-like operations. We will see later that Quasi-Gossip networks cost more than Gossip networks. First we define a *t-Broadcast*:

Definition 6 (t-Broadcast) *A (K_n, c) network is called a **t-Broadcast** network if the capacity function c allows to perform a flow with a set of t distinct sinks, each sink receiving one unit of flow, and a set of sources distinct from the sinks.*

Remark : Note that, in a t-Broadcast, the capacity c can alternatively be seen as a flow from an hypothetical unique sender vertex to the sinks by using the set of sources as intermediary vertices.

Definition 7 (Quasi-Gossip) *A (K_n, c) network is called a (n, s, t)-Quasi-Gossip if there exists c_0 and c_1 such that*

- $c = c_0 + c_1$
- $c_0^+(x) \geq n - 1$ *for any x.*

- $c_1 = \sum_{i=1}^{s} c_{1,i}$ *where* $c_{1,i}$ *is the capacity function of a t-Broadcast. (ie:* c_1 *can be decomposed in s t-Broadcast.)*

Notation : We denote $QG_{\mathcal{C}}(n,s,t)$ the minimum cost of a (n,s,t)-Quasi-Gossip network for the cost function $\mathcal{C}$.

Lemma 8 *The cost of an optimal n-Gossip network is at least* $QG_{\mathcal{C}}(n, N_\Delta, n-1-\Delta)$.

$$\mathcal{K}_{\mathcal{C}}(n) \geq QG_{\mathcal{C}}(n, N_\Delta, n-1-\Delta)$$

PROOF. In a n-Gossip network, all N_Δ vertices x of degree less than Δ induce a t_x-broadcast with $t_x \geq n-1-\Delta$. Let the $c_{1,x}$ be the needed capacity for each t_x-broadcast and c_0 the needed capacity for the flow sent through their neighbors which has to be $c_0^+(x) \geq n-1$ for all x. Then the cost of a n-Gossip network is more than $QG_{\mathcal{C}}(n, N_\Delta, n-1-\Delta)$. □

4.2 Cost of a Quasi-Gossip

Definition 9 *We say that a directed Graph is a* ***forest*** *if it is acyclic and for each vertex v, there is at most one arc* (u,v) *directed to v.*

From this definition, we say that a vertex with no arc directed to it (but eventually some leaving it) is a *root.*

Lemma 10 *There exists a* (n,s,t)*-Quasi-Gossip with minimal cost where all the t-broadcasts are the same, moreover one can assume that the broadcast is performed using a forest.*

PROOF. Consider (K_n, c) an optimal Quasi-Gossip network and (c_0, c_1) a decomposition of c as described in Definition 7. By definition: $c_1 = c_{1,1} + c_{1,2} + \ldots + c_{1,s}$ where each function $c_{1,i}$ is a t-Broadcast. Since by definition the capacity c_1 of a Quasi-Gossip network needs only to be the sum of s t-Broadcasts, we note that any $c_0 + sc_{1,i}$, for $i = 1,2\ldots s$, is a Quasi-Gossip, so c is a convex combination of the Quasi-Gossips $c_0 + sc_{1,i}$. Since the cost function is concave, we conclude that, there exists an optimal Quasi-Gossip (K_n, c), where c_1 is s times the same t-Broadcast. We next consider such a particular optimal (n,s,t)-Quasi-Gossip. Assume that the undirected graph supporting the flow c_1 does not induce a forest, then there exists $\varepsilon > 0$ such that some vertex v receives a flow ε along at least two distinct dipaths P_1 and P_2. Let $\chi(P_i)$ (for $i = 1,2$), be the function that equals 1 on the arcs of P_i and 0 otherwise, and consider $c' = c_0 + c_1 + \varepsilon\chi(P_1) - \varepsilon\chi(P_2)$ and $c'' = c_0 + c_1 + \varepsilon\chi(P_2) - \varepsilon\chi(P_1)$. Both c', c'' are Quasi-Gossips, and c is their mean, so, as f is concave, we conclude that c' and c'' are also optimal Quasi-Gossips. □

We now evaluate the cost of an optimal Quasi-Gossip of the above form. Note that since f is increasing we can assume that the capacity corresponds to the capacity necessary to broadcast some information to t vertices using a forest.

Lemma 11 *The minimal cost of* (n,s,t)*-Quasi-Gossip,* $QG_{\mathcal{C}}(n,s,t)$*, is at least*

$$t \times \min_{L=1,2,\ldots,t} \frac{2\mathcal{C}(Ls) + \sum_{i=1}^{i=L-1} \mathcal{C}(is)}{L}$$

PROOF. We consider an optimal (K_n, c) Quasi-Gossip with $c = c_0 + c_1$ where c_1 describes a forest F as in Lemma 10. Assume that one knows c_1, then c_0 is almost completely determined, indeed the only constrain on c_0 is that $c_0^+(x) \geq n-1$ for any $x \in V$. Since f is concave the cheapest Quasi-Gossip is obtained by choosing $c_0(x,y) = n-1$ for any arc leaving x such that $c_1(x,y)$ is maximum. Note that there may exist several such arcs but in this case all the choices are equivalent, and one can compute the cost of any Quasi-Gossip simply looking at c_1. We consider one tree of the forest with root r. If there exists a vertex u (distinct from the root) with degree greater than 2, one can attach all of its subtrees directly to the root, excepted one with maximum $c_1(u,v)$ for all v's. This operation decreases the cost since the capacity function decreases if one consider that a capacity function c is greater than a capacity c' if, up to a permutation, $c - c' \geq 0$ (see Figure 3). In order

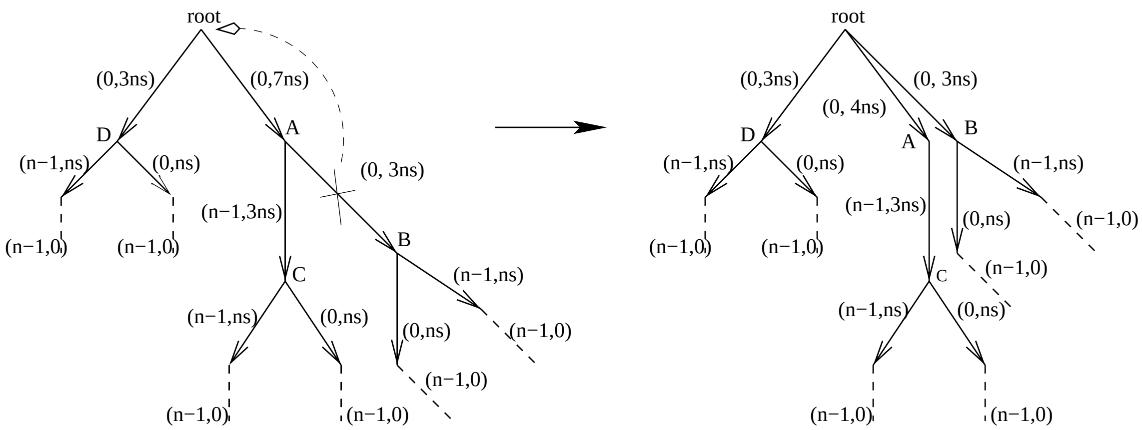

Figure 3: Moving a subtree to the root decreases the cost, each arc a is labeled with a couple $(c_0(a), c_1(a))$.

to simplify the counting we will never take into account the capacity c_0 for arcs leaving the root. Consider a branch with length l, the load on the arc attached to the root is ls, the load of the arc at distance $i > 0$ from the root is $n - 1 + (l - i)s \geq (l - i + 1)s$. So the total cost induced by the branch is at least $\mathcal{C}(ls) + \sum_{i=1}^{i=l} \mathcal{C}(is) = 2\mathcal{C}(ls) + \sum_{i=1}^{i=l-1} \mathcal{C}(is)$. Since this branch contains l sinks, we pay $\frac{1}{l}(2\mathcal{C}(ls) + \sum_{i=1}^{i=l-1} \mathcal{C}(is))$ per sink. Let $opt(s,t) = \min_{L=1,2,\ldots,t} \frac{2\mathcal{C}(Ls)+\sum_{i=1}^{i=L-1} \mathcal{C}(is)}{L}$ be the minimum cost per sink of a branch, the cost of any branch with l sinks is then at least $l \times opt(s,t)$, since there are t sinks, and since the cost of the Quasi-Gossip is the sum of the cost of the branches, we conclude that $QG_{\mathcal{C}}(n,s,t) \geq t \times opt(s,t)$. □

Theorem 12 *Let* $f(n) = n - 1 - \Delta(n)$,

$$\mathcal{K}_{\mathcal{C}}(n) \geq f(n) \min_{L=1,\ldots,f(n)} \left(\frac{2\mathcal{C}(Lf(n)) + \sum_{i=1}^{i=L-1} \mathcal{C}(if(n))}{L} \right)$$

Proof. Applying Lemma 8 and Corollary 11 we have

$$\mathcal{K}_{\mathcal{C}}(n) \geq (n - 1 - \Delta) \min_{L=1,2,\ldots,n-1-\Delta} \left(\frac{2\mathcal{C}(LN_\Delta) + \sum_{i=1}^{i=L-1} \mathcal{C}(iN_\Delta)}{L} \right)$$

applying it with $\Delta(n)$, Lemma 5 gives the result. □

Remark : Note that formula of Theorem 12 can be computed in polynomial time, moreover if one wishes for estimates, one can for large values L use the fact that $\int_{x=0}^{x=L-1} \mathcal{C}(f(n)x)\,dx \leq \sum_{i=1,2,\ldots,L} \mathcal{C}(f(n)x) \leq \int_{x=1}^{x=L} \mathcal{C}(f(n)x)$.

4.3 Proposed solutions

Proposition 13 $\mathcal{K}_{\mathcal{C}}(n) \leq (n-1)\min_{1 \leq k \leq n-1} \frac{k+1}{k}\mathcal{C}\left(nk - \frac{k(k+1)}{2}\right)$

PROOF. A solution where the digraph of non-zero capacity arcs is formed by cycles of the same size connected by one vertex seems to be a good construction. In this solution, routing using elementary paths is unique and so the capacity of each arc is the same and then the cost of this network can be easily estimated : let $k+1$ be the length of each cycle (suppose that $n-1$ is multiple of k), then the load on each cycle is uniform and is equal to $nk - k(k+1)/2$. Then each cycle costs $(k+1)\mathcal{C}(nk - k(k+1)/2)$ and as the number of cycle is $(n-1)/k$, the total cost of this network is $(n-1)\frac{k+1}{k}\mathcal{C}\left(nk - \frac{k(k+1)}{2}\right)$. Figure 4 gives some examples of this class of solutions. If $\mathcal{C}(c) = c^\alpha$ with $0 \leq \alpha \leq 1$, the cost is minimal when $1/\alpha = (k+1)(1-\frac{k}{2n-k-1})$, i.e. for $k = 1/\alpha - 1$. As $k+1$ is the length of each cycle, k has to be greater than 1 and then the optimal construction of that type, if $\alpha \geq 1/2$, is the star network (all cycles of length 2). Then the minimal cost of this kind of construction is $\frac{1}{1-\alpha}\left(\frac{1-\alpha}{\alpha}\right)^\alpha n^{1+\alpha} + o(n^{1+\alpha})$. □

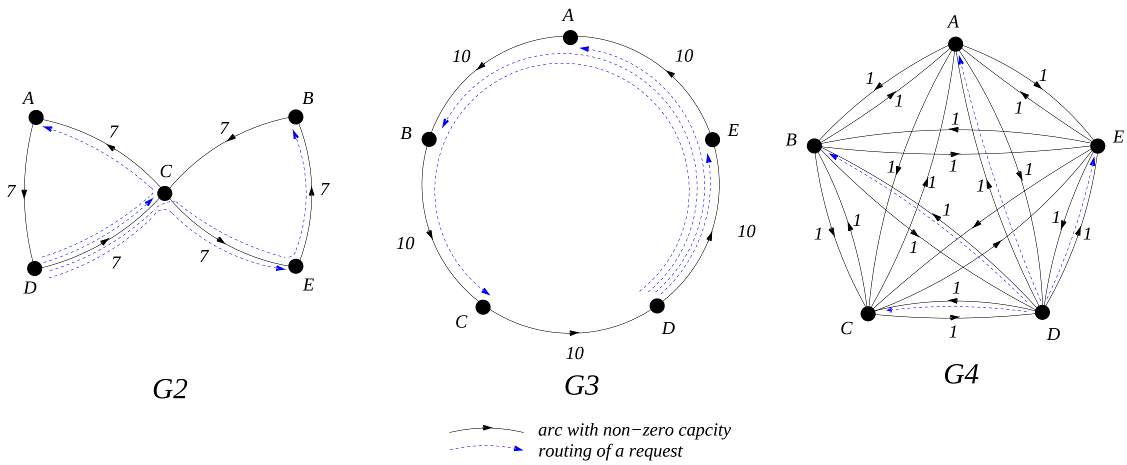

Figure 4: $G2$ and $G3$: constructions using cycles of same length. $G4$: optimal construction for $\mathcal{C}(n) = n$.

Now we only compare our bound when $\mathcal{C}(n)$ is of the form n^α with $\alpha \in [0,1]$, for this we study how varies the efficiency of a branch.

4.4 Gossip network cost with $\mathcal{C}(n) = n^\alpha$

4.4.1 Lower bound

When the cost function is $\mathcal{C} : n \mapsto n^\alpha$ we have $\frac{2\mathcal{C}(Lf(n)) + \sum_{i=1}^{i=L-1}\mathcal{C}(if(n))}{L} = f^\alpha(n)\frac{2L^\alpha + \sum_{i=1}^{i=L-1} i^\alpha}{L}$, simple analysis of $\frac{2L^\alpha + \sum_{i=1}^{i=L-1} i^\alpha}{L}$ shows that the value L_0 for which the minimum is attained increase when α decrease. For $\alpha \in [\log_2(\frac{3}{2}), 1]$ the minimum is 2, attained for $L = 1$; for $\alpha \in [0.42, \log_2\frac{3}{2} \sim 0.58]$, the minimum is $\frac{2^{\alpha+1}+1}{2}$, attained for $L = 2$ and as long as the minimum is not attained for $L = 3$, which happens for $\alpha \sim 0.42$; and so on, Table 1 gives the intervals of α and its corresponding optimal value L_0.

4.4.2 Upper bound

We can do the same computation for the upper bound and find the best size of the cycle length $(k+1)$ according to the value of α. Table 2 gives the intervals of α for first values of k.

α	L_0
$[0.584\ldots,1]$	1
$[0.421\ldots,0.584\ldots]$	2
$[0.321\ldots,0.421\ldots]$	3
$[0.255\ldots,0.321\ldots]$	4
$[0.210\ldots,0.255\ldots]$	5
$[0.177\ldots,0.210\ldots]$	6
$[0.153\ldots,0.177\ldots]$	7
$[0.134\ldots,0.153\ldots]$	8
$\ldots$	$\ldots$

Table 1: Intervals of α for which the value L_0 minimizes $\frac{2L^\alpha+\sum_{i=1}^{i=L-1} i^\alpha}{L}$.

α	k
$[0.415\ldots,1]$	1
$[0.290\ldots,0.415\ldots]$	2
$[0.224\ldots,0.290\ldots]$	3
$[0.183\ldots,0.224\ldots]$	4
$[0.157\ldots,0.183\ldots]$	5
$[0.134\ldots,0.157\ldots]$	6
$[0.118\ldots,0.134\ldots]$	7
$[0.105\ldots,0.118\ldots]$	8
$\ldots$	$\ldots$

Table 2: Intervals of α for which the value k minimizes $\min_{1\leq k\leq n-1} \frac{k+1}{k}k^\alpha$.

4.4.3 Proposed solutions versus lower bound

As $f(n)=n^\alpha+o(n^\alpha)$ the lower bound is

$$n^{(1+\alpha)} \min_{1\leq L\leq f(n)} \frac{2L^\alpha+\sum_{i=1}^{i=L-1} i^\alpha}{L} + o(n^{(1+\alpha)})$$

and the upper bound, which is $(n-1)\min_{1\leq k\leq n-1} \frac{k+1}{k}\left(nk-\frac{k(k+1)}{2}\right)^\alpha$, is less than

$$n^{(1+\alpha)} \min_{k\geq 1} \frac{k+1}{k}k^\alpha + o(n^{(1+\alpha)})$$

Both of these minimum do not depend of n if n is large enough. So, for each value of α, the asymptotical ratio between the upper and the lower bound is bounded by a constant. Numerical analysis in Figure 5 indicates that this ratio is less than 1.12.

5 Conclusion

In this paper we have investigated an important special case of network design with concave costs. The results obtained show that the general shape of the optimal network varies significantly with the type of concavity we study. In terms of network design, this result impacts the understanding of much more than the SDH networks. Of course a natural question concerns other types of demands than all-to-all requests, and other types of costs than uniform costs over all the edges. Further insight in that field would greatly help the telecommunication operators and other actors in the transportation industry to have a more efficient management of their resources.

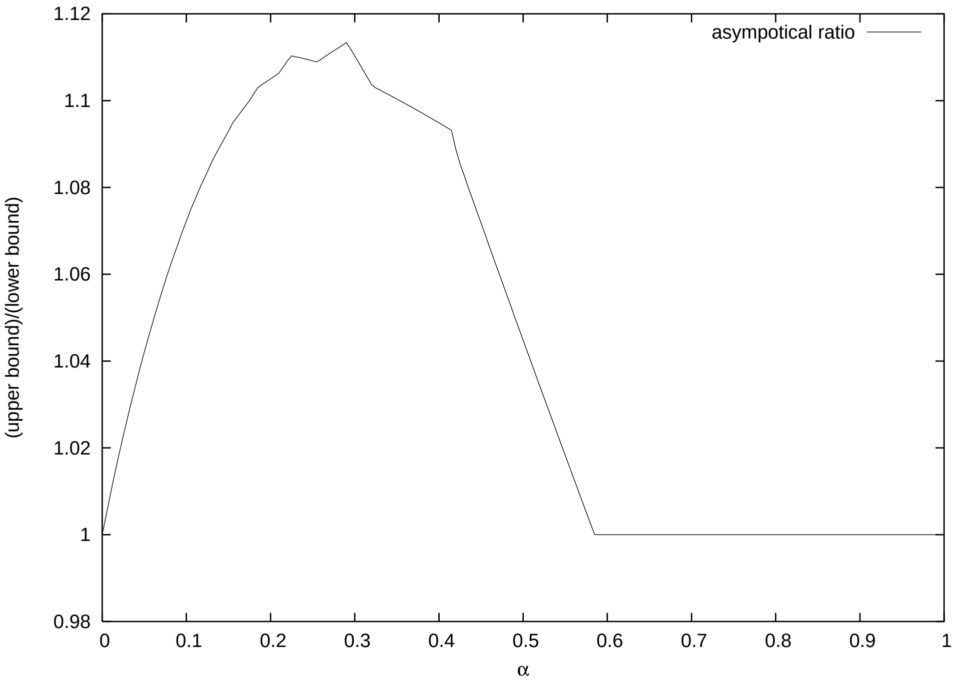

Figure 5: Asymptotical ratio (upper bound)/(lower bound) according to α.

References

[1] D. Bienstock, S. Chopra, O. Günlük, and C.-Y. Tsai. Minimum cost capacity installation for multicommodity network flows. *Mathematical Programming*, 81:177–199, 1998.

[2] B. Brockmüller, O. Günlük, and L.A. Wolsey. Designing Private Line Networks, 1999.

[3] V. Gabrel and M. Minoux. Large Scale LP Relaxations for Minimum Cost Multicommodity Flow Problems with Step Increasing Cost Functions and Computational Results. Technical Report 96/17, Laboratoire MASI, Université Paris 6, France, 1996.

[4] A. Ouorou, P. Mahey, and J.-P. Vial. A survey of algorithms for convex multicommodity flow problems. *Management Science vol.46, 1*, pages 126–147, 2000.

[5] M. Prytz. Lagrangian decomposition for general step cost single path network design. Technical Report TRITA-MAT-2002-OS03, Royal Institute of Technology, Swedish, 2002.

[6] A. Schrijver, P. Seymour, and P. Winkler. The ring loading problem. *SIAM Journal on Discrete Mathematics*, 11(1):1–14, 1998.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399